National Curriculum (Vocational) Mathematics Level 4

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DYLAN BUSA AND NATASHIA BEARAM-EDMUNDS

Contents

SUBJECT OUTCOME I

COMPLEX NUMBERS: WORKING WITH COMPLEX NUMBERS

Subject outcome

Subject outcome 1.1: Work with complex numbers

Learning outcomes

- Perform addition, subtraction, multiplication and division on complex numbers in standard form (includes i-notation).
  Note: Leave answers with positive argument.
- Perform multiplication and division on complex numbers in polar form.
- Use De Moivre's theorem to raise complex numbers to powers (excluding fractional powers).
- Convert the form of complex numbers where needed to enable performance of advanced operations on complex numbers (a combination of standard and polar form may be assessed in one expression).

Unit 1 outcomes

By the end of this unit you will be able to:

- Add complex numbers in standard form.
- Subtract complex numbers in standard form.
- Multiply complex numbers in standard form.
- Divide complex numbers in standard form through the use of a suitable conjugate.

Unit 2 outcomes

By the end of this unit you will be able to:

- Plot a complex number on the complex plan.
- Find the absolute value of a complex number.
- Convert a complex number from standard (or rectangular) form to polar form.

- Convert a complex number from polar form to standard (or rectangular) form.
- Understand what is meant by the abbreviation when dealing with complex numbers in polar

Unit 3 outcomes

By the end of this unit you will be able to:

- Multiply complex numbers in polar form.
- Divide complex numbers in polar form.

Unit 4 outcomes

By the end of this unit you will be able to:

- Find the powers of complex numbers in polar form.
- Simplify complex expressions with powers.

Unit 1: Revise the basic operations with complex numbers in standard form

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Unit outcomes

By the end of this unit you will be able to:

- Add complex numbers in standard form.
- Subtract complex numbers in standard form.
- Multiply complex numbers in standard form.
- Divide complex numbers in standard form through the use of a suitable conjugate.

What you should know

Before you start this unit, make sure you can:

- Define complex numbers. Refer to level 3 subject outcome 1.1 unit 1 if you need help with this.
- Represent complex numbers in standard rectangular coordinate form. Refer to level 3 subject outcome 1.1 unit 1 if you need help with this.
- Perform basic operations on imaginary numbers. Refer to level 3 subject outcome 1.1 unit 1 if you need help with this.
- Perform addition, subtraction and multiplication on complex numbers in standard/rectangular form. Refer to level 3 subject outcome 1.2 unit 1 if you need help with this.
- Perform division on complex numbers in standard form introducing the concept of conjugate. Refer to level 3 subject outcome 1.2 unit 1 if you need help with this.

Introduction

This subject outcome and this unit revise and build on the work you did on complex numbers in level 3 subject outcomes 1.1 and 1.2. It is important that you complete these subject outcomes before continuing.

By this stage, you should recognise the different types of numbers, as shown in Figure 1. It shows the classification of all the different kinds of **real numbers** beginning with the counting or natural numbers, expanding these to include zero (the whole numbers), the negative counting numbers (the integers), the fractions (rational numbers), and finally, those numbers that cannot be written as fractions (the irrational numbers).

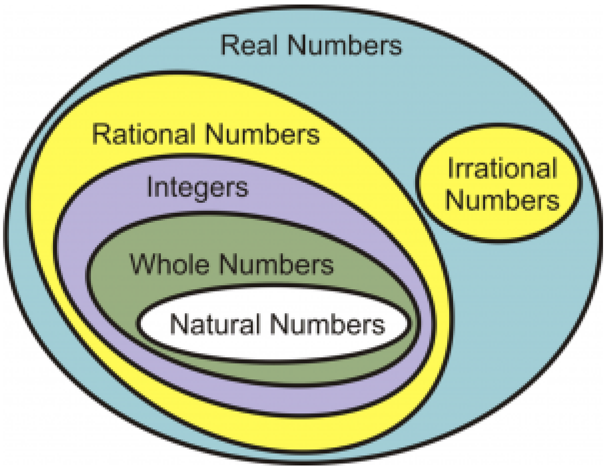


Figure 1: The real numbers

But you know that this is not the whole story. There is an even bigger set of numbers that includes the so-called imaginary numbers. This set is called the **complex numbers** (see Figure 2). As far as we know, the complex numbers do include everything.

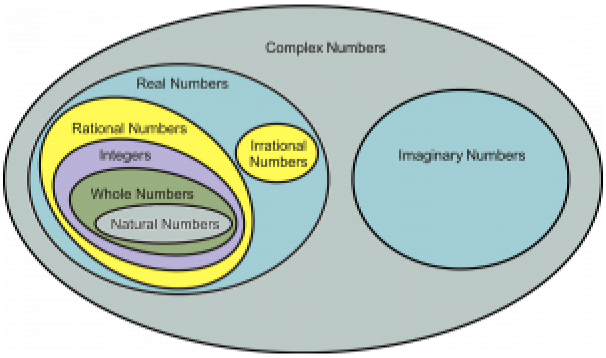


Figure 2: The complex numbers

The name 'imaginary numbers' is actually quite unfortunate because these numbers crop up in all sorts of very 'real' places, from electricity to bridge-building, from car design to the flow of liquids.

Note

Before going further, if you have an internet connection, watch this short video explaining why complex numbers are awesome; "Complex Numbers are Awesome".

Complex Numbers are Awesome (Duration: 3.45)

When it comes to working with complex numbers (adding, subtracting, multiplying and dividing them), the good news is that all of the rules we have learnt about adding, subtracting, multiplying and dividing real numbers work in much the same way. We don't need to learn a whole new set of mathematical techniques.

Imaginary numbers

In level 3 subject outcome 1.1 unit 1, we learnt about imaginary numbers and that the imaginary number i is defined as $\sqrt{-1}$. In other words, $\sqrt{-1} = i$. This means that $i^2 = \left(\sqrt{-1}\right)^2 = -1$.

This definition allows us to determine the value of numbers like $\sqrt{-49}$.

$$\begin{aligned}\sqrt{-49} &= \sqrt{49 \times (-1)} \\ &= \sqrt{49} \times \sqrt{-1} \\ &= 7i\end{aligned}$$

We say that $7i$ is an imaginary number.

Exercise 1.1

Write the following negative roots as multiples of i:

1. $\sqrt{-25}$
2. $\sqrt{-2}$
3. $\sqrt{-12}$
4. $\sqrt{-400}$
5. $\sqrt{-24}$

The full solutions are at the end of the unit.

What do you think happens when we square an imaginary number? Try it by working through some examples.

Example 1.1

Simplify $(8i)^2$.

Solution

$$\begin{aligned}(8i)^2 &= 8i \times 8i \\ &= 8 \times 8 \times i \times i \\ &= 64 \times i^2\end{aligned}$$

But $i^2 = -1$. Therefore, $64 \times i^2 = 64 \times (-1) = -64$.

Example 1.2

Simplify $\frac{4i^3 \times 3i^2}{6i^4}$.

Solution

$$\begin{aligned}\frac{4i^3 \times 3i^2}{6i^4} &= \frac{4 \times 3 \times i^3 \times i^2}{6 \times i^4} && \text{We can use exponent laws: } i^3 \times i^2 = i \times i \times i \times i \times i = i^{3+2} = i^5 \\ &= \frac{12 \times i^5}{6i^4} && \text{We can use exponent laws: } \frac{i^5}{i^4} = \frac{i \times i \times i \times i \times i}{i \times i \times i \times i} = i^{5-4} = i \\ &= 2i\end{aligned}$$

Exercise 1.2

Simplify the following:

1. $(9i)^2$
2. $\left(\sqrt{12}i\right)^2$
3. $\left(3\sqrt{3}i\right)^2$
4. $(4i)^3$
5. $\left(2\sqrt{3}i\right)^3$
6. $\left(2\sqrt[3]{2}i\right)^4$
7. $\frac{7i^3.3i^8.2i^5}{21i^5.3i^5}$
8. $\frac{\sqrt{12}i^2.(-4i).3i^5}{\sqrt{-24}.6i^4}$

The full solutions are at the end of the unit.

Complex numbers

A complex number is simply the sum of a real number and an imaginary number. Therefore, it has two parts to it – a real part and an imaginary part. Figure 3 shows an example of a complex number.

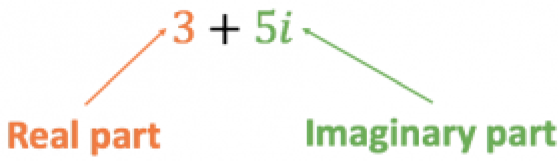


Figure 3: A complex number

Take note!

A **complex number** is a number of the form $a + bi$ where:

- a is the real part of the complex number
- bi is the imaginary part of the complex number.

If $b = 0$, then $a + bi$ becomes just a and the number is a pure **real number**.

If $a = 0$ and $b \neq 0$, then $a + bi$ becomes just bi and the number is a pure **imaginary number**.

Complex numbers written in the form $a + bi$ are said to be written in **standard form**.

We can convert complex numbers from **root form** to **standard form** and back again by remembering that $i = \sqrt{-1}$ and that $i^2 = -1$.

Example 1.3

Write $5 + \sqrt{-3}$ in standard form.

Solution

$$\begin{aligned} 5 + \sqrt{-3} &= 5 + \sqrt{3 \times (-1)} \\ &= 5 + \sqrt{3} \times \sqrt{-1} \\ &= 5 + \sqrt{3}i \end{aligned}$$

Example 1.4

Write $7 - 7i$ in root form.

Solution

$$\begin{aligned} 7-7i &= 7-\sqrt{49}\cdot\sqrt{-1} \\ &= 7-\sqrt{49\cdot(-1)} \\ &= 7-\sqrt{-49} \end{aligned}$$

Some people refer to think of imaginary numbers as **non-real numbers** to set them apart from the real numbers, but imaginary numbers are actually very real. The name 'imaginary numbers' is quite an unfortunate mistake of history.

Note

If you have an internet connection, watch this excellent playlist of videos called "Imaginary Numbers are Real" (13 videos) to learn how this happened.

Imaginary Numbers are Real (13 videos)

Add and subtract complex numbers

Adding and subtracting complex numbers is actually pretty simple. The basic rule is that we have to add or subtract the real parts and then add or subtract the imaginary parts. Have a look at the next example.

Example 1.5

Simplify $(3-4i)+(2+5i)$.

Solution

$$\begin{aligned} (3-4i)+(2+5i) &= 3-4i+2+5i \\ &= 3+2-4i+5i \\ &= (3+2)+(-4+5)i \\ &= 5+i \end{aligned}$$

Note that all our standard expansion rules apply, for example $-(5+4i)=-5-4i$.

Example 1.6

Simplify $(-5+2i)-(-11+7i)$.

Solution

$$\begin{aligned}(-5+2i)-(-11+7i) &= -5+2i+11-7i\\ &= -5+11+2i-7i\\ &= (-5+11)+(2i-7i)\\ &= 6-5i\end{aligned}$$

Exercise 1.3

Add or subtract the following complex numbers:

1. $(-2-4i)+(1+6i)$
2. $(-5+3i)-(6-i)$
3. $(2-3i)-(3+2i)$
4. $(-4+4i)-(-6+9i)$
5. $(5+4i)-(3+2i)-(8-7i)$
6. $\sqrt{-12}+\sqrt{-27}$
7. $\sqrt{-32}-\sqrt{-48}$

The full solutions are at the end of the unit.

Adding complex numbers:
$(a+bi)+(c+di)=(a+c)+(b+d)i$

Subtracting complex numbers:
$(a+bi)-(c+di)=(a-c)+(b-d)i$

Multiply complex numbers

Multiplying complex numbers is very similar to expanding binomials (see Figure 4). The only difference is that we work with the real and imaginary parts separately just like we do when adding and subtracting complex numbers.

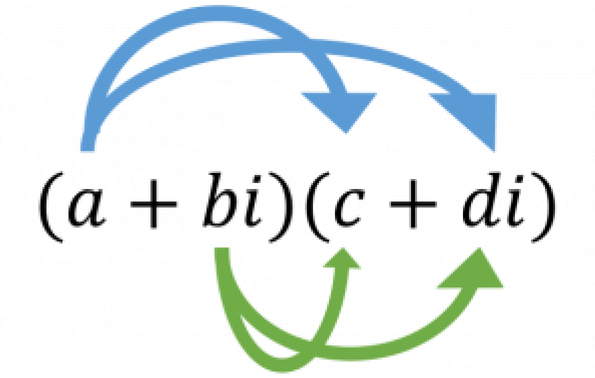


Figure 4: Multiplying complex numbers

Example 1.7

Simplify $(3+2i)(6+3i)$.

Solution

$$\begin{aligned}(3+2i)(6+3i) &= (3\times 6)+(3\times 3i)+(2i\times 6)+(2i\times 3i)\\ &= 18+9i+12i+6i^2 \qquad \text{Remember } i^2=-1\\ &= 18+9i+12i+6(-1)\\ &= (18-6)+(9+12)i\\ &= 12+21i\end{aligned}$$

Example 1.8

Simplify $(4+3i)(2-5i)$.

Solution

$$\begin{aligned}(4+3i)(2-5i) &= (4\times 2)+(4\times(-5i))+(3i\times 2)+(3i\times(-5i))\\ &= 8-20i+6i-15i^2 \qquad \text{Remember } i^2=-1\\ &= 8-20i+6i+15\\ &= (8+15)+(-20+6)i\\ &= 23-14i\end{aligned}$$

Exercise 1.4

Simplify the following:

1. $(3-4i)(2+3i)$
2. $(2+3i)(4-i)$
3. $(-1+2i)(-2+3i)$
4. $(4+3i)^2$
5. $(3+4i)(3-4i)$
   What kind of expression does this remind you of?
6. $\left(3-\sqrt{-12}\right)\left(5+\sqrt{-27}\right)$

The full solutions are at the end of the unit.

Divide complex numbers

Dividing complex numbers is the most challenging of the complex number operations. However, it is still reasonably simple, so long as you know what a **conjugate** is. We first came across conjugates when simplifying algebraic expressions with binomial denominators.

Have a look at question 5 in Exercise 1.4 again. We were asked to multiply $(3+4i)(3-4i)$ and found that the answer was a real number. There was no imaginary part. We call $(3-4i)$ the complex conjugate of $(3+4i)$.

A complex conjugate is the complex number that we need to multiply another complex number by to get rid of the imaginary part. The complex conjugate of $(a+bi)$ is $(a-bi)$.

Therefore, finding complex conjugates is easy. You just need to change the sign in the complex number. The complex conjugate of $(4-i)$ is $(4+i)$. Quickly multiply these two complex numbers together to make sure that the answer is a pure real number. Did you get 17?

When you divide by a complex number, you need to multiply it by its complex conjugate. However, to keep the value of the fraction the same, you must multiply the numerator by the same complex conjugate as well.

Example 1.9

Simplify $\frac{2+5i}{4-i}$.

Solution

We are dividing by a complex number. Therefore, we need to multiply the denominator by its complex conjugate $(4+i)$. However, to keep the value of the fraction the same we need to multiply the numerator by the same complex conjugate. We will, therefore, multiply the fraction by $\frac{(4+i)}{(4+i)}=1$.

$$\begin{aligned}\frac{2+5i}{4-i} &= \frac{(2+5i)}{(4-i)} \times \frac{(4+i)}{(4+i)} \\ &= \frac{8+2i+20i+5i^2}{16+4i-4i-i^2} \quad \text{Remember that } i^2 = -1 \\ &= \frac{8+22i-5}{16+1} \\ &= \frac{3+22i}{17}\end{aligned}$$

The final step is to write the answer in standard $a+bi$ form.

$$\frac{3+22i}{17} = \frac{3}{17} + \frac{22}{17}i$$

Example 1.10

Simplify $\frac{2+5i}{5-2i}$.

Solution

The complex conjugate of $(5-2i)$ is $(5+2i)$.

$$\begin{aligned}\frac{2+5i}{5-2i} &= \frac{(2+5i)}{(5-2i)} \times \frac{(5+2i)}{(5+2i)} \\ &= \frac{10+4i+25i+10i^2}{25-4i^2} \\ &= \frac{10+29i-10}{25+4} \\ &= \frac{29i}{29} \\ &= i\end{aligned}$$

Exercise 1.5

Simplify the following, leaving your answer in standard form:

1. $\frac{3}{5+2i}$
2. $\frac{5+3i}{4i}$
3. $\frac{3+4i}{2-i}$

4. $\dfrac{2+\sqrt{-12}}{\sqrt{-5}}$
5. $\dfrac{-\sqrt{-4}-4\sqrt{-25}}{5+i}$
6. $\dfrac{(1+3i)(2-4i)}{1+2i}$
7. $\dfrac{4+i}{i}+\dfrac{3-4i}{1-i}$

The full solutions are at the end of the unit.

Summary

In this unit you have learnt the following:

- What an imaginary number is.
- How to add, subtract, multiply and divide imaginary numbers.
- What a complex number is.
- How to add, subtract and multiply complex numbers.
- How to divide complex numbers using a complex conjugate.

Unit 1: Assessment

Suggested time to complete: 45 minutes

Simplify the following expressions, leaving your answer in standard form:

1. $(3-4i)+(2+5i)$
2. $(5-\sqrt{-50})-(3-\sqrt{-8})$
3. $(5-\sqrt{-50})(3-\sqrt{-8})$
4. $(-5+7i)-(-11+2i)+(-3-6i)^2$
5. $(-3-5i)^2\cdot 6i$
6. $\dfrac{4-8i}{3-\sqrt{3}i}$
7. $(2+i)^2-(3-i)^2$
8. $(-1+\sqrt{-3})^2$
9. $i^2(2+7i)+i(3-6i)-16-i$
10. $\dfrac{3-2i}{1+i}-\dfrac{1-3i}{1-i}$

The full solutions are at the end of the unit.

Unit 1: Solutions

Exercise 1.1

1. $\sqrt{-25} = \sqrt{25 \times (-1)} = \sqrt{25} \times \sqrt{-1} = 5i$
2. $\sqrt{-2} = \sqrt{2 \times (-1)} = \sqrt{2} \times \sqrt{-1} = \sqrt{2}i$
3. $\sqrt{-12} = \sqrt{12 \times (-1)} = \sqrt{12} \times \sqrt{-1} = \sqrt{12}i = 2\sqrt{3}i$
4. $\sqrt{-400} = \sqrt{400 \times (-1)} = \sqrt{400} \times \sqrt{-1} = 20i$
5. $\sqrt{-24} = \sqrt{24 \times (-1)} = \sqrt{24} \times \sqrt{-1} = 2\sqrt{6}i$

Back to Exercise 1.1

Exercise 1.2

1.
$$\begin{aligned}(9i)^2 &= 81i^2\\ &= 81 \times (-1)\\ &= -81\end{aligned}$$

2.
$$\begin{aligned}\left(\sqrt{12}i\right)^2 &= 12i^2\\ &= 12 \times (-1)\\ &= -12\end{aligned}$$

3.
$$\begin{aligned}\left(3\sqrt{3}i\right)^2 &= 9 \cdot 3 \cdot i^2\\ &= 27i^2\\ &= 27 \cdot (-1)\\ &= -27\end{aligned}$$

4.
$$\begin{aligned}(4i)^3 &= 64 \cdot i^3\\ &= 64 \cdot i^2 \cdot i\\ &= -64i\end{aligned}$$

5.
$$\begin{aligned}\left(2\sqrt{3}i\right)^3 &= 8 \cdot 9\sqrt{3} \cdot i^3\\ &= 72\sqrt{3} \cdot i^2 \cdot i\\ &= -72\sqrt{3}i\end{aligned}$$

6.
$$\begin{aligned}\left(2\sqrt[3]{2}i\right)^4 &= 2^4 \cdot \left(\sqrt[3]{2}\right)^4 \cdot i^4\\ &= 2^4 \cdot 2^{\frac{4}{3}} \cdot i^2 \cdot i^2 \qquad \left[i^2 \times i^2 = (-1) \times (-1) = 1\right]\\ &= 2^{4+\frac{4}{3}}\\ &= 2^{\frac{16}{3}}\end{aligned}$$

7.

$$\begin{aligned}\frac{7i^3.3i^8.2i^5}{21i^5.3i^5} &= \frac{42i^{16}}{63i^{10}} \\ &= \frac{2i^4}{3} \quad \left[i^2 \times i^2 = (-1) \times (-1) = 1\right] \\ &= \frac{2}{3}\end{aligned}$$

8.

$$\begin{aligned}\frac{\sqrt{12}i^2 \cdot (-4i) \cdot 3i^5}{\sqrt{-24} \cdot 6i^4} &= \frac{2\sqrt{3}i^2 \cdot (-12i^6)}{\sqrt{24}i \cdot 6i^4} \\ &= \frac{-24\sqrt{3}i^8}{12\sqrt{6}i^5} \\ &= \frac{-2\sqrt{3}i^3}{\sqrt{2}\sqrt{3}} \\ &= \frac{2i}{\sqrt{2}} \quad \text{Multiply by } \frac{\sqrt{2}}{\sqrt{2}} \\ &= \frac{2\sqrt{2}i}{2} \\ &= \sqrt{2}i\end{aligned}$$

Back to Exercise 1.2

Exercise 1.3

1.

$$\begin{aligned}(-2-4i)+(1+6i) &= (-2+1)+(-4+6)i \\ &= -1+2i\end{aligned}$$

2.

$$\begin{aligned}(-5+3i)-(6-i) &= -5+3i-6+i \\ &= (-5-6)+(3+1)i \\ &= -1-5i\end{aligned}$$

3. .

$$\begin{aligned}(2-3i)-(3+2i) &= 2-3i-3-2i \\ &= (2-3)+(-3-2)i \text{ /li>} \\ &= -1+2i\end{aligned}$$

4.

$$\begin{aligned}(-4+4i)-(-6+9i) &= -4+4i+6-9i \\ &= (-4+6)+(4-9)i \\ &= 2-5i\end{aligned}$$

5.

$$\begin{aligned}(5+4i)-(3+2i)-(8-7i) &= 5+4i-3-2i-8+7i \\ &= (5-3-8)+(4-2+7)i \\ &= -6+9i\end{aligned}$$

6.

$$\begin{aligned}\sqrt{-12}+\sqrt{-27} &= \sqrt{12}i+\sqrt{27}i \\ &= 2\sqrt{3}i+3\sqrt{3}i \\ &= 5\sqrt{3}i\end{aligned}$$

7.

$$\begin{aligned}\sqrt{-32}-\sqrt{-48} &= \sqrt{32}i-\sqrt{48}i \\ &= 4\sqrt{2}i-4\sqrt{3}i\end{aligned}$$

Back to Exercise 1.3

Exercise 1.4

1.
$$\begin{aligned}(3-4i)(2+3i) &= 6+9i-8i-12i^2\\ &= 6+i+12\\ &= 18+i\end{aligned}$$

2.
$$\begin{aligned}(2+3i)(4-i) &= 8-2i+12i-3i^2\\ &= 8+10i+3\\ &= 11+10i\end{aligned}$$

3.
$$\begin{aligned}(-1+2i)(-2+3i) &= 2-3i-4i+6i^2\\ &= 2-7i-6\\ &= -4-7i\end{aligned}$$

4.
$$\begin{aligned}(4+3i)^2 &= (4+3i)(4+3i)\\ &= 16+12i+12i+9i^2\\ &= 16+24i-9\\ &= 7+24i\end{aligned}$$

5.
$$\begin{aligned}(3+4i)(3-4i) &= 9-12i+12i-16i^2\\ &= 9+16\\ &= 25\end{aligned}$$

This is similar to a difference of two squares.

6.
$$\begin{aligned}\left(3-\sqrt{-12}\right)\left(5+\sqrt{-27}\right) &= \left(3-2\sqrt{3}i\right)\left(5+3\sqrt{3}i\right)\\ &= 15+9\sqrt{3}i-10\sqrt{3}i-18i^2\\ &= 15-\sqrt{3}i+18\\ &= 33-\sqrt{3}i\end{aligned}$$

Back to Exercise 1.4

Exercise 1.5

1.
$$\begin{aligned}\frac{3}{5+2i} &= \frac{3}{5+2i}\times\frac{5-2i}{5-2i}\\ &= \frac{15-6i}{25-4i^2}\\ &= \frac{15-6i}{29}\\ &= \frac{15}{29}-\frac{6}{29}i\end{aligned}$$

2.

$$\frac{5+3i}{4i} = \frac{5+3i}{4i} \times \frac{4i}{4i}$$
$$= \frac{20i+12i^2}{16i^2}$$
$$= \frac{-12+20i}{-16}$$
$$= \frac{3}{4} - \frac{5}{4}i$$

3.

$$\frac{2+\sqrt{-12}}{\sqrt{-5}} = \frac{2+2\sqrt{3}i}{\sqrt{5}i}$$
$$= \frac{\left(2+2\sqrt{3}i\right)}{5i} \times \frac{5i}{5i}$$
$$= \frac{2\sqrt{5}i+2\sqrt{15}i^2}{5i^2}$$
$$= \frac{-2\sqrt{15}+2\sqrt{5}i}{-5}$$
$$= \frac{2\sqrt{15}}{5} - \frac{2\sqrt{5}i}{5}$$

4.

Formula does not parse

5.

$$\frac{-\sqrt{-4}-4\sqrt{-25}}{5+i} = \frac{-2i-20i}{5+i}$$
$$= \frac{(-22i)}{(5+i)} \times \frac{(5-i)}{(5-i)}$$
$$= \frac{-110i+22i^2}{25-i^2}$$
$$= \frac{-22-110i}{26}$$
$$= -\frac{11}{13} - \frac{55}{13}i$$

6.

$$\frac{(1+3i)(2-4i)}{1+2i} = \frac{2-4i+6i-12i^2}{1+2i}$$
$$= \frac{2+2i+12}{1+2i}$$
$$= \frac{(14+2i)}{(1+2i)} \times \frac{(1-2i)}{(1-2i)}$$
$$= \frac{14-28i+2i-4i^2}{1-4i^2}$$
$$= \frac{14-26i+4}{5}$$
$$= \frac{18-26i}{5}$$
$$= \frac{18}{5} - \frac{26}{5}i$$

7.

$$\begin{aligned}\frac{4+i}{i}+\frac{3-4i}{1-i} &= \frac{(4+i)(1-i)+i(3-4i)}{i(1-i)} \qquad \text{LCD: } i(1-i)\\ &= \frac{4-4i+i-i^2+3i-4i^2}{i-i^2}\\ &= \frac{4-3i+1+3i+4}{i+1}\\ &= \frac{9}{(1+i)}\times\frac{(1-i)}{(1-i)}\\ &= \frac{9-9i}{1-i^2}\\ &= \frac{9-9i}{2}\\ &= \frac{9}{2}-\frac{9}{2}i\end{aligned}$$

Back to Exercise 1.5

Unit 1: Assessment

1.
$$\begin{aligned}(3-4i)+(2+5i) &= 3-4i+2+5i\\ &= 5+i\end{aligned}$$

2.
$$\begin{aligned}\left(5-\sqrt{-50}\right)-\left(3-\sqrt{-8}\right) &= 5-\sqrt{50}i-3+\sqrt{8}i\\ &= 5-5\sqrt{2}i-3+2\sqrt{2}i\\ &= 2-3\sqrt{2}i\end{aligned}$$

3.
$$\begin{aligned}\left(5-\sqrt{-50}\right)\left(3-\sqrt{-8}\right) &= \left(5-\sqrt{50}i\right)\left(3-\sqrt{8}i\right)\\ &= \left(5-5\sqrt{2}i\right)\left(3-2\sqrt{2}i\right)\\ &= 15-10\sqrt{2}i-15\sqrt{2}i+10\cdot 2i^2\\ &= 15-25\sqrt{2}i-20\\ &= -5-25\sqrt{2}i\end{aligned}$$

4.
$$\begin{aligned}(-5+7i)-(-11+2i)+(-3-6i)^2 &= -5+7i+11=2i+(-3-6i)(-3-6i)\\ &= 6+5i+9+18i+18i+36i^2\\ &= 15+41i-36\\ &= -21+41i\end{aligned}$$

5.
$$\begin{aligned}(-3-5i)^2\cdot 6i &= (-3-5i)(-3-5i)\cdot 6i\\ &= (9+15i+15i+25i^2)\cdot 6i\\ &= (9+30i-25)\cdot 6i\\ &= (-16+30i)\cdot 6i\\ &= -96i+180i^2\\ &= 180-96i\end{aligned}$$

6.

$$\begin{aligned}\frac{4-8i}{3-\sqrt{3}i} &= \frac{(4-8i)}{\left(3-\sqrt{3}i\right)} \times \frac{\left(3+\sqrt{3}i\right)}{\left(3+\sqrt{3}i\right)} \\ &= \frac{12+4\sqrt{3}i-24i-8\sqrt{3}i^2}{9-3i^2} \\ &= \frac{12+4\sqrt{3}i-24i+8\sqrt{3}}{9+3} \\ &= \frac{12+8\sqrt{3}+4\sqrt{3}i-24i}{12} \\ &= \frac{4\left(3+2\sqrt{3}\right)+4\left(\sqrt{3}-6\right)i}{12} \\ &= \frac{4\left(3+2\sqrt{3}\right)}{12}+\frac{4\left(\sqrt{3}-6\right)}{12}i \\ &= \frac{3+2\sqrt{3}}{3}+\frac{\left(\sqrt{3}-6\right)}{3}i\end{aligned}$$

7.
$$\begin{aligned}(2+i)^2-(3-i)^2 &= (2+i)(2+i)-(3-i)(3-i) \\ &= 4+2i+2i+i^2-(9-3i-3i+i^2) \\ &= 4+4i-1-9+6i+1 \\ &= -5+10i\end{aligned}$$

8.
$$\begin{aligned}\left(-1+\sqrt{-3}\right)^2 &= \left(-1+\sqrt{-3}\right)\left(-1+\sqrt{-3}\right) \\ &= 1-\sqrt{3}i-\sqrt{3}i+3i^2 \\ &= 1-2\sqrt{3}i-3 \\ &= -2-2\sqrt{3}i\end{aligned}$$

9.
$$\begin{aligned}i^2(2+7i)+i(3-6i)-16-i &= 2i^2+7i^3+3i-6i^2-16-i \\ &= -2-7i+3i+6-16-i \\ &= -12-5i\end{aligned}$$

10.
$$\begin{aligned}&\frac{3-2i}{1+i}-\frac{1-3i}{1-i} \quad \text{LCD: } (1+i)(1-i) \\ &= \frac{(3-2i)(1-i)-(1-3i)(1+i)}{(1+i)(1-i)} \\ &= \frac{3-3i-2i+2i^2-(1+i-3i-3i^2)}{1-i^2} \\ &= \frac{3-3i-2i+2i^2-1-i+3i+3i^2}{1+1} \\ &= \frac{2-3i-2-3}{2} \\ &= \frac{-3-3i}{2} \\ &= -\frac{3}{2}-\frac{3}{2}i\end{aligned}$$

Back to Unit 1: Assessment

Media Attributions

Unit 2: Revise the polar form of complex numbers

DYLAN BUSA

Unit outcomes

By the end of this unit you will be able to:

- Plot a complex number on the complex plan.
- Find the absolute value of a complex number.
- Convert a complex number from standard (or rectangular) form to polar form.
- Convert a complex number from polar form to standard (or rectangular) form.
- Understand what is meant by the abbreviation when dealing with complex numbers in polar

What you should know

Before you start this unit, make sure you can:

- Represent complex numbers using an Argand diagram. Refer to level 3 subject outcome 1.1 unit 2 if you need help with this.
- Find the modulus argument of a complex number. Refer to level 3 subject outcome 1.1 unit 2 if you need help with this.
- Express complex numbers in polar form. Refer to level 3 subject outcome 1.1 unit 3 if you need help with this.
- Simplify complex numbers in polar form. Refer to level 3 subject outcome 1.1 unit 3 if you need help with this.
- Convert complex numbers from standard/rectangular form to polar form. Refer to level 3 subject outcome 1.1 unit 3 if you need help with this.
- Convert complex numbers from polar form to standard/rectangular form. Refer to level 3 subject outcome 1.1 unit 3 if you need help with this.

Introduction

This unit revises the polar form of complex numbers covered in level 3 subject outcome 1.1 units 2 and 3. It is important that you complete these subject outcomes before continuing.

To understand what the polar form of a complex number is and where it comes from, we need to understand how to plot complex numbers on the complex plane.

The complex plane

We know that we can plot the position of any real number on a number line as shown in Figure 1.

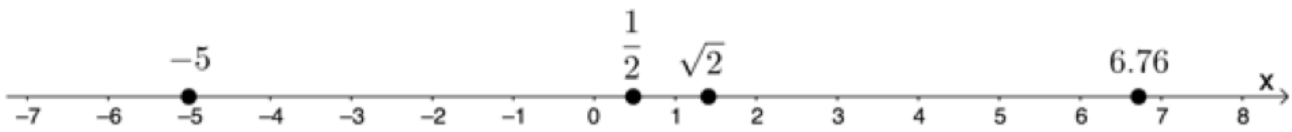


Figure 1: Real numbers on a number line

But complex numbers have a real and an imaginary part, therefore we need two numbers to plot them – one for the real part (the x-axis) and one for the imaginary part (the y-axis) placed at right angles to each other to create the complex plane. This is a coordinate system like the Cartesian plane and complex numbers are points on the plane, expressed as ordered pairs.

Take the complex number $2 - 3i$, for example. We can plot it on the complex plane as shown in Figure 2. It is the point $(2, -3)$.

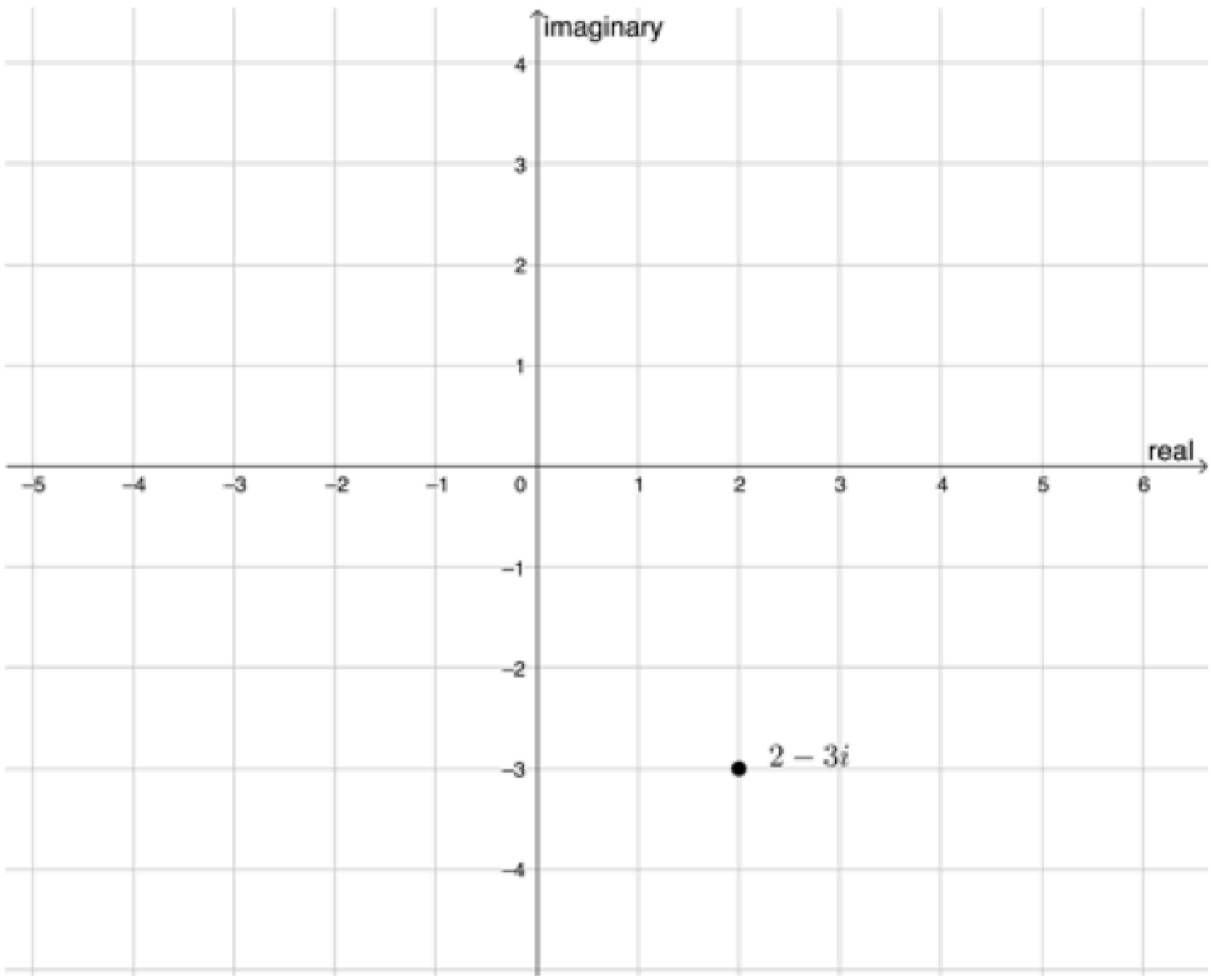


Figure 2: $2 - 3i$ plotted on the complex plane

Exercise 2.1

Plot the following complex numbers on the same complex plane:

1. $-4 - 5i$
2. $4 + 3i$
3. -3
4. $-4i$

The full solutions are at the end of the unit.

Take note!

An **Argand diagram** is a plot of complex numbers as points on the complex plane using the x-axis as the real axis and y-axis as the imaginary axis.

The modulus and argument

We often refer to complex numbers as z and the complex plane as the **z-plane**. We can say that $z = 3 + 4i$.

Figure 3 shows the complex number $z = 3 + 4i$. The distance of this point from the origin is called the **modulus** and is designated as $|z|$.

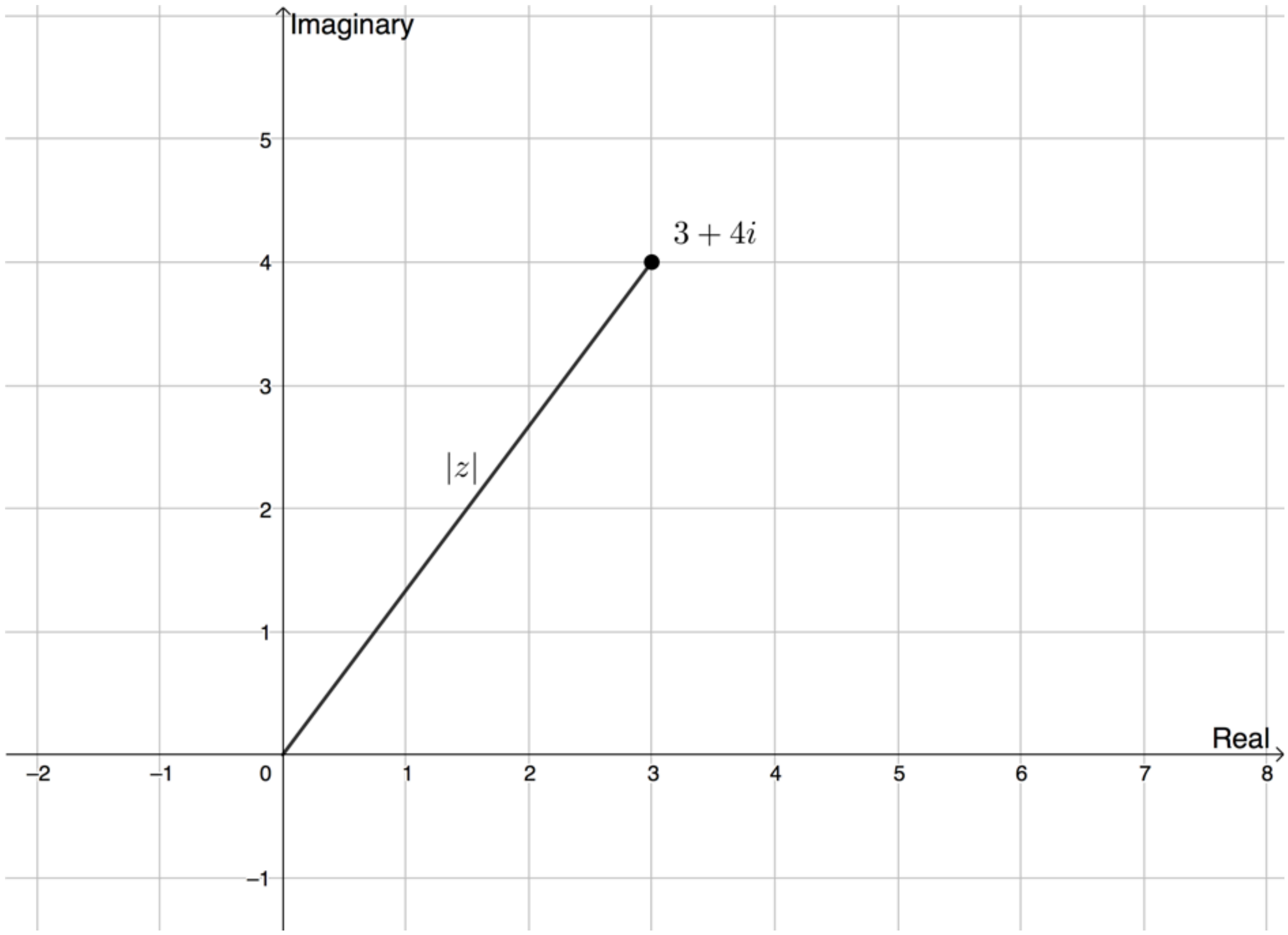


Figure 3: $3 + 4i$ *plotted on the complex plane*

We can determine the modulus by dropping a perpendicular from this point (see Figure 4) and using Pythagoras' theorem.

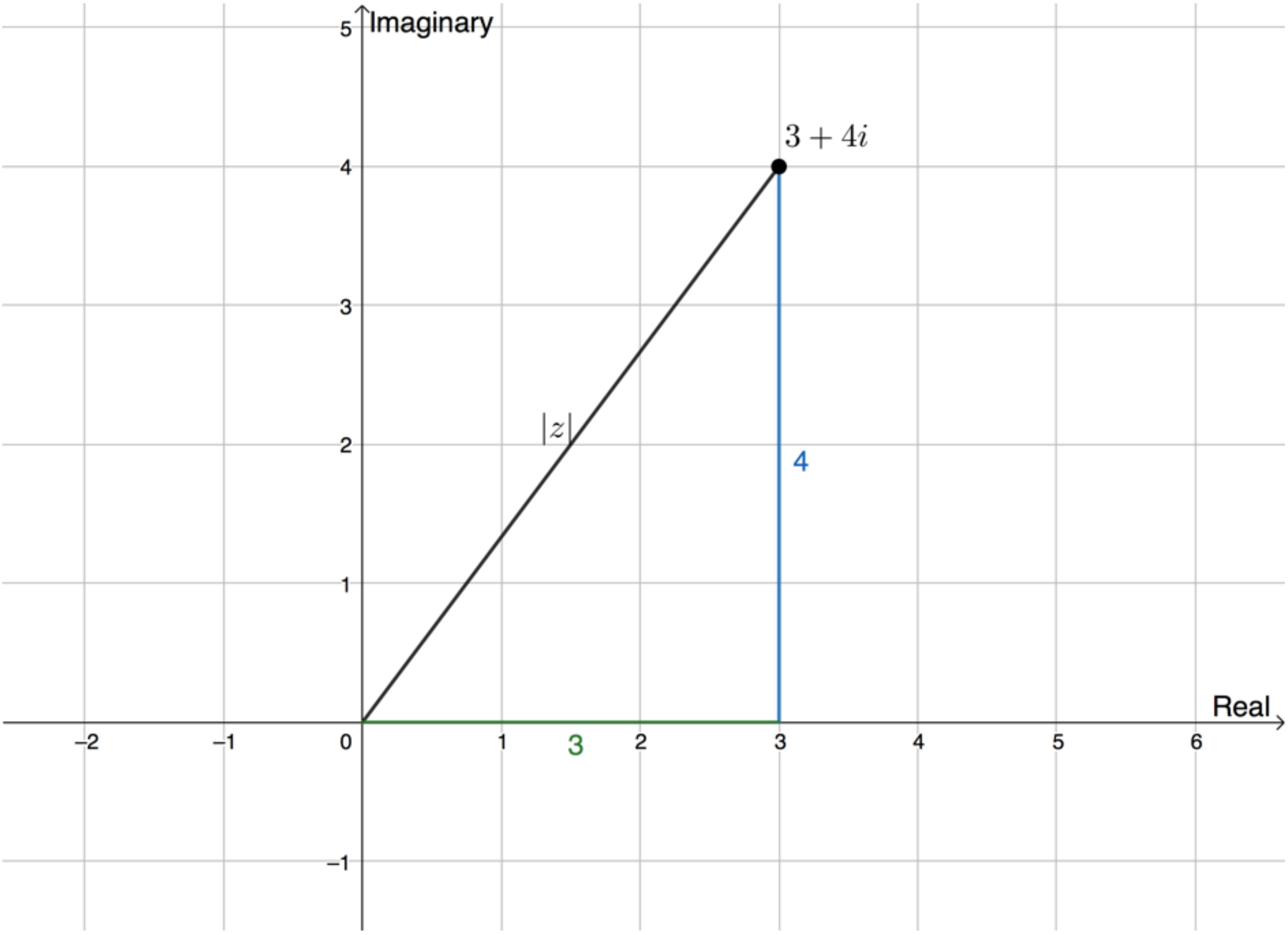


Figure 4: Determining $|z|$ using Pythagoras' theorem

$$\begin{aligned} z^2 &= 3^3 + 4^2 \\ &= 9 + 16 = 25 \\ \therefore |z| &= \sqrt{25} = 5 \end{aligned}$$

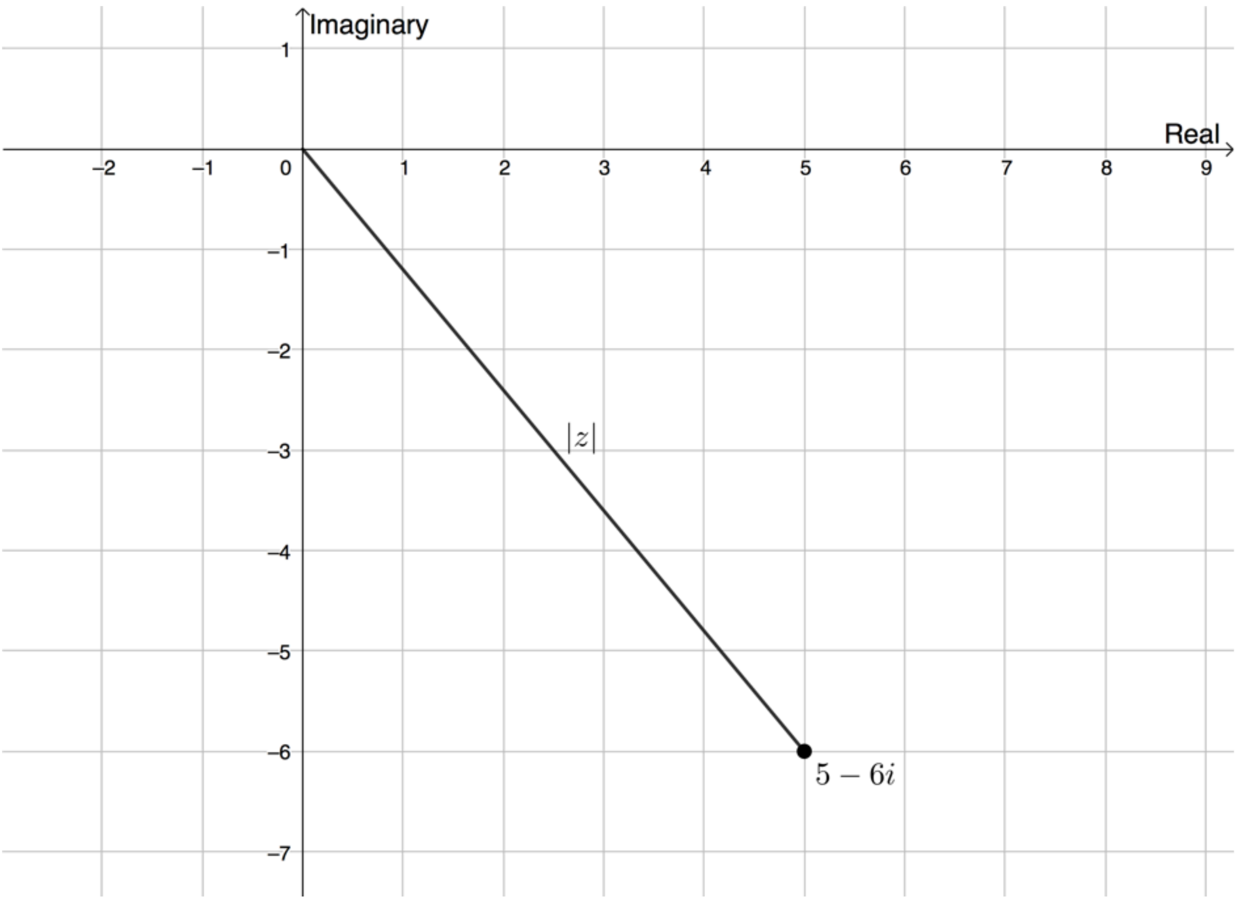
The modulus is always represented as an absolute value with $||$ signs because it is a length and so always taken as the postive value. In the calculation of $|z|$ above, we ignore -5 as a possible solution.

Example 2.1

Given $z = 5 - 6i$, find $|z|$.

Solution

We can plot the complex number $z = 5 - 6i$ on the complex plane.



$$\begin{aligned}|z| &= \sqrt{x^2+y^2} \quad x \text{ is the real part and } y \text{ is the imaginary part}\\ &= \sqrt{5^2+(-6)^2}\\ &= \sqrt{25+36}\\ &= \sqrt{61}\end{aligned}$$

Exercise 2.2

Find $|z|$ in each case:

1. $z = 1 + 7i$
2. $z = -3 - 5i$
3. $z = -4 + \frac{3}{2}i$
4. $z = -\sqrt{5} - \sqrt{-6}$

The full solutions are at the end of the unit.

The modulus is not enough to fully define the position of a complex number on the complex plane. Figure 5 shows two complex numbers, $z_1 = 4 + 3i$ and $z_2 = 3 - 4i$. For each, $|z| = 5$.

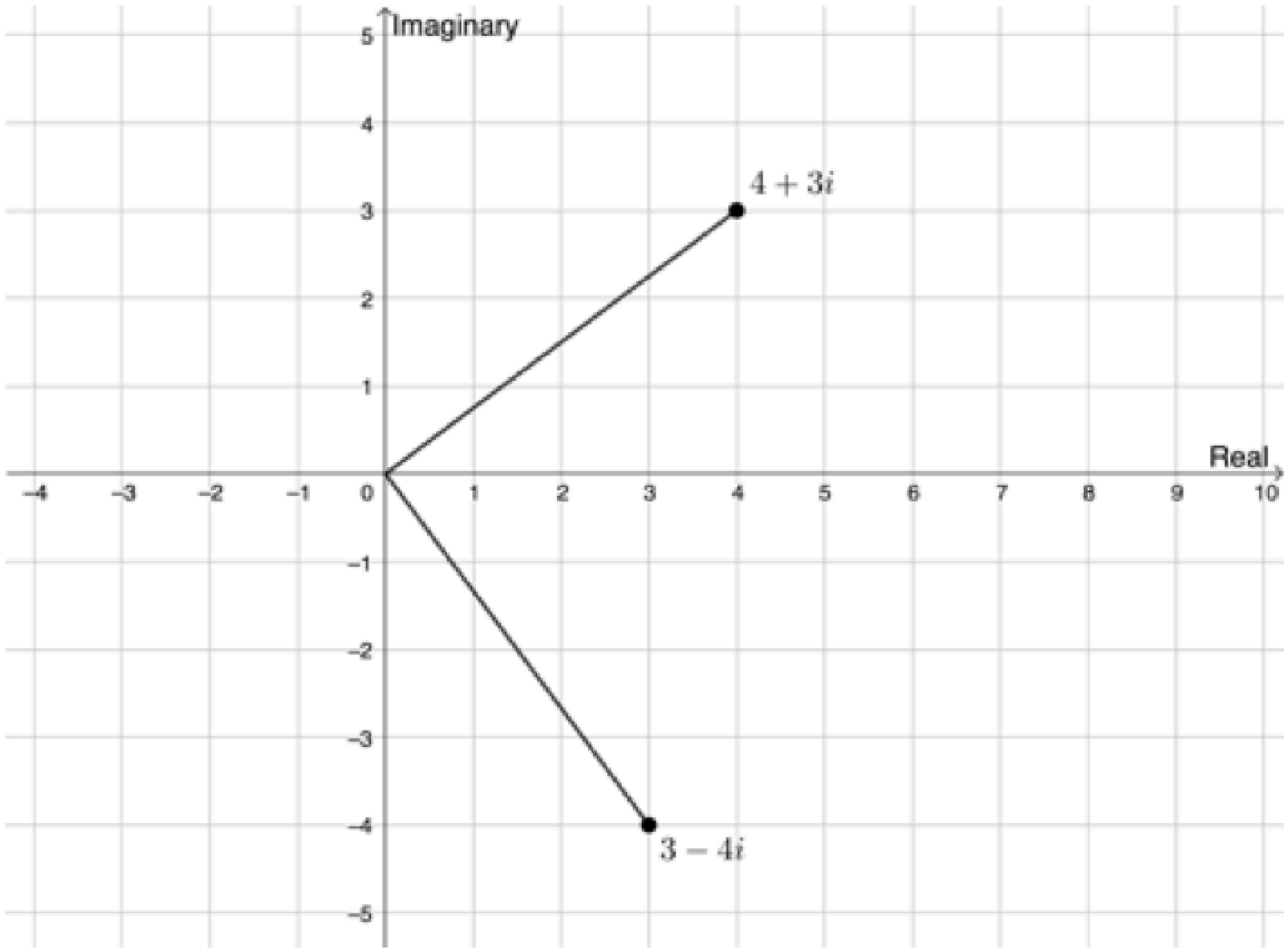


Figure 5: $z_1 = 4 + 3i$ *and* $z_2 = 3 - 4i$ *on the complex plane*

We also need to know the **argument**, the angle the line representing the modulus makes with the positive x-axis (see Figure 6).

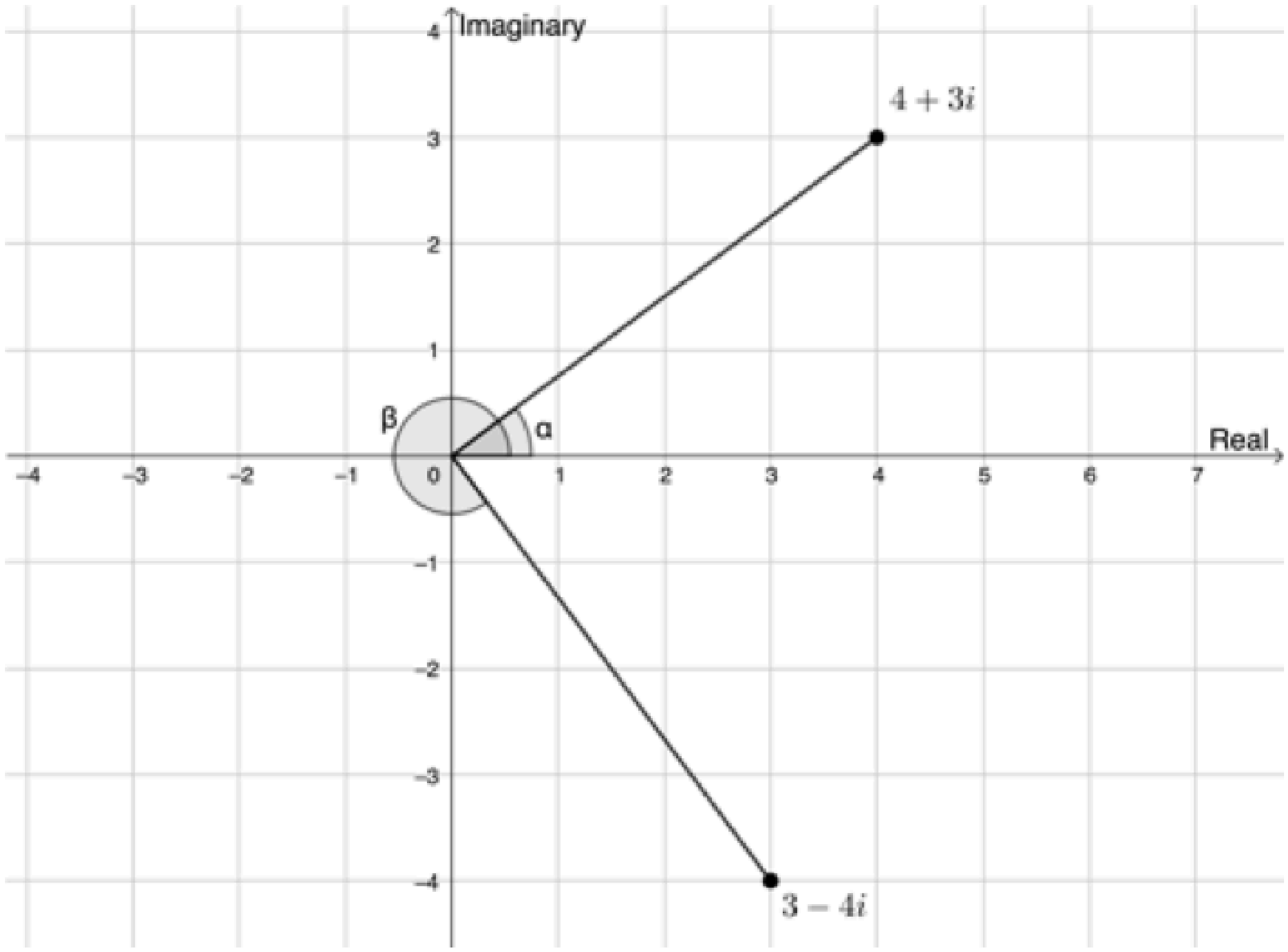


Figure 6: Angles made by the lines representing the moduli of $z_1 = 4 + 3i$ and $z_2 = 3 - 4i$ with the positive x-axis

In Figure 6 we see that $\sin\alpha = \frac{y}{r} = \frac{3}{5}$ or that $\cos\alpha = \frac{x}{r} = \frac{4}{5}$. Therefore, $\alpha = 36.870°$. This answer makes sense because $z_1 = 4 + 3i$ is in the first quadrant.

Also, we can say that $\sin\beta = \frac{y}{r} = \frac{-4}{5}$ or that $\cos\beta = \frac{x}{r} = \frac{3}{5}$. Therefore, $\beta = -53.130°$. This makes sense because $z_2 = 3 - 4i$ is in the fourth quadrant.

With the modulus and the argument, we can uniquely specify the position of any complex number on the complex plane.

Take note!

The value of the modulus $|z|$ is the same as the value of r in $\sin\theta = \frac{y}{r}$ or $\cos\theta = \frac{x}{r}$.

Take note!

When calculating the argument, it is important that you draw a sketch of the position of z on the complex plane to determine which quadrant the point is in. The three trigonometric ratios still follow the CAST diagram on the complex plane (see Figure 7).

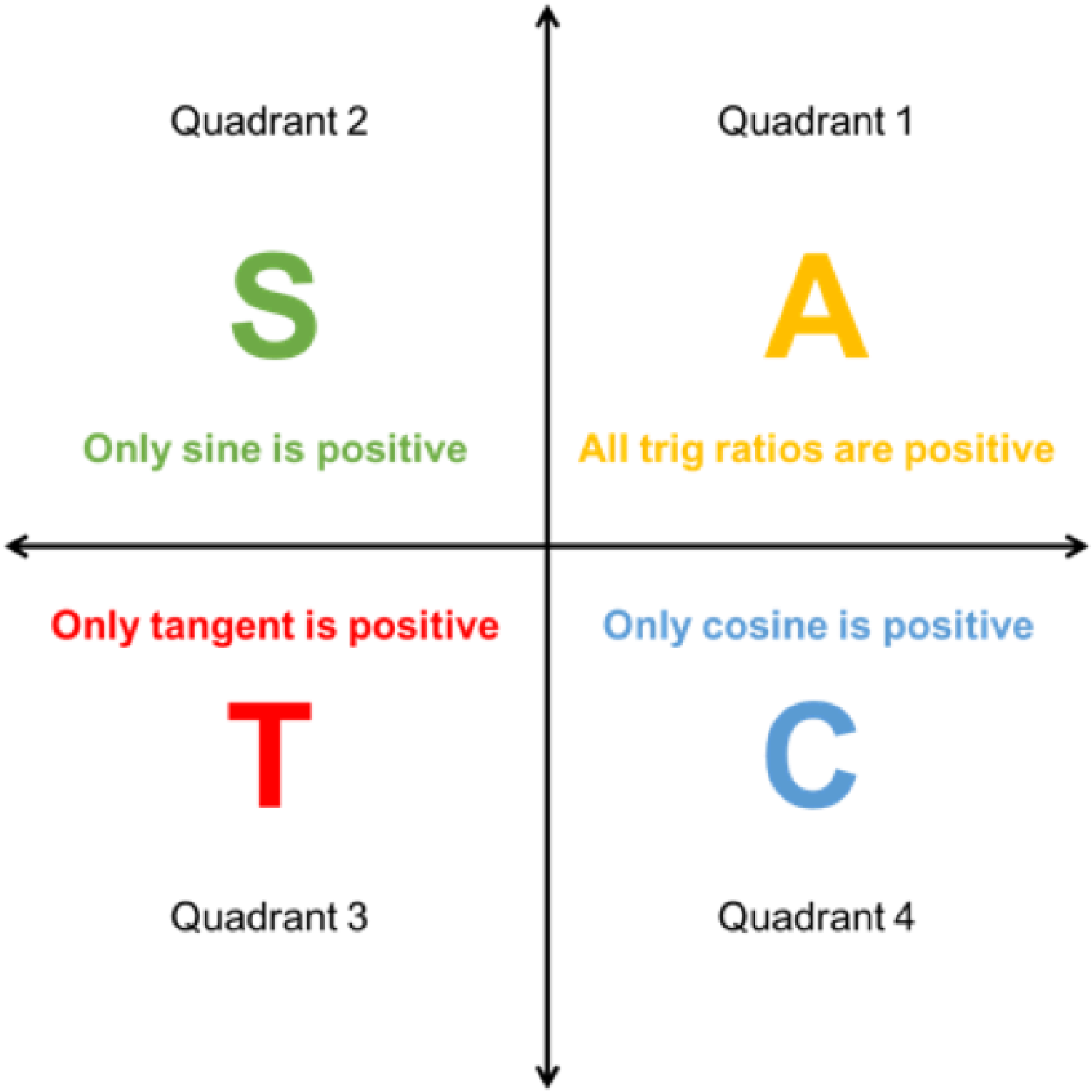


Figure 7: The CAST diagram

Example 2.2

Determine the modulus and the argument of $z = -2 - \sqrt{-3}$.

Solution

First, write the complex number in standard form $a + bi$.
$z = -2 - \sqrt{-3} = -2 - \sqrt{3}i$

z lies in the third quadrant because x (-2) and y $(-\sqrt{3})$ are both negative.

$$\begin{aligned} |z| &= \sqrt{x^2 + y^2} \\ &= \sqrt{(-2)^2 + (-\sqrt{3})^2} \\ &= \sqrt{4+3} \\ &= \sqrt{7} \end{aligned}$$

To find the argument, start by finding a **reference angle** (α) using a positive ratio of sine or cosine.

$$\sin\alpha = \frac{y}{r} = \frac{\sqrt{3}}{\sqrt{7}}$$
$$\therefore \alpha = 40.89^\circ$$

But z lies in the third quadrant. Therefore, $\theta = 180^\circ + 40.89^\circ = 220.89^\circ$.

Take note!

When finding the argument, remember that you need to pay attention to which quadrant the complex number is in. A useful strategy can be to make the ratio for $\sin\theta = \frac{y}{r}$ or $\cos\theta = \frac{x}{r}$ positive initially, in order to find the **acute reference angle** θ, and then to transfer this angle into the necessary quadrant as indicated in Figure 8. To transfer these angles, use the following identities:

- Second quadrant: $180^\circ - \theta$
- Third quadrant: $180^\circ + \theta$
- Fourth quadrant: $360^\circ - \theta$

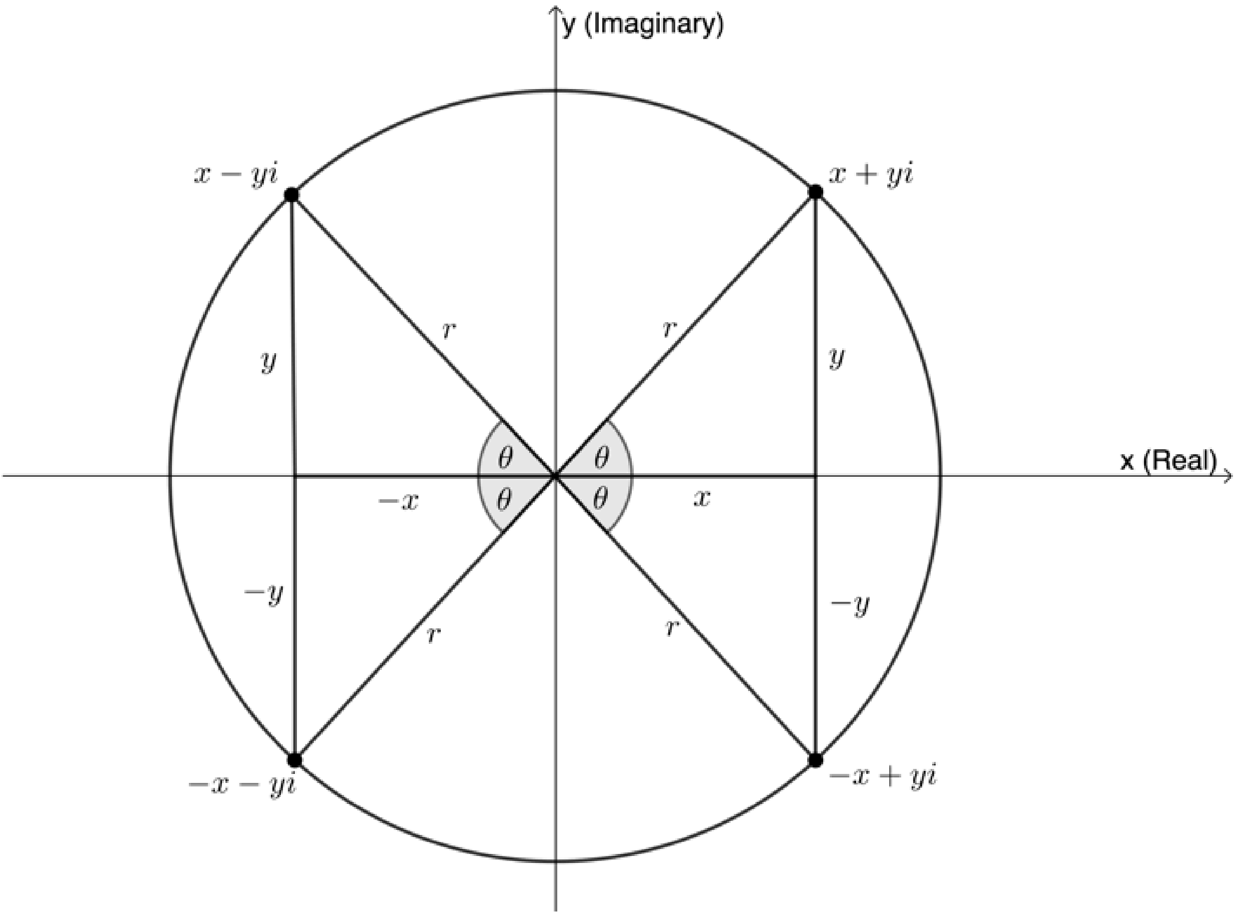


Figure 8: Transferring the reference angle into the necessary quadrant

Exercise 2.3

Determine the modulus and the argument of the following complex numbers, leaving your moduli in surd form:

1. $z = -1 - 8i$
2. $z = -2 + 2i$
3. $z = 4 - 3i$

The full solutions are at the end of the unit.

Polar form

We say that $z = 2 - 3i$ is written in standard or rectangular form where the number is expressed in terms of a real and imaginary component. The **polar form** of a complex number expresses the number in terms of its modulus and argument.

Suppose we have a complex number $z = x + yi$. We can represent this with an Argand diagram (see Figure 9).

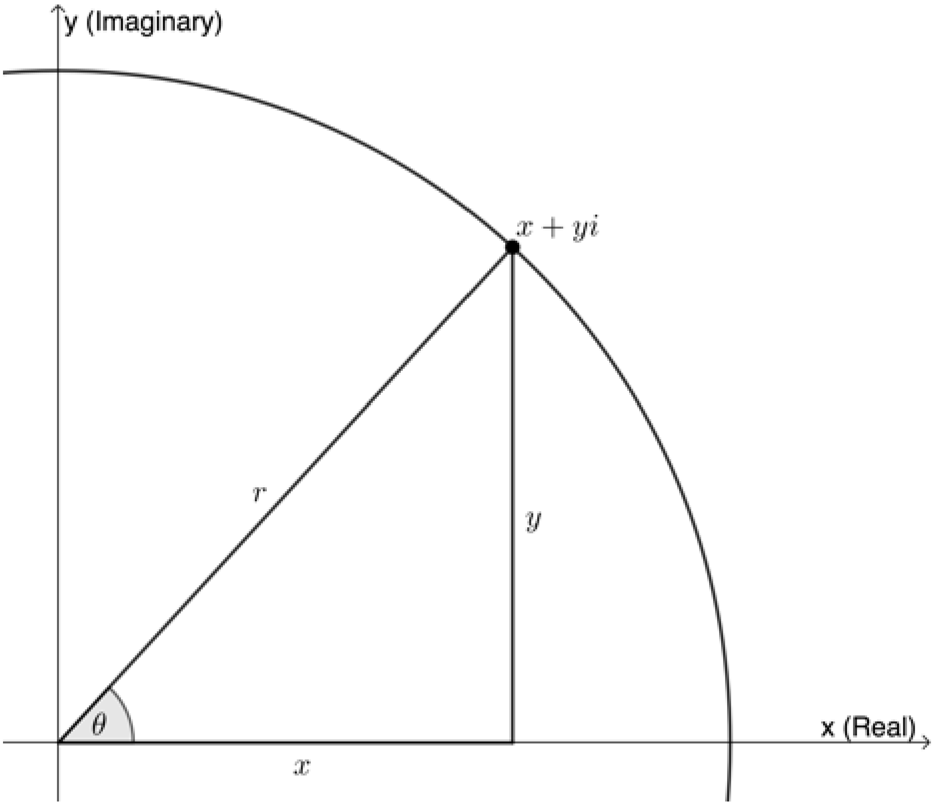


Figure 9: Argand diagram for $z = x + yi$

Now we know that $\cos\theta = \dfrac{x}{r}$ and $\sin\theta = \dfrac{y}{r}$. Therefore, $x = r\cos\theta$ and $y = r\sin\theta$.

So, the point (x, y) has coordinates given by $x = r\cos\theta$ and $y = r\sin\theta$ where $r = \sqrt{x^2 + y^2}$. Therefore:

$$\begin{aligned} z &= x + yi \\ &= r\cos\theta + (r\sin\theta)i \\ &= r(\cos\theta + i\sin\theta) \end{aligned}$$

Take note!

We often use the abbreviation $r\text{cis}\theta$ to represent $r(\cos\theta + i\sin\theta)$.

$$r(\cos\theta + i\sin\theta)$$

$$r\text{cis}\theta$$

The polar form of a complex number:

$$\begin{aligned} z &= x + yi \\ &= r\cos\theta + (r\sin\theta)i \\ &= r(\cos\theta + i\sin\theta) \\ &= r\text{cis}\theta \end{aligned}$$

where $r = |z|$ and θ is the argument.

Example 2.3

Find the polar form of $z = -4 - 4i$.

Solution

Step 1: Determine r (or $|z|$)

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ &= \sqrt{(-4)^2 + (-4)^2} \\ &= \sqrt{16 + 16} \\ &= \sqrt{32} \\ &= 4\sqrt{2} \end{aligned}$$

Step 2: Determine θ

z is in the third quadrant. Find reference angle α.

$$\cos\alpha = \frac{x}{r} = \frac{4}{4\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\therefore \alpha = 45^\circ$$

Therefore $\theta = 180^\circ + 45^\circ = 225^\circ$.

Step 3: Write the solution

$$\begin{aligned} z &= r(\cos\theta + i\sin\theta) \\ &= 4\sqrt{2}(\cos 225^\circ + i\sin 225^\circ) \\ &= 4\sqrt{2}\text{cis}225^\circ \end{aligned}$$

Exercise 2.4

Write the following complex numbers in polar form:

1. $z = 5 + 12i$
2. $z = -7 - 3i$
3. $z = \sqrt{5} - \sqrt{-6}$

The full solutions are at the end of the unit.

Convert polar form to rectangular form

Sometimes it is necessary to convert a complex number in polar form into standard or rectangular form. To do this, we need to first evaluate the trigonometric functions $\cos\theta$ and $\sin\theta$, then multiply through by r.

Example 2.4

Convert $z = 12\text{cis}45^\circ$ into standard/rectangular form.

Solution

Step 1: Evaluate $\cos\theta$ and $\sin\theta$
We can use the fact that 45° is a special angle to evaluate without a calculator.

$$\cos 45^\circ = \frac{1}{\sqrt{2}}$$
$$\sin 45^\circ = \frac{1}{\sqrt{2}}$$

Step 2: Find the standard form
z is in the first quadrant where cosine and sine are positive.

$$\begin{aligned} z &= 12(\cos 45^\circ + i\sin 45^\circ) \\ &= 12\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) \\ &= \frac{12}{\sqrt{2}} + \frac{12}{\sqrt{2}}i \\ &= \frac{12\sqrt{2}}{2} + \frac{12\sqrt{2}}{2}i \\ &= 6\sqrt{2} + 6\sqrt{2}i \end{aligned}$$

The complex number in standard form is $z = 6\sqrt{2} + 6\sqrt{2}i$.

Example 2.5

Convert $z = 4\text{cis}150^\circ$ into standard/rectangular form.

Solution

Step 1: Evaluate $\cos\theta$ and $\sin\theta$
120° is not a special angle but $180^\circ - 150^\circ = 30^\circ$ is a special angle. Therefore, we can evaluate $\cos\theta$ and $\sin\theta$ without a calculator and transfer the angle into the correct quadrant, in this case, the second quadrant.

$$\cos 30^\circ = \frac{\sqrt{3}}{2}$$
$$\sin 30^\circ = \frac{1}{2}$$

Step 2: Find the standard form
We know that the complex number is in the second quadrant where cosine is negative and sine is positive.

$$\begin{aligned} z &= 4(\cos 150^\circ + i\sin 150^\circ) \\ &= 4\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) \quad \text{Remember that cosine is negative in the second quadrant} \\ &= -2\sqrt{3} + 2i \end{aligned}$$

The complex number in standard form is $z = -2\sqrt{3} + 2i$.

Example 2.6

Convert $z = 3\text{cis}243^\circ$ into standard/rectangular form.

Solution

Step 1: Evaluate $\cos\theta$ and $\sin\theta$
243° is not a special angle and cannot be reduced to a special angle. Therefore, we have to evaluate $\cos\theta$ and $\sin\theta$ with a calculator. The complex number is in the third quadrant.

$$\cos 243^\circ = -0.454$$
$$\sin 243^\circ = -0.891$$

Step 2: Find the standard form
We know that the complex number is in the third quadrant where cosine and sine are negative.

$$\begin{aligned} z &= 3(\cos 243^\circ + i\sin 243^\circ) \\ &= 3(-0.454 - 0.891i) \\ &= -1.362 - 2.673i \end{aligned}$$

The complex number in standard form is $z = -1.362 - 2.673i$.

Exercise 2.5

Convert the following complex numbers into standard/rectangular form:

1. $z = \sqrt{3}\text{cis}45^\circ$
2. $z = \sqrt{7}\text{cis}210^\circ$
3. $z = 3\text{cis}340^\circ$

The full solutions are at the end of the unit.

Summary

In this unit you have learnt the following:

- How to plot a complex number on the complex plane.
- How to find the absolute value or modulus of a complex number.
- How to convert a complex number from standard (or rectangular) form to polar form.
- How to convert a complex number from polar form to standard (or rectangular) form.
- How to use the cis shorthand for the polar form.

Unit 2: Assessment

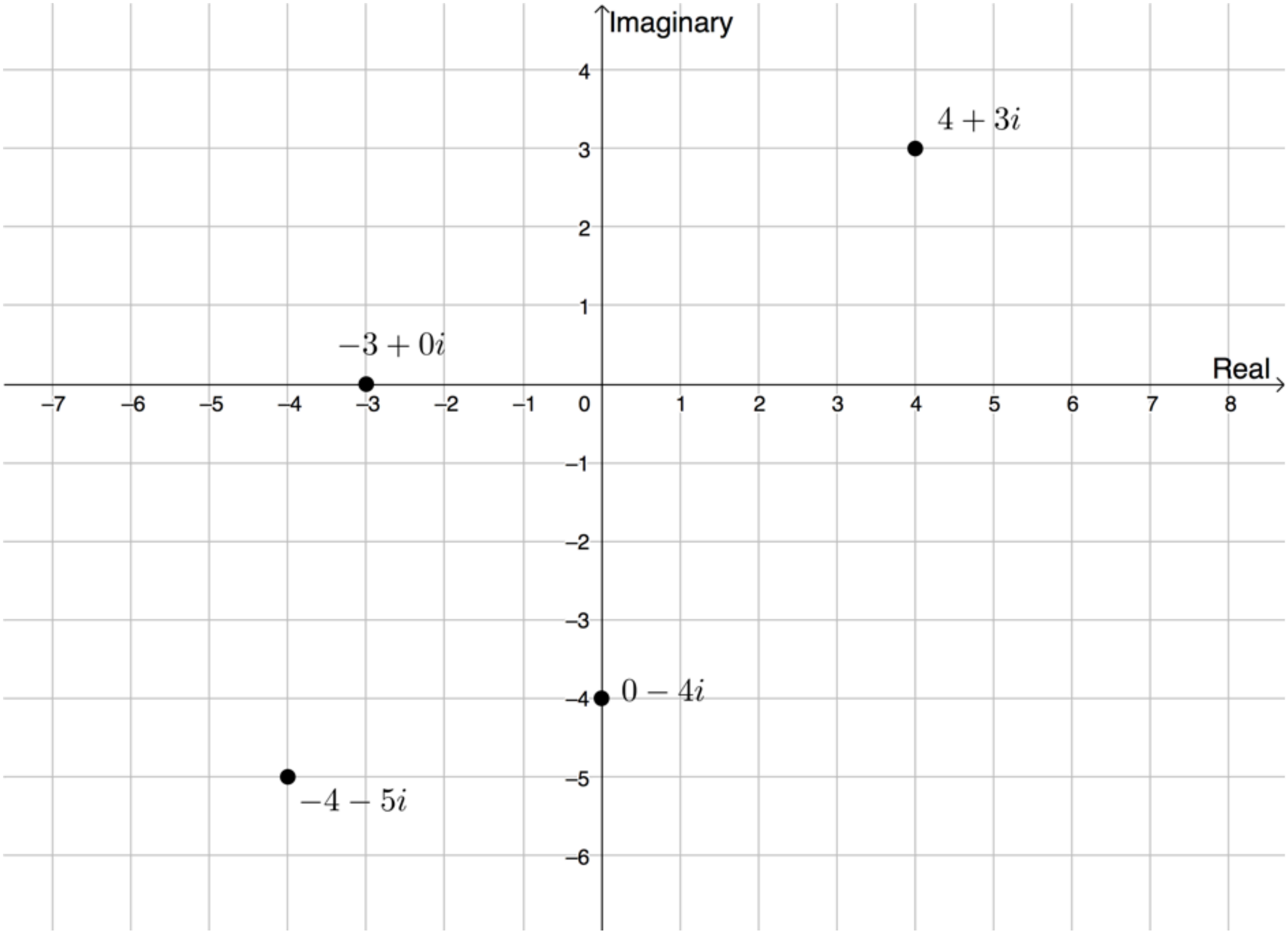
Suggested time to complete: 25 minutes

1. Given $z = -1 + \sqrt{3}i$:
   a. Write down the conjugate of z.
   b. Calculate the modulus of z.
   c. Determine the argument of z.
   d. Write z in polar form.
2. Write the following complex numbers in polar form:
   a. $z = \frac{3}{2} - 7i$
   b. $z = -4 + \sqrt{-8}$
   c. $z = \sqrt{13} - \sqrt{-13}$
   d. $z = -6 + 3.543i$
3. Write the following complex numbers in standard form:
   a. $z = 13\text{cis}330^\circ$
   b. $z = \sqrt{7}\text{cis}225^\circ$
   c. $z = \frac{2}{\sqrt{3}}\text{cis}120^\circ$
   d. $z = 4\text{cis}160^\circ$

The full solutions are at the end of the unit.

Unit 2: Solutions

Exercise 2.1



Back to Exercise 2.1

Exercise 2.2

1.
$$\begin{aligned}|z| &= \sqrt{1^2+7^2}\\ &= \sqrt{1+49}\\ &= \sqrt{50}\\ &= 5\sqrt{2}\end{aligned}$$

2.
$$\begin{aligned}|z| &= \sqrt{(-3)^2+(-5)^2}\\ &= \sqrt{9+25}\\ &= \sqrt{34}\end{aligned}$$

3.

$$|z| = \sqrt{(-4)^2 + \left(\frac{3}{2}\right)^2}$$
$$= \sqrt{16 + \frac{9}{4}}$$
$$= \sqrt{\frac{64+9}{4}}$$
$$= \sqrt{\frac{73}{4}}$$
$$= \frac{\sqrt{73}}{2}$$

4.
$$z = -\sqrt{5} - \sqrt{-6} = -\sqrt{5} - \sqrt{6}i$$
$$|z| = \sqrt{\left(-\sqrt{5}\right)^2 + \left(-\sqrt{6}\right)^2}$$
$$= \sqrt{5+6}$$
$$= \sqrt{11}$$

Back to Exercise 2.2

Exercise 2.3

1. $z = -1 - 8i$
$$|z| = \sqrt{(-1)^2 + (-8)^2}$$
$$= \sqrt{1+64}$$
$$= \sqrt{65}$$
z is in the third quadrant. Find reference angle α.
$$\sin\alpha = \frac{8}{\sqrt{65}}$$
$$\therefore \alpha = 82.87^\circ$$
Therefore, $\theta = 180^\circ + \alpha = 262.87^\circ$.

2. $z = -2 + 2i$
$$|z| = \sqrt{(-2)^2 + 2^2}$$
$$= \sqrt{4+4}$$
$$= \sqrt{8}$$
$$= 2\sqrt{2}$$
z is in the second quadrant. Find reference angle α.
$$\sin\alpha = \frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}}$$
$$\therefore \alpha = 45^\circ$$
Therefore, $\theta = 180^\circ - \alpha = 135^\circ$.

3. $z = 4 - 3i$
$$|z| = \sqrt{(4)^2 + (-3)^2}$$
$$= \sqrt{16+9}$$
$$= \sqrt{25}$$
$$= 5$$
z is in the fourth quadrant. Find reference angle α.
$$\sin\alpha = \frac{3}{5}$$
$$\therefore \alpha = 36.87^\circ$$
Therefore, $\theta = 360^\circ - \alpha = 323.13^\circ$.

Back to Exercise 2.3

Exercise 2.4

1. $z = 5 + 12i$
   $r = \sqrt{(5)^2 + (12)^2}$
   $= \sqrt{25 + 144}$
   $= \sqrt{169}$
   $= 13$
   z is in the first quadrant.
   $\sin\theta = \dfrac{12}{13}$
   $\therefore \theta = 67.38^\circ$
   $z = r(\cos\theta + i\sin\theta)$
   $= 13(\cos 67.38^\circ + i\sin 67.38^\circ)$
   $= 13\text{cis}67.38^\circ$
2. $z = -7 - 3i$
   $r = \sqrt{(-7)^2 + (-3)^2}$
   $= \sqrt{49 + 9}$
   $= \sqrt{58}$
   z is in the third quadrant.
   $\sin\alpha = \dfrac{3}{\sqrt{58}}$
   $\therefore \alpha = 23.20^\circ$
   Therefore, $\theta = 180^\circ + 23.20^\circ = 203.20^\circ$
   $z = r(\cos\theta + i\sin\theta)$
   $= \sqrt{58}(\cos 203.30^\circ + i\sin 203.20^\circ)$
   $= \sqrt{58}\text{cis}203.20^\circ$
3. $z = \sqrt{5} - \sqrt{-6} = \sqrt{5} - \sqrt{6}i$
   $r = \sqrt{\left(\sqrt{5}\right)^2 + \left(\sqrt{6}\right)^2}$
   $= \sqrt{5 + 6}$
   $= \sqrt{11}$
   z is in the fourth quadrant.
   $\sin\alpha = \dfrac{\sqrt{6}}{\sqrt{11}}$
   $\therefore \alpha = 47.61^\circ$
   Therefore, $\theta = 360^\circ - 47.61^\circ = 312.39^\circ$.
   $z = r(\cos\theta + i\sin\theta)$
   $= \sqrt{11}(\cos 312.39^\circ + i\sin 312.39^\circ)$
   $= \sqrt{11}\text{cis}312.39^\circ$

Back to Exercise 2.4

Exercise 2.5

1. $z = \sqrt{3}\text{cis}45^\circ$
   $\cos 45^\circ = \dfrac{1}{\sqrt{2}}$

$$\sin 45^\circ = \frac{1}{\sqrt{2}}$$

$$\begin{aligned} z &= \sqrt{3}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) \\ &= \frac{\sqrt{3}}{\sqrt{2}} + \frac{\sqrt{3}}{\sqrt{2}}i \\ &= \frac{\sqrt{2}\sqrt{3}}{2} + \frac{\sqrt{2}\sqrt{3}}{2}i \end{aligned}$$

2. $z = \sqrt{7}\text{cis}210^\circ$

$$\cos 210^\circ = \cos(180^\circ + 30^\circ) = -\cos 30^\circ = -\frac{\sqrt{3}}{2}$$

$$\sin 210^\circ = \sin(180^\circ + 30^\circ) = -\sin 30^\circ = -\frac{1}{2}$$

$$\begin{aligned} z &= \sqrt{7}\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) \\ &= \frac{\sqrt{21}}{2} - \frac{\sqrt{7}}{2}i \end{aligned}$$

3. $z = 3\text{cis}340^\circ$

$\cos 340^\circ = 0.940$

$\sin 340^\circ = -0.342$

$$\begin{aligned} z &= 3\,(0.940 - 0.342i) \\ &= 2.82 - 1.026i \end{aligned}$$

Back to Exercise 2.5

Unit 2: Assessment

1. $z = -1 + \sqrt{3}i$
   a. $-1 - \sqrt{3}i$
   b.
   $$\begin{aligned} |z| &= \sqrt{(-1)^2 + (\sqrt{3})^2} \\ &= \sqrt{1+3} \\ &= \sqrt{4} \\ &= 2 \end{aligned}$$
   c. z is in the second quadrant.
   $$\sin\alpha = \frac{\sqrt{3}}{2}$$
   $$\therefore \alpha = 60^\circ$$
   $$\theta = 180^\circ - 60^\circ = 120^\circ$$
   d.
   EQU
2.
   a. $z = \frac{3}{2} - 7i$

$$|z| = \sqrt{\left(\frac{3}{2}\right)^2 + (-7)^2}$$
$$= \sqrt{\frac{9}{4} + 49}$$
$$= \sqrt{\frac{9 + 196}{4}}$$
$$= \sqrt{\frac{205}{4}}$$
$$= \frac{\sqrt{205}}{2}$$

z is in the fourth quadrant.

$$\sin\alpha = \frac{7}{\frac{\sqrt{205}}{2}}$$
$$\therefore \alpha = 77.91^\circ$$
$$\theta = 360^\circ - 77.91^\circ = 282.09^\circ$$
$$z = \frac{\sqrt{205}}{2}(\cos 282.09^\circ + i\sin 282.09^\circ)$$
$$= \frac{\sqrt{205}}{2}\text{cis}282.09^\circ$$

b. $z = -4 + \sqrt{-8} = -4 + \sqrt{8}i$

$$|z| = \sqrt{(-4)^2 + 8^2}$$
$$= \sqrt{16 + 64}$$
$$= \sqrt{80}$$
$$= 4\sqrt{5}$$

z is in the second quadrant.

$$\sin\alpha = \frac{\sqrt{8}}{4\sqrt{5}}$$
$$\therefore \alpha = 18.43^\circ$$
$$\theta = 180^\circ - 18.43^\circ = 161.57^\circ$$
$$z = 4\sqrt{5}(\cos 161.57^\circ + i\sin 161.57^\circ)$$
$$= 4\sqrt{5}\text{cis}161.57^\circ$$

c. $z = \sqrt{13} - \sqrt{-13} = \sqrt{13} - \sqrt{13}i$

$$|z| = \sqrt{\left(\sqrt{13}\right)^2 + \left(-\sqrt{13}\right)^2}$$
$$= \sqrt{13 + 13}$$
$$= \sqrt{26}$$

z is in the fourth quadrant.

$$\sin\alpha = \frac{\sqrt{13}}{\sqrt{26}} = \frac{\sqrt{13}}{\sqrt{2}\sqrt{13}} = \frac{1}{\sqrt{2}}$$
$$\therefore \alpha = 45^\circ$$
$$\theta = 360^\circ - 45^\circ = 315^\circ$$
$$z = \sqrt{26}(\cos 315^\circ + i\sin 315^\circ)$$
$$= \sqrt{26}\text{cis}315^\circ$$

d. $z = -6 + 3.543i$

$$|z| = \sqrt{(-6)^2 + 3.543^2}$$
$$= \sqrt{36 + 12.55}$$
$$= \sqrt{48.55}$$

z is in the second quadrant.

$$\sin\alpha = \frac{3.543}{\sqrt{48.55}}$$
$$\therefore \alpha = 30.56^\circ$$
$$\theta = 180^\circ - 30.56^\circ = 149.44^\circ$$
$$\begin{aligned} z &= \sqrt{48.55}(\cos 149.44^\circ + i\sin 149.44^\circ) \\ &= \sqrt{48.55}\text{cis}149.44^\circ \end{aligned}$$

3.

a. $z = 13\text{cis}330^\circ$

$$\cos 330^\circ = \cos(360^\circ - 30^\circ) = \cos 30^\circ = \frac{\sqrt{3}}{2}$$
$$\sin 330^\circ = \sin(360^\circ - 30^\circ) = -\sin 30^\circ = -\frac{1}{2}$$
$$\begin{aligned} z &= 13\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) \\ &= \frac{13\sqrt{3}}{2} - \frac{13}{2}i \end{aligned}$$

b. $z = \sqrt{7}\text{cis}225^\circ$

$$\cos 225^\circ = \cos(180^\circ + 45^\circ) = -\cos 45^\circ = -\frac{1}{\sqrt{2}}$$
$$\sin 225^\circ = \sin(180^\circ + 45^\circ) = -\sin 45^\circ = -\frac{1}{\sqrt{2}}$$
$$\begin{aligned} z &= \sqrt{7}\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) \\ &= -\frac{\sqrt{7}}{\sqrt{2}} - \frac{\sqrt{7}}{\sqrt{2}}i \end{aligned}$$

c. $z = \frac{2}{\sqrt{3}}\text{cis}120^\circ$

$$\cos 120^\circ = \cos(180^\circ - 60^\circ) = -\cos 60^\circ = -\frac{1}{2}$$
$$\sin 120^\circ = \sin(180^\circ - 60^\circ) = \sin 60^\circ = \frac{\sqrt{3}}{2}$$
$$\begin{aligned} z &= \frac{2}{\sqrt{3}}\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) \\ &= -\frac{1}{\sqrt{3}} + i \end{aligned}$$

d. $z = 4\text{cis}160^\circ$

$$\cos 160^\circ = -0.940$$
$$\sin 160^\circ = 0.342$$
$$\begin{aligned} z &= 4(-0.940 + 0.342i) \\ &= -3.76 + 1.368i \end{aligned}$$

Back to Unit 2: Assessment

Media Attributions

Unit 3: Revise the multiplication and division of complex numbers in polar form

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Unit outcomes

By the end of this unit you will be able to:

- Multiply complex numbers in polar form.
- Divide complex numbers in polar form.

What you should know

Before you start this unit, make sure you can:

- Convert a complex number from standard (or rectangular) form to polar form.
- Convert a complex number from polar form to standard (or rectangular) form.
- Understand what is meant by the abbreviation when dealing with complex numbers in polar form.

Refer to unit 2 in this subject outcome if you need help with any of the above.

Introduction

In the previous unit we revised the polar form of complex numbers and how to convert between the standard (or rectangular) form and the polar form. One of the benefits of polar form is that it makes the multiplication and division of complex numbers quite easy.

Multiply complex numbers in polar form

When we multiply complex numbers in polar form, we have to **multiply** the moduli and **add** the arguments.

If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ then $z_1 z_2 = r_1 r_2\left(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right)$

We could write the product using the polar form shorthand as $z_1 z_2 = r_1 r_2 \text{cis}(\theta_1+\theta_2)$.

Example 3.1

Find the product of z_1 and z_2 if $z_1 = 3\text{cis}50^\circ$ and $z_2 = 4\text{cis}70^\circ$.

Solution

$z_1z_2 = r_1(\cos\theta_1 + i\sin\theta_1) \times r_2(\cos\theta_2 + i\sin\theta_2) = r_1r_2\left(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right)$
$z_1 = 3\text{cis}50^\circ = 3(\cos 50^\circ + i\sin 50^\circ)$ and $z_2 = 4\text{cis}70^\circ = 4(\cos 70^\circ + i\sin 70^\circ)$

$$\begin{aligned} z_1 \times z_2 &= (3\times 4)\left(\cos(50^\circ + 70^\circ) + i\sin(50^\circ+70^\circ)\right) \\ &= 12(\cos 120^\circ + i\sin 120^\circ) \\ &= 12\text{cis}120^\circ \end{aligned}$$

We can leave our answer in polar form or we can convert it into standard/rectangular form.

$\cos 120^\circ = \cos(180^\circ - 60^\circ) = -\cos 60^\circ = -\frac{1}{2}$

$\sin 120^\circ = \sin(180^\circ - 60^\circ) = \sin 60^\circ = \frac{\sqrt{3}}{2}$

$$\begin{aligned} 12\text{cis}120^\circ &= 12\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) \\ &= -6 + 6\sqrt{3}i \end{aligned}$$

Example 3.2

Find the product z_1z_2, given $z_1 = 4(\cos 75^\circ + i\sin 75^\circ)$and $z_2 = 2(\cos 155^\circ + i\sin 155^\circ)$.

Solution

$z_1z_2 = r_1(\cos\theta_1 + i\sin\theta_1) \times r_2(\cos\theta_2 + i\sin\theta_2) = r_1r_2\left(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right)$
$z_1 = 4(\cos 75^\circ + i\sin 75^\circ)$and $z_2 = 2(\cos 155^\circ + i\sin 155^\circ)$

$$\begin{aligned} z_1 \times z_2 &= (4\times 2)\left(\cos(75^\circ + 155^\circ) + i\sin(75^\circ+155^\circ)\right) \\ &= 8(\cos 230^\circ + i\sin 230^\circ) \\ &= 8\text{cis}230^\circ \end{aligned}$$

We can leave our answer in polar form or we can convert it into standard/rectangular form.
$\cos 230^\circ = -0.643$
$\sin 230^\circ = -0.766$

$$\begin{aligned} 8(\cos 230^\circ + i\sin 230^\circ) &= 8(-0.643 - 0.766i) \\ &= -5.144 - 6.128i \end{aligned}$$

Products of complex numbers in polar form:
If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ then $z_1z_2 = r_1r_2\left(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right)$
$z_1z_2 = r_1r_2\text{cis}(\theta_1+\theta_2)$

Exercise 3.1

1. Find z_1z_2 leaving your answer in polar form:

a. $z_1 = 2\sqrt{3}\text{cis}116^\circ;\ z_2 = 2\text{cis}109^\circ$

b. $z_1 = \sqrt{2}\text{cis}210^\circ;\ z_2 = 2\sqrt{2}\text{cis}100^\circ$

2. Find $z_1 z_2$ leaving your answer in standard form:

a. $z_1 = \frac{1}{3}\text{cis}120^\circ;\ z_2 = 4\text{cis}15^\circ$

b. $z_1 = \sqrt{3}\text{cis}120^\circ;\ z_2 = \frac{1}{\sqrt{2}}\text{cis}60^\circ$

The full solutions are at the end of the unit.

Divide complex numbers in polar form

Dividing complex numbers in polar form is very similar to finding the product except that we find the **quotient** of the two moduli and the **difference** of the two arguments.

If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ then
$\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2))$

We could write the product using the polar form shorthand as $\frac{z_1}{z_2} = \frac{r_1}{r_2}\text{cis}(\theta_1 - \theta_2)$.

Example 3.3

Determine $\frac{z_1}{z_2}$ if $z_1 = \sqrt{3}\text{cis}213^\circ$ and $z_2 = 2\text{cis}33^\circ$.

Solution

$$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} = \frac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2))$$

$z_1 = \sqrt{3}\text{cis}213^\circ = \sqrt{3}(\cos 213^\circ + i\sin 213^\circ)$ and $z_2 = 2\text{cis}33^\circ = 2(\cos 33^\circ + i\sin 33^\circ)$

$$\begin{aligned}\frac{z_1}{z_2} &= \frac{\sqrt{3}}{2}(\cos(213^\circ - 33^\circ) + i\sin(213^\circ - 33^\circ)) \\ &= \frac{\sqrt{3}}{2}(\cos 180^\circ + i\sin 180^\circ) \\ &= \frac{\sqrt{3}}{2}\text{cis}180^\circ\end{aligned}$$

We can leave our answer in polar form or we can convert it into standard/rectangular form.

$$\begin{aligned}\frac{\sqrt{3}}{2}(\cos 180^\circ + i\sin 180^\circ) &= \frac{\sqrt{3}}{2}(-1 + i0) \qquad \cos 180^\circ = -1,\ \sin 180^\circ = 0 \\ &= -\frac{\sqrt{3}}{2} + 0i \\ &= -\frac{\sqrt{3}}{2}\end{aligned}$$

Example 3.4

Find the quotient $\frac{z_1}{z_2}$, given $z_1 = 2\sqrt{3}(\cos 352^\circ + i\sin 352^\circ)$and $z_2 = 2(\cos 52^\circ + i\sin 52^\circ)$.

Solution

$$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} = \frac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2))$$

$z_1 = 2\sqrt{3}(\cos 352^\circ + i\sin 352^\circ)$ and $z_2 = 2(\cos 52^\circ + i\sin 52^\circ)$

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{2\sqrt{3}}{2}(\cos(352^\circ - 52^\circ) + i\sin(352^\circ - 52^\circ)) \\ &= \sqrt{3}(\cos 300^\circ + i\sin 300^\circ) \\ &= \sqrt{3}\text{cis}300^\circ \end{aligned}$$

We can leave our answer in polar form or we can convert it into standard/rectangular form.

$$\cos 300^\circ = \cos(360^\circ - 60^\circ) = \cos 60^\circ = \frac{1}{2}$$

$$\sin 300^\circ = \sin(360^\circ + 60^\circ) = -\sin 60^\circ = -\frac{\sqrt{3}}{2}$$

$$\begin{aligned} \sqrt{3}(\cos 300^\circ + i\sin 300^\circ) &= \sqrt{3}\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) \\ &= \frac{\sqrt{3}}{2} - \frac{3}{2}i \end{aligned}$$

Quotients of complex numbers in polar form:

If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ then

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2))$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\text{cis}(\theta_1 - \theta_2)$$

Exercise 3.2

1. Find $\frac{z_1}{z_2}$ leaving your answer in polar form:
   a. $z_1 = 7\text{cis}155^\circ;\ z_2 = 3\text{cis}20^\circ$
   b. $z_1 = \sqrt{2}\text{cis}215^\circ;\ z_2 = 2\sqrt{2}\text{cis}100^\circ$
2. Find $\frac{z_1}{z_2}$ leaving your answer in standard form:
   a. $z_1 = \sqrt{3}\text{cis}90^\circ;\ z_2 = \sqrt{3}\text{cis}60^\circ$
   b. $z_1 = 12\text{cis}270^\circ;\ z_2 = 3\sqrt{2}\text{cis}45^\circ$

The full solutions are at the end of the unit.

Summary

In this unit you have learnt the following:

- How to multiply complex numbers in polar form.
- How to divide complex numbers in polar form.

Unit 3: Assessment

Suggested time to complete: 45 minutes

1. Determine $8\text{cis}70^\circ \times 3\text{cis}25^\circ$, leaving your answer in polar form.
2. Find the product of $z_1 = 4\text{cis}80^\circ$ and $z_2 = 2\text{cis}145^\circ$, leaving your answer in polar form.
3. Find the quotient of $z_1 = 2\text{cis}223^\circ$ and $z_2 = 4\text{cis}43^\circ$, leaving your answer in rectangular form.
4. Simplify the following complex expression, leaving your answer in polar form:
$$\frac{2.5\text{cis}35^\circ \times 4\text{cis}45^\circ}{2\text{cis}60^\circ}$$
5. Simplify the following complex expression without a calculator, leaving your answer in standard form:
$$\frac{5\text{cis}175^\circ \times 3\sqrt{2}\text{cis}145^\circ}{\sqrt{6}\text{cis}5^\circ}$$
6. An electrical circuit has impedances that are expressed as complex numbers $Z_A = 3 - 3i$ and $Z_B = 5 + 3i$. Calculate the value of the impedance Z_{AB} if $Z_{AB} = \dfrac{Z_A \times Z_B}{Z_A + Z_B}$ and leave your answer in polar form.

The full solutions are at the end of the unit.

Unit 3: Solutions

Exercise 3.1

1.

a. $z_1 = 2\sqrt{3}\text{cis}116^\circ;\ z_2 = 2\text{cis}109^\circ$

$$\begin{aligned} z_1 z_2 &= r_1 r_2 \left(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\right) \\ &= 4\sqrt{3}(\cos 225^\circ + i\sin 225^\circ) \\ &= 4\sqrt{3}\text{cis}225^\circ \end{aligned}$$

b. $z_1 = \sqrt{2}\text{cis}210^\circ;\ z_2 = 2\sqrt{2}\text{cis}100^\circ$

$$
\begin{aligned}
z_1 z_2 &= r_1 r_2\left(\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2)\right)\\
&= 4(\cos 310^\circ + i\sin 310^\circ)\\
&= 4\text{cis}310^\circ
\end{aligned}
$$

2.

a. $z_1 = \frac{1}{3}\text{cis}120^\circ;\ z_2 = 4\text{cis}15^\circ$

$$
\begin{aligned}
z_1 z_2 &= r_1 r_2\left(\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2)\right)\\
&= \frac{4}{3}(\cos 135^\circ + i\sin 135^\circ)\\
&= \frac{4}{3}\text{cis}135^\circ
\end{aligned}
$$

$$\cos 135^\circ = \cos(180^\circ - 45^\circ) = -\cos 45^\circ = -\frac{1}{\sqrt{2}}$$

$$\sin 135^\circ = \sin(180^\circ - 45^\circ) = \sin 45^\circ = \frac{1}{\sqrt{2}}$$

$$
\begin{aligned}
z &= \frac{4}{3}\left(-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)\\
&= -\frac{4}{3\sqrt{2}}+\frac{4}{3\sqrt{2}}i\\
&= -\frac{4\sqrt{2}}{6}+\frac{4\sqrt{2}}{6}i\\
&= -\frac{2\sqrt{2}}{3}+\frac{2\sqrt{2}}{3}i
\end{aligned}
$$

b. $z_1 = \sqrt{3}\text{cis}120^\circ;\ z_2 = \frac{1}{\sqrt{2}}\text{cis}60^\circ$

$$
\begin{aligned}
z_1 z_2 &= r_1 r_2\left(\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2)\right)\\
&= \frac{\sqrt{3}}{\sqrt{2}}(\cos 180^\circ + i\sin 180^\circ)\\
&= \frac{\sqrt{3}}{\sqrt{2}}\text{cis}180^\circ
\end{aligned}
$$

$$\cos 180^\circ = -1$$
$$\sin 180^\circ = 0$$

$$
\begin{aligned}
z &= \frac{\sqrt{3}}{\sqrt{2}}(-1+0i)\\
&= -\frac{\sqrt{3}}{\sqrt{2}} \quad \text{Rationalise the denominator by multiplying by } \frac{\sqrt{2}}{\sqrt{2}}\\
&= -\frac{\sqrt{2}\sqrt{3}}{2}\\
&= -\frac{\sqrt{6}}{2}
\end{aligned}
$$

Back to Exercise 3.1

Exercise 3.2

1.

a. $z_1 = 7\text{cis}155^\circ;\ z_2 = 3\text{cis}20^\circ$

$$\begin{aligned}\frac{z_1}{z_2} &= \frac{r_1}{r_2}(\cos(\theta_1-\theta_2)+i\sin(\theta_1-\theta_2))\\ &= \frac{7}{3}(\cos 135^\circ + i\sin 135^\circ)\\ &= \frac{7}{3}\text{cis}135^\circ\end{aligned}$$

b. $z_1 = \sqrt{2}\text{cis}215^\circ;\ z_2 = 2\sqrt{2}\text{cis}100^\circ$

$$\begin{aligned}\frac{z_1}{z_2} &= \frac{r_1}{r_2}(\cos(\theta_1-\theta_2)+i\sin(\theta_1-\theta_2))\\ &= \frac{\sqrt{2}}{2\sqrt{2}}(\cos 115^\circ + i\sin 115^\circ)\\ &= \frac{1}{2}\text{cis}115^\circ\end{aligned}$$

2.

a. $z_1 = \sqrt{3}\text{cis}90^\circ;\ z_2 = \sqrt{3}\text{cis}60^\circ$

$$\begin{aligned}\frac{z_1}{z_2} &= \frac{r_1}{r_2}(\cos(\theta_1-\theta_2)+i\sin(\theta_1-\theta_2))\\ &= \frac{\sqrt{3}}{\sqrt{3}}(\cos 30^\circ + i\sin 30^\circ)\\ &= 1\text{cis}30^\circ\end{aligned}$$

$$\cos 30^\circ = \frac{\sqrt{3}}{2}$$

$$\sin 30^\circ = \frac{1}{2}$$

$$\begin{aligned}z &= 1\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)\\ &= \frac{\sqrt{3}}{2}+\frac{1}{2}i\end{aligned}$$

b. $z_1 = 12\text{cis}270^\circ;\ z_2 = 3\sqrt{2}\text{cis}45^\circ$

$$\begin{aligned}\frac{z_1}{z_2} &= \frac{r_1}{r_2}(\cos(\theta_1-\theta_2)+i\sin(\theta_1-\theta_2))\\ &= \frac{12}{3\sqrt{2}}(\cos 225^\circ + i\sin 225^\circ)\\ &= \frac{4}{\sqrt{2}}\text{cis}225^\circ\\ &= \frac{4\sqrt{2}}{2}\text{cis}225^\circ\\ &= 2\sqrt{2}\text{cis}225^\circ\end{aligned}$$

$$\cos 225^\circ = \cos(180^\circ + 45^\circ) = -\cos 45^\circ = -\frac{1}{\sqrt{2}}$$

$$\sin 225^\circ = \sin(180^\circ + 45^\circ) = -\sin 45^\circ = -\frac{1}{\sqrt{2}}$$

$$\begin{aligned}z &= 2\sqrt{2}\left(-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i\right)\\ &= -\frac{2\sqrt{2}}{\sqrt{2}}-\frac{2\sqrt{2}}{\sqrt{2}}i\\ &= -2-2i\end{aligned}$$

Back to Exercise 3.2

Unit 3: Assessment

1. $8\text{cis}70^\circ \times 3\text{cis}25^\circ = 24\text{cis}95^\circ$

2. $z_1 = 4\text{cis}80^\circ$ and $z_2 = 2\text{cis}145^\circ$

$$\begin{aligned} z_1 z_2 &= r_1 r_2\left(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right) \\ &= 8(\cos 225^\circ + i\sin 225^\circ) \\ &= 8\text{cis}225^\circ \end{aligned}$$

3. $z_1 = 2\text{cis}223^\circ$ and $z_2 = 4\text{cis}43^\circ$

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{r_1}{r_2}(\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)) \\ &= \frac{1}{2}(\cos(223^\circ - 43^\circ) + i\sin(223^\circ - 43^\circ)) \\ &= \frac{1}{2}(\cos 180^\circ + i\sin 180^\circ) \\ &= \frac{1}{2}\text{cis}180^\circ \end{aligned}$$

$$\cos 180^\circ = -1$$
$$\sin 180^\circ = 0$$
$$\begin{aligned} z &= \frac{1}{2}(-1+0i) \\ &= -\frac{1}{2} \end{aligned}$$

4.

$$\begin{aligned} \frac{2.5\text{cis}35^\circ \times 4\text{cis}45^\circ}{2\text{cis}60^\circ} &= \frac{10\text{cis}80^\circ}{2\text{cis}60^\circ} \\ &= 5\text{cis}20^\circ \end{aligned}$$

5.

$$\begin{aligned} \frac{5\text{cis}175^\circ \times 3\sqrt{2}\text{cis}145^\circ}{\sqrt{6}\text{cis}5^\circ} &= \frac{15\sqrt{2}\text{cis}320^\circ}{\sqrt{6}\text{cis}5^\circ} \\ &= \frac{15\sqrt{2}}{\sqrt{2}\sqrt{3}}\text{cis}315^\circ \\ &= \frac{15}{\sqrt{3}}\text{cis}315^\circ \end{aligned}$$

$$\cos 315^\circ = \cos(360^\circ - 45^\circ) = \cos 45^\circ = \frac{1}{\sqrt{2}}$$
$$\sin 315^\circ = \sin(360^\circ - 45^\circ) = -\sin 45^\circ = -\frac{1}{\sqrt{2}}$$

$$\begin{aligned} z &= \frac{15}{\sqrt{3}}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) \\ &= \frac{15}{\sqrt{6}} - \frac{15}{\sqrt{6}}i \quad \text{Rationalise denominators by multiply by } \frac{\sqrt{6}}{\sqrt{6}} \\ &= \frac{15\sqrt{6}}{6} - \frac{15\sqrt{6}}{6}i \\ &= \frac{5\sqrt{6}}{2} - \frac{5\sqrt{6}}{2}i \end{aligned}$$

6. $Z_A = 3 - 3i$ and $Z_B = 5 + 3i$

$$Z_{AB} = \frac{Z_A \times Z_B}{Z_A + Z_B}$$
$$= \frac{(3-3i)(5+3i)}{(3-3i)+(5+3i)}$$
$$= \frac{15+9i-15i-9i^2}{8}$$
$$= \frac{15-6i+9}{8}$$
$$= \frac{24-6i}{8}$$
$$= 3 - \frac{3}{4}i$$
$$|Z_{AB}| = \sqrt{3^2 + \left(-\frac{3}{4}\right)^2}$$
$$= \sqrt{9 + \frac{9}{16}}$$
$$= \sqrt{\frac{144+9}{16}}$$
$$= \frac{\sqrt{153}}{4}$$

Z_{AB} is in the fourth quadrant.

$$\sin\alpha = \frac{\frac{3}{4}}{\frac{\sqrt{153}}{4}} = \frac{3}{4} \times \frac{4}{\sqrt{153}} = \frac{3}{\sqrt{153}}$$
$$\therefore \alpha = 14.04^\circ$$
$$\theta = 360^\circ - 14.04^\circ = 345.96^\circ$$
$$Z_{AB} = \frac{\sqrt{153}}{4}\text{cis}345.96^\circ$$

Back to Unit 3: Assessment

Unit 4: Raise complex numbers to exponents using De Moivre's Theorem

DYLAN BUSA

Unit outcomes

By the end of this unit you will be able to:

- Find the powers of complex numbers in polar form.
- Simplify complex expressions with powers.

What you should know

Before you start this unit, make sure you can:

- Multiply complex numbers in polar form. Refer to unit 3 if you need help with this.
- Multiply complex numbers in polar form. Refer to unit 3 if you need help with this.
- Convert between the rectangular form and polar form of complex numbers. Refer to unit 2 if you need help with this.

Introduction

In this unit we are going to learn how to deal with the powers of complex numbers. As you will see, the polar form is extremely useful in simplifying what would otherwise be very messy and difficult calculations.

De Moivre's theorem

Abraham de Moivre (pictured in Figure 1) was a French mathematician of the late 17th and early 18th century. He spent most of his life in England having escaped religious persecution in France, and was a friend of Isaac Newton and James Stirling.

He is most famous for his theorem that links the complex numbers to trigonometry, but perhaps contributed most significantly to the study of probability.

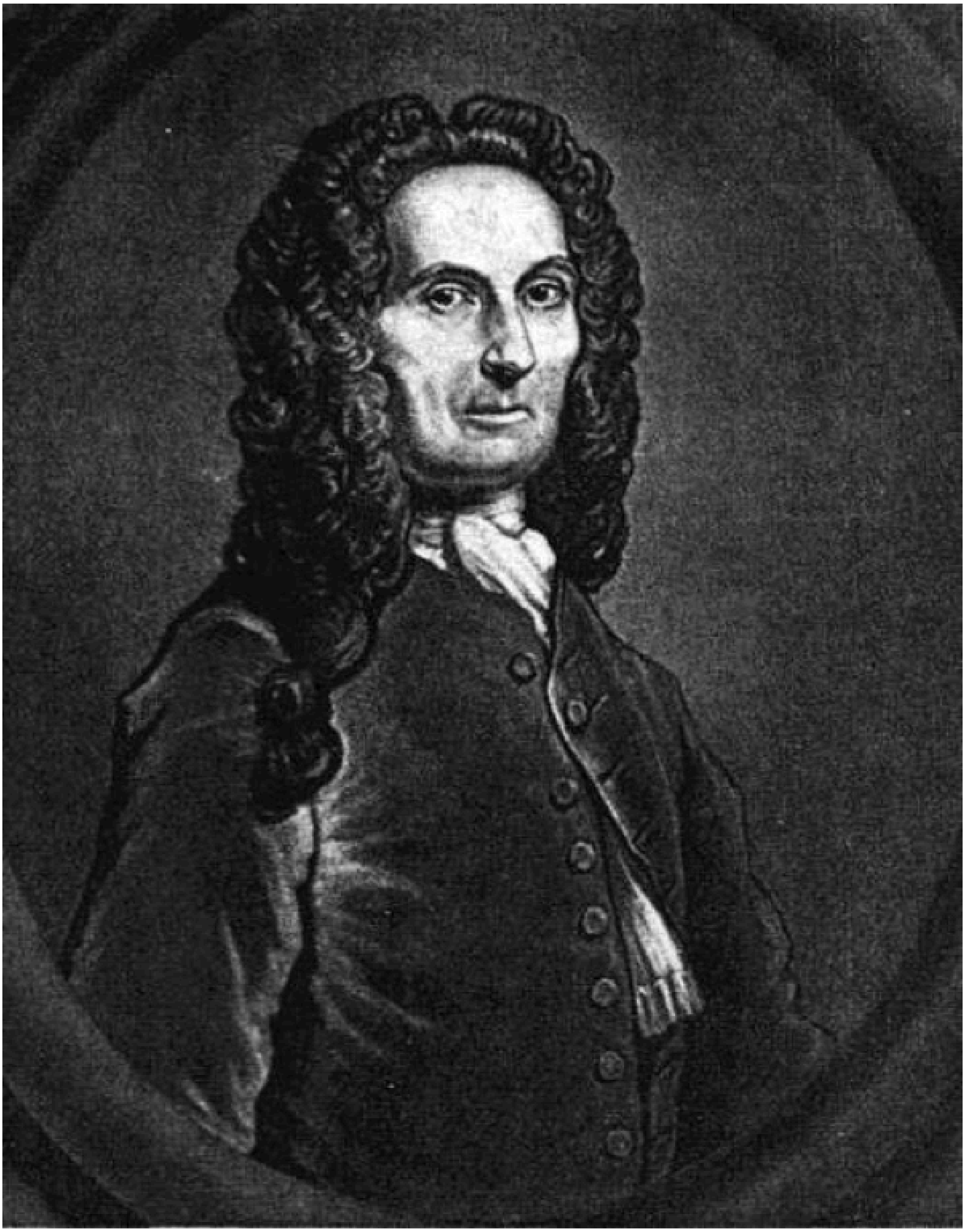

Figure 1: Abraham de Moivre

In Activity 4.1, we will discover de Moivre's theorem for ourselves.

Activity 4.1: De Moivre's theorem

Time required: 15 minutes

What you need:

- a pen or pencil
- a blank piece of paper

What to do:

1. If $z = r(\cos\theta + i\sin\theta)$ write down an expression for z^2. Hint: What is $z_1 z_2$?
2. Using the fact that $z^3 = z^2 \cdot z$, write down an expression for z^3.
3. Using the fact that $z^4 = z^3 \cdot z$, write down an expression for z^4.
4. Write down an expression for z^5.
5. Write down an expression for z^n.

What did you find?

1. If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then we know that $z_1 z_2 = r_1 r_2\,(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2))$. Therefore, we can say that if $z = r(\cos\theta + i\sin\theta)$, then
$$\begin{aligned} z^2 &= r \cdot r\,[\cos(\theta+\theta) + i\sin(\theta+\theta)] \\ &= r^2(\cos 2\theta + i\sin 2\theta) \end{aligned}$$
2. $z^3 = z^2 \cdot z$, therefore:
$$\begin{aligned} z^3 &= r^2(\cos 2\theta + i\sin 2\theta) \times r(\cos\theta + i\sin\theta) \\ &= r^2 \cdot r\,[\cos(2\theta+\theta) + i\sin(2\theta+\theta)] \\ &= r^3(\cos 3\theta + i\sin 3\theta) \end{aligned}$$
3. $z^4 = z^3 \cdot z$, therefore:
$$\begin{aligned} z^4 &= r^3(\cos 3\theta + i\sin 3\theta) \times r(\cos\theta + i\sin\theta) \\ &= r^3 \cdot r\,[\cos(3\theta+\theta) + i\sin(3\theta+\theta)] \\ &= r^4(\cos 4\theta + i\sin 4\theta) \end{aligned}$$
4. $z^5 = z^4 \cdot z$, therefore:
$$\begin{aligned} z^5 &= r^4(\cos 4\theta + i\sin 4\theta) \times r(\cos\theta + i\sin\theta) \\ &= r^4 \cdot r\,[\cos(4\theta+\theta) + i\sin(4\theta+\theta)] \\ &= r^5(\cos 5\theta + i\sin 5\theta) \end{aligned}$$
5. $z^n = r^n\,[\cos(n\theta) + i\sin(n\theta)]$

Congratulations! You have just discovered de Moivre's theorem for yourself. You are a mathematician! The process we used to derive de Moivre's theorem is called mathematical induction and is a technique used to prove results statements for natural numbers using a domino effect. We show that the first case is true and that if this is true then the next case must also be true.

De Moivre's theorem:

If $z = r(\cos\theta + i\sin\theta)$ is a complex number then
$z^n = r^n\,[\cos(n\theta) + i\sin(n\theta)]$
$z^n = r^n \operatorname{cis}(n\theta)$
Where n is a positive integer.

Example 4.1

Evaluate $(\sqrt{3}+i)^5$ using de Moivre's theorem, leaving your answer in standard form.

Solution

De Moivre's theorem applies to complex numbers in polar form, so we first need to write our number in polar form.

$$z=\sqrt{3}+i$$
$$\therefore |z|=\sqrt{(\sqrt{3})^2+1^2}$$
$$=\sqrt{3+1}$$
$$=2$$

z is in the first quadrant.

$$\sin\theta=\frac{1}{2}$$
$$\therefore \theta=30^\circ$$
$$z=2(\cos 30^\circ+i\sin 30^\circ)$$

Now we can use de Moivre's theorem to evaluate $(\sqrt{3}+i)^5$.

$$\begin{aligned}(\sqrt{3}+i)^5 &= [2(\cos 30^\circ+i\sin 30^\circ)]^5\\ &= 2^5[\cos(5\times 30^\circ)+i\sin(5\times 30^\circ)]\\ &= 32(\cos 150^\circ+i\sin 150^\circ)\\ &= 32\left(-\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)\\ &= -16\sqrt{3}+16i\end{aligned}$$

Example 4.2

Evaluate $(2-2i)^8$ using de Moivre's theorem, leaving your answer in standard form.

Solution

De Moivre's theorem applies to complex numbers in polar form, so we first need to write our number in polar form.

$$z=2-2i$$
$$\therefore |z|=\sqrt{2^2+(-2)^2}$$
$$=\sqrt{4+4}$$
$$=\sqrt{8}$$
$$=2\sqrt{2}$$

z is in the fourth quadrant.

$$\sin\alpha=\frac{2}{2\sqrt{2}}=\frac{1}{\sqrt{2}}$$
$$\therefore \alpha=45^\circ$$
$$\theta=360^\circ-45^\circ=315^\circ$$
$$z=2\sqrt{2}\text{cis}315^\circ$$

Now we can use de Moivre's theorem to evaluate $(2 - 2i)^8$.

$$\begin{aligned}(2-2i)^8 &= \left(2\sqrt{2}\text{cis}315^\circ\right)^8 \\ &= \left(2\sqrt{2}\right)^8\left[\text{cis}(8 \times 315^\circ)\right] \\ &= 256 \times 16\text{cis}2\ 520^\circ \\ &= 4\ 096\text{cis}2\ 520^\circ\end{aligned}$$

It is better to leave your answer with an angle that is between zero and 360° by adding or subtracting the necessary multiple of 360° from it.

Therefore, $(2 - 2i)^8 = 4\ 096\text{cis}0^\circ$.

Example 4.3

Use de Moivre's theorem to evaluate $\dfrac{(3+2i)^6}{5-3i}$, leaving your answer in polar form.

Solution

Remember that to use de Moivre's theorem, the complex number must be in polar form. We need to convert $3 + 2i$ to polar form to evaluate the numerator. In questions like this, where we need to work separately with different complex numbers in the expression, it is a good idea to designate each part of the expression as a differently numbered complex number.

In $\dfrac{(3+2i)^6}{5-3i}$, let $z_1 = 3 + 2i$ and $z_2 = 5 - 3i$

$$\begin{aligned}z_1 &= 3 + 2i \\ |z_1| &= \sqrt{3^2 + 2^2} \\ &= \sqrt{9+4} \\ &= \sqrt{13}\end{aligned}$$

z_1 is in the first quadrant.

$$\begin{aligned}\sin\theta_1 &= \frac{2}{\sqrt{13}} \\ \therefore \theta_1 &= 33.69^\circ \\ z_1 &= \sqrt{13}\text{cis}33.69^\circ\end{aligned}$$

Now we can evaluate $(3 + 2i)^6$.

$$\begin{aligned}(z_1)^6 &= (3+2i)^6 \\ &= \left[\sqrt{13}\text{cis}33.69^\circ\right]^6 \\ &= \left(\sqrt{13}\right)^6\left[\text{cis}(6 \times 33.69^\circ)\right] \\ &= 2\ 197\text{cis}202.14^\circ \\ &= 2\ 197(\cos 202.14^\circ + i\sin 202.14^\circ)\end{aligned}$$

At this point, we can either convert the numerator into standard form, or convert the denominator into polar form. Dividing complex numbers in polar form tends to be easier so converting everything to polar form is normally the best option.

$z_2 = 5 - 3i$

$$
\begin{aligned}
|z_2| &= \sqrt{5^2 + (-3)^2} \\
&= \sqrt{25 + 9} \\
&= \sqrt{34}
\end{aligned}
$$

z_2 is in the fourth quadrant.

$$
\sin\alpha = \frac{3}{\sqrt{34}}
$$

$$
\therefore \alpha = 30.96^\circ
$$

$$
\theta_2 = 360^\circ - 30.96^\circ = 329.04^\circ
$$

$$
z_2 = \sqrt{34}\text{cis}329.04^\circ
$$

Therefore:

$$
\begin{aligned}
\frac{(3+2i)^6}{5-3i} &= \frac{2\ 197\text{cis}202.14^\circ}{\sqrt{34}\text{cis}329.04^\circ} \\
&= \frac{2\ 197}{\sqrt{34}}\text{cis}(202.14^\circ - 329.04^\circ) \\
&= \frac{2\ 197}{\sqrt{34}}\text{cis}(-126.9^\circ)
\end{aligned}
$$

You can leave you answer with a negative angle, but it is better to change it into a positive angle by adding the necessary multiple of 360°.

$$
\frac{(3+2i)^6}{5-3i} = \frac{2\ 197}{\sqrt{34}}\text{cis}233.1^\circ
$$

Exercise 4.1

Use de Moivre's theorem to evaluate the following, leaving your answer in polar form:

1. $\left(\sqrt{3} - \sqrt{3}i\right)^7$
2. $\dfrac{(4-8i)^2}{\left(-3-\sqrt{3}i\right)^3}$

The full solutions are at the end of the unit.

Did you know?

De Moivre's theorem can be expanded to deal with roots of complex numbers as well. The symmetry is truly beautiful.

If $z = r(\cos\theta + i\sin\theta)$ is a complex number, then:

$$
z^{\frac{1}{n}} = r^{\frac{1}{n}}\left[\cos\left(\frac{\theta}{n}\right) + i\sin\left(\frac{\theta}{n}\right)\right]
$$

$$z^{\frac{1}{n}} = r^{\frac{1}{n}} \operatorname{cis}\left(\frac{\theta}{n}\right)$$

Where n is a positive integer.

However, fractional powers are excluded from the NC(V) curriculum.

Summary

In this unit you have learnt the following:

- What de Moivre's theorem is.
- How to use de Moivre's theorem to raise complex numbers to exponents

Unit 4: Assessment

Suggested time to complete: 45 minutes

Use de Moivre's theorem to evaluate the following and leave your answer in polar form:

1. $\left(-\sqrt{2}+\sqrt{2}i\right)^6$
2. $\dfrac{(2+5i)^3}{(1-2i)}$
3. $\dfrac{(4-8i)^3}{\left(\sqrt{3}+3i\right)^4}$
4. $\dfrac{(2+4i)}{\left(\sqrt{2}-3i\right)\left(\sqrt{2}+3i\right)^3}$

The full solutions are at the end of the unit.

Unit 4: Solutions

Exercise 4.1

1. $z = \sqrt{3} - \sqrt{3}i$
$$\begin{aligned} |z| &= \sqrt{3^2 + (-3)^2} \\ &= \sqrt{9+9} \\ &= \sqrt{18} \\ &= 3\sqrt{2} \end{aligned}$$
z is in the fourth quadrant.

$\sin\alpha = \dfrac{\sqrt{3}}{3\sqrt{2}}$

$\therefore \alpha = 24.09^\circ$

$\theta = 360^\circ - 24.09^\circ = 335.91^\circ$

$z = 3\sqrt{2}\text{cis}335.91^\circ$

$$\begin{aligned} z^7 &= \left(3\sqrt{2}\text{cis}335.91^\circ\right)^7 \\ &= \left(3\sqrt{2}\right)^7 \text{cis}(7 \times 335.91^\circ) \\ &= \left(2\ 187 \times 8\sqrt{2}\right)\text{cis}(2\ 351.37^\circ) \\ &= 17\ 496\sqrt{2}\text{cis}191.37^\circ \end{aligned}$$

2. $\dfrac{(4-8i)^2}{\left(-3-\sqrt{3}i\right)^3}$

Let $z_1 = 4 - 8i$ and $z_2 = 3 - \sqrt{3}i$

$z_1 = 4 - 8i$

$$\begin{aligned} |z_1| &= \sqrt{4^2 + (-8)^2} \\ &= \sqrt{16+64} \\ &= \sqrt{80} \\ &= 4\sqrt{5} \end{aligned}$$

z_1 is in the fourth quadrant.

$\sin\alpha = \dfrac{8}{4\sqrt{5}}$

$\therefore \alpha = 63.43^\circ$

$\theta = 360^\circ - 63.43^\circ = 296.57^\circ$

$z_1 = 4\sqrt{5}\text{cis}296.57^\circ$

$$\begin{aligned} (z_1)^2 &= \left(4\sqrt{5}\text{cis}296.57^\circ\right)^2 \\ &= \left(4\sqrt{5}\right)^2\text{cis}(2 \times 296.57^\circ) \\ &= (16 \times 5)\,\text{cis}593.14^\circ \\ &= 80\text{cis}593.14^\circ \end{aligned}$$

$z_2 = -3 - \sqrt{3}i$

$$\begin{aligned} |z_2| &= \sqrt{(-3)^2 + \left(-\sqrt{3}\right)^2} \\ &= \sqrt{9+3} \\ &= \sqrt{12} \\ &= 2\sqrt{3} \end{aligned}$$

z_2 is in the third quadrant.

$\sin\alpha = \dfrac{\sqrt{3}}{2\sqrt{3}} = \dfrac{1}{2}$

$\therefore \alpha = 30^\circ$

$\theta = 180^\circ + 30^\circ = 210^\circ$

$z_2 = 2\sqrt{3}\text{cis}210^\circ$

$$\begin{aligned} (z_2)^3 &= \left(2\sqrt{3}\text{cis}210^\circ\right)^3 \\ &= \left(2\sqrt{3}\right)^3\text{cis}(3 \times 210^\circ) \\ &= \left(8 \times 3\sqrt{3}\right)\text{cis}630^\circ \\ &= 24\sqrt{3}\text{cis}630^\circ \end{aligned}$$

$$\frac{(4-8i)^2}{\left(-3-\sqrt{3}i\right)^3}=\frac{80\text{cis}593.14^\circ}{24\sqrt{3}\text{cis}630^\circ}$$
$$=\frac{10}{3\sqrt{3}}\text{cis}(593.14^\circ-630^\circ)$$
$$=\frac{10\sqrt{3}}{9}\text{cis}(-36.86^\circ)$$
$$=\frac{10\sqrt{3}}{9}\text{cis}323.14^\circ$$

Back to Exercise 4.1

Unit 4: Assessment

1. $z=-\sqrt{2}+\sqrt{2}i$
   $$|z|=\sqrt{\left(-\sqrt{2}\right)^2+\left(\sqrt{2}\right)^2}$$
   $$=\sqrt{2+2}$$
   $$=\sqrt{4}$$
   $$=2$$
   z is in the second quadrant.
   $$\sin\alpha=\frac{\sqrt{2}}{2}=\frac{1}{\sqrt{2}}$$
   $$\therefore\alpha=45^\circ$$
   $$\theta=180^\circ-45^\circ=135^\circ$$
   $$z=2\text{cis}135^\circ$$
   $$(z)^6=(2\text{cis}135^\circ)^6$$
   $$=(2)^6\text{cis}(6\times135^\circ)$$
   $$=(64)\,\text{cis}810^\circ$$
   $$=64\text{cis}90^\circ$$

2. $\dfrac{(2+5i)^3}{(1-2i)}$
   Let and $z_2=1-2i$
   $$|z_1|=\sqrt{2^2+5^2}$$
   $$=\sqrt{4+25}$$
   $$=\sqrt{29}$$
   z_1 is in the first quadrant.
   $$\sin\theta=\frac{5}{\sqrt{29}}$$
   $$\therefore\theta=68.20^\circ$$
   $$z_1=\sqrt{29}\text{cis}68.20^\circ$$
   $$(z_1)^3=\left(\sqrt{29}\text{cis}68.20^\circ\right)^3$$
   $$=\left(\sqrt{29}\right)^3\text{cis}(3\times68.20^\circ)$$
   $$=29\sqrt{29}\text{cis}204.6^\circ$$

   $$|z_2|=\sqrt{1^2+(-2)^2}$$
   $$=\sqrt{1+4}$$
   $$=\sqrt{5}$$
   z_2 is in the fourth quadrant.

$\sin\alpha = \dfrac{2}{\sqrt{5}}$

$\therefore \alpha = 63.43^\circ$

$\theta = 360^\circ - 63.43^\circ = 296.57^\circ$

$z_2 = \sqrt{5}\text{cis}296.57^\circ$

$$\begin{aligned}\frac{(2+5i)^3}{(1-2i)} &= \frac{29\sqrt{29}\text{cis}204.6^\circ}{\sqrt{5}\text{cis}296.57^\circ} \\ &= \frac{29\sqrt{29}}{\sqrt{5}}\text{cis}(204.6^\circ - 296.57^\circ) \\ &= \frac{29\sqrt{29}}{\sqrt{5}}\text{cis}(-91.97^\circ) \\ &= \frac{29\sqrt{29}}{\sqrt{5}}\text{cis}268.03^\circ\end{aligned}$$

3. $\dfrac{(4-8i)^3}{\left(\sqrt{3}+3i\right)^4}$

Let $z_1 = 4 - 8i$ and $z_2 = \sqrt{3} + 3i$

$$\begin{aligned}|z_1| &= \sqrt{4^2 + (-8)^2} \\ &= \sqrt{16+64} \\ &= \sqrt{80} \\ &= 4\sqrt{5}\end{aligned}$$

z_1 is in the fourth quadrant.

$\sin\alpha = \dfrac{8}{4\sqrt{5}}$

$\therefore \alpha = 63.43^\circ$

$\theta = 360^\circ - 63.43^\circ = 296.57^\circ$

$z_1 = 4\sqrt{5}\text{cis}296.57^\circ$

$$\begin{aligned}(z_1)^3 &= \left(4\sqrt{5}\text{cis}296.57^\circ\right)^3 \\ &= \left(4\sqrt{5}\right)^3\text{cis}(3 \times 296.57^\circ) \\ &= 320\sqrt{5}\text{cis}889.71^\circ\end{aligned}$$

$$\begin{aligned}|z_2| &= \sqrt{\left(\sqrt{3}\right)^2 + 3^2} \\ &= \sqrt{3+9} \\ &= \sqrt{12} \\ &= 2\sqrt{3}\end{aligned}$$

z_2 is in the first quadrant.

$\sin\theta = \dfrac{3}{2\sqrt{3}} = \dfrac{3\sqrt{3}}{6} = \dfrac{\sqrt{3}}{2}$

$\therefore \theta = 60^\circ$

$z_2 = 2\sqrt{3}\text{cis}60^\circ$

$$\begin{aligned}(z_2)^4 &= \left(2\sqrt{3}\text{cis}60^\circ\right)^4 \\ &= \left(2\sqrt{3}\right)^4\text{cis}(4 \times 60^\circ) \\ &= 144\text{cis}240^\circ\end{aligned}$$

$$\frac{(4-8i)^3}{\left(\sqrt{3}+3i\right)^4} = \frac{320\sqrt{5}\text{cis}889.71^\circ}{144\text{cis}240^\circ}$$
$$= \frac{20\sqrt{5}}{9}\text{cis}(889.71^\circ - 240^\circ)$$
$$= \frac{20\sqrt{5}}{9}\text{cis}649.71^\circ$$
$$= \frac{20\sqrt{5}}{9}\text{cis}289.71^\circ$$

4.

$$\frac{(2+4i)}{\left(\sqrt{2}-3i\right)\left(\sqrt{2}+3i\right)^3} = \frac{(2+4i)}{\left(\sqrt{2}-3i\right)\left(\sqrt{2}+3i\right)\left(\sqrt{2}+3i\right)^2}$$
$$= \frac{(2+4i)}{\left(2-9i^2\right)\left(\sqrt{2}+3i\right)^2}$$
$$= \frac{(2+4i)}{11\left(\sqrt{2}+3i\right)^2}$$

Let $z_1 = 2+4i$ and $z_2 = \sqrt{2}+3i$.

$$|z_1| = \sqrt{2^2+4^2}$$
$$= \sqrt{4+16}$$
$$= \sqrt{20}$$
$$= 2\sqrt{5}$$

z_1 is in the first quadrant.

$$\sin\theta = \frac{4}{2\sqrt{5}}$$
$$\therefore \theta = 63.43^\circ$$
$$z_1 = 2\sqrt{5}\text{cis}63.43^\circ$$

$$|z_2| = \sqrt{\left(\sqrt{2}\right)^2+3^2}$$
$$= \sqrt{2+9}$$
$$= \sqrt{11}$$

z_2 is in the first quadrant.

$$\sin\theta = \frac{3}{\sqrt{11}}$$
$$\therefore \theta = 64.76^\circ$$
$$z_2 = \sqrt{11}\text{cis}64.76^\circ$$
$$(z_2)^2 = \left(\sqrt{11}\text{cis}64.76^\circ\right)^2$$
$$= \left(\sqrt{11}\right)^2\text{cis}(2\times 64.76^\circ)$$
$$= 11\text{cis}129.52^\circ$$

$$\frac{(2+4i)}{11\left(\sqrt{2}+3i\right)^2} = \frac{2\sqrt{5}\text{cis}63.43^\circ}{11\times 11\text{cis}129.52^\circ}$$
$$= \frac{2\sqrt{5}\text{cis}63.43^\circ}{144\text{cis}129.52^\circ}$$
$$= \frac{\sqrt{5}}{72}\text{cis}(63.43^\circ - 129.52^\circ)$$
$$= \frac{\sqrt{5}}{72}\text{cis}(-66.09^\circ)$$
$$= \frac{\sqrt{5}}{72}\text{cis}293.91^\circ$$

Back to Unit 4: Assessment

Media Attributions

SUBJECT OUTCOME II

COMPLEX NUMBERS: SOLVE PROBLEMS WITH COMPLEX NUMBERS

Subject outcome

Subject outcome 1.2: Solve problems with complex numbers

Learning outcomes

- Solve identical complex numbers in rectangular/standard form using the concept of simultaneous equations.
- Use complex numbers to solve equations that cannot be solved using the real number system by applying:
  - Factorisation
  - The quadratic formula.

Unit 1 outcomes

By the end of this unit you will be able to:

- Solve for unknowns in equivalent complex numbers using simultaneous equation techniques.
- Solve quadratic equations that have complex roots.

Unit 1: Solve complex number problems

DYLAN BUSA

Unit 1 outcomes

By the end of this unit you will be able to:

- Solve for unknowns in equivalent complex numbers using simultaneous equation techniques.
- Solve quadratic equations that have complex roots.

What you should know

Before you start this unit, make sure you can:

- Solve quadratic equations by factorising. Refer to level 3 subject outcome 2.3 unit 1 if you need help with this.
- Solve quadratic equations using the quadratic formula. Refer to level 3 subject outcome 2.3 unit 1 if you need help with this.
- Solve systems of two linear equations. Refer to level 3 subject outcome 2.3 unit 4 if you need help with this.

Introduction

The quadratic function $f(x) = x^2 - 2x + 3$ (shown in Figure 1) does not intersect the x-axis and therefore has no real roots. But it does have roots. They are non-real or complex roots. What are the complex roots of the function?

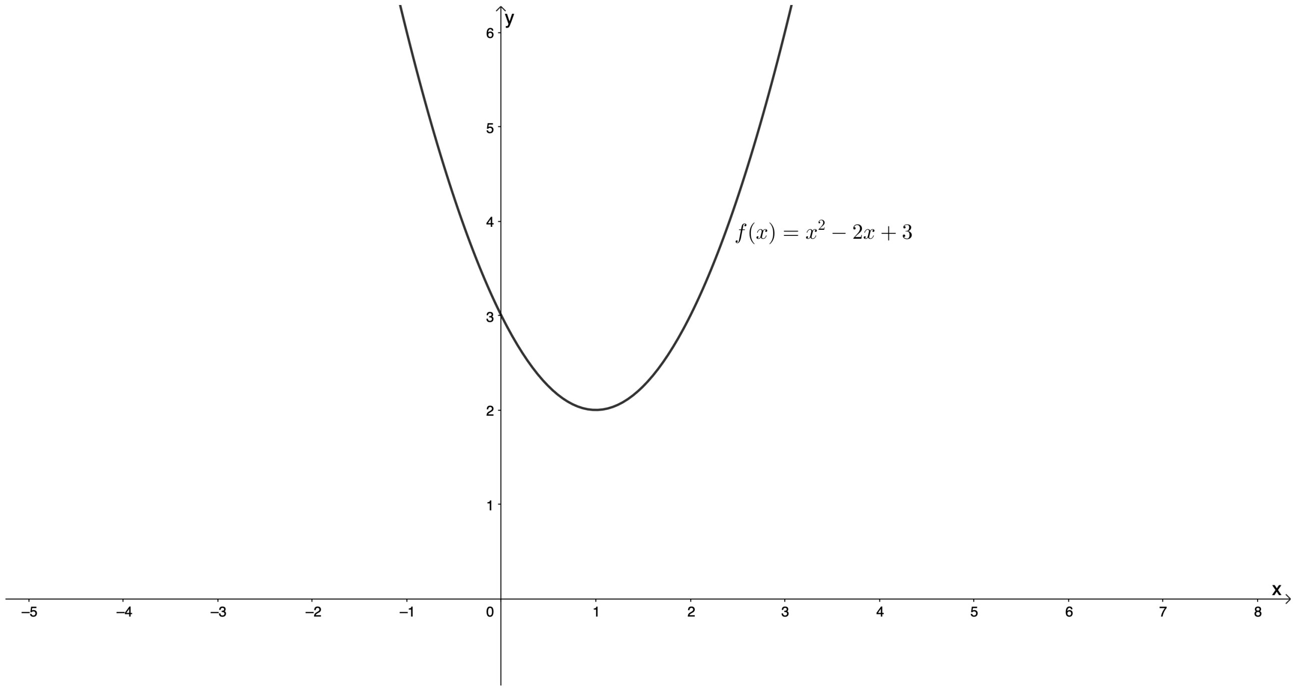

Figure 1: Graph of $f(x) = x^2 - 2x + 3$

To find the roots of $f(x)$, we know we need to solve for x where $f(x) = 0$. In other words, we need to solve the quadratic equation $x^2 - 2x + 3 = 0$. Unfortunately, the quadratic expression on the left-hand side does not factorise easily. However, we can use the quadratic formula to solve for x.

Solve equations with complex roots

Let's look at Example 1.1 to see how to solve for x using the quadratic formula.

Example 1.1

Solve for x in $x^2 - 2x + 3 = 0$ using the quadratic formula.

Solution

$a = 1, b = -2, c = 3$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(3)}}{2(1)}$$
$$= \frac{2 \pm \sqrt{4 - 12}}{2}$$
$$= \frac{2 \pm \sqrt{-8}}{2}$$
$$= \frac{2 \pm \sqrt{8}i}{2}$$
$$= \frac{2 \pm 2\sqrt{2}i}{2}$$
$$= 1 \pm \sqrt{2}i$$

Therefore, $x = 1 + \sqrt{2}i$ or $x = 1 - \sqrt{2}i$.

We still get two roots (or solutions) as we expect from a quadratic equation. However, both roots are complex.

Note

The part of the quadratic formula under the square root sign ($b^2 - 4ac$) is called the **discriminant**. It is given the symbol Δ (Delta), which is the Greek letter D.

- If $\Delta > 0$, then the roots of the quadratic equation are real and different.
- If $\Delta < 0$, then the roots of the quadratic equation are complex and different.
- If $\Delta = 0$, then the roots of the quadratic equation are real and the same.

Example 1.2

Solve for m in $m^2 - 2m + 5 = 0$.

Solution

The quadratic expression $m^2 - 2m + 5$ does not factorise easily, so we will use the quadratic formula.
$a = 1, b = -2, c = 5$

$$
\begin{aligned}
m &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(5)}}{2(1)} \\
&= \frac{2 \pm \sqrt{4 - 20}}{2} \\
&= \frac{2 \pm \sqrt{-16}}{2} \\
&= \frac{2 \pm 4i}{2} \\
&= 1 \pm 2i
\end{aligned}
$$

Therefore, $m = 1 + 2i$ or $m = 1 - 2i$.

Example 1.3

Solve for x:

$$\frac{3}{x+3} - \frac{2}{x+2} = 1 \qquad x \neq -3, x \neq -2$$

Solution

We follow all the same steps as usual when solving equations, starting with multiplying through by the LCD.

$$
\begin{aligned}
\frac{3}{x+3} - \frac{2}{x+2} &= 1 \qquad \text{LCD: } (x+3)(x+2) \\
\therefore 3(x+2) - 2(x+3) &= (x+3)(x+2) \\
\therefore 3x + 6 - 2x - 6 &= x^2 + 5x + 6 \\
\therefore x &= x^2 + 5x + 6 \\
\therefore x^2 + 4x + 6 &= 0
\end{aligned}
$$

The quadratic expression does not factorise easily, so we will use the quadratic formula.

$a = 1, b = 4, c = 6$

$$
\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{-(4) \pm \sqrt{(4)^2 - 4(1)(6)}}{2(1)} \\
&= \frac{-4 \pm \sqrt{16 - 24}}{2} \\
&= \frac{-4 \pm \sqrt{-8}}{2} \\
&= \frac{-4 \pm 2\sqrt{2}i}{2} \\
&= -2 \pm \sqrt{2}i
\end{aligned}
$$

Therefore, $x = -2 + \sqrt{2}i$ or $x = -2 - \sqrt{2}i$.

Exercise 1.1

Solve for x, expressing any complex roots in standard form:

1. $x^2 + x + 1 = 0$
2. $5x^2 - 8x = -6$
3. $2x^2 - 12x + 19 = 0$
4. $\frac{x}{(x-2)} + \frac{3}{(x-3)} = 3$
5. $x^2 + (x+1)^2 + (x+2)^2 = -1$
6. $x^4 = -8x^2$

The full solutions are at the end of the unit.

Solve complex simultaneous equations

We know that complex numbers have a real and imaginary part. If we have an equation such as $2x + 3yi = 1 + i$, we can easily find the values for x and y that will make the equation true. We simply need to equate the real and imaginary parts.

$$2x = 1$$
$$\therefore x = \frac{1}{2}$$
$$3y = 1$$
$$\therefore y = \frac{1}{3}$$

Sometimes we find x and y in both the real and imaginary parts. For example, if we wanted to find the values of x and y in $(x - y) - (x + y)i = 2 + 3i$, we could set up a system of two simultaneous equations and then solve for the unknowns.

Have a look at the next example to see how to answer this question.

Example 1.4

Solve for x and y in the following:
$(x - y) - (x + y)i = 2 + 3i$

Solution

On the left-hand side (LHS), the real part is $(x - y)$ and the imaginary part is $-(x + y)i$. Therefore $a = (x - y)$ and $b = -(x + y)$.

On the right-hand side (RHS) we have another complex number where $a = 2$ and $b = 3$.

But because the complex number on the LHS is equal to the complex number on the RHS, we know that real and imaginary parts must be equal, in other words, that $(x-y)=2$ and $-(x+y)=3$. We have a system of simultaneous equations that we can use to solve for x and y.

$(x-y)=2 \qquad (1)$
$-(x+y)=3 \qquad (2)$

From (1):
$(x-y)=2$
$\therefore x=2+y \qquad (3)$
Substitute (3) into (2):
$-((2+y)+y)=3$
$\therefore -(2+2y)=3$
$\therefore -2-2y=3$
$\therefore 2y=-5$
$\therefore y=-\frac{5}{2}$
Substitute $y=-\frac{5}{2}$ into (3):
$x=2-\frac{5}{2}$
$\therefore x=-\frac{1}{2}$

$x=-\frac{1}{2}$ and $y=-\frac{5}{2}$

Example 1.5

Solve for x and y in $\frac{2i-4}{3-2i}=\frac{i-x}{y-1}$.

Solution

We need to simplify first in order to set up a system of simultaneous equations.

$$\frac{2i-4}{3-2i}=\frac{i-x}{y-1} \qquad y\neq 1 \text{ LCD: } (3-2i)(y-1)$$
$$\therefore (2i-4)(y-1)=(i-x)(3-2i)$$
$$\therefore 2iy-2i-4y+4=3i-2i^2-3x-2xi \qquad \text{Collect the real and imaginary parts on each side}$$
$$\therefore (4-4y)+(2y-2)i=(2-3x)+(3-2x)i$$

Now we can set up a system of simultaneous equations by equating the real and imaginary parts of the complex numbers.
$(4-4y)=(2-3x)$
$\therefore 4-4y=2-3x$
$\therefore 4y-4=3x-2$
$\therefore 4y=3x+2$
$\therefore y=\frac{3}{4}x+\frac{1}{2} \qquad (1)$

$$(2y-2)=(3-2x)$$
$$\therefore 2y-2=3-2x$$
$$\therefore 2y=5-2x$$
$$\therefore y=\frac{5}{2}-x \qquad (2)$$

Substitute (1) into (2):

$$\frac{3}{4}x+\frac{1}{2}=\frac{5}{2}-x$$
$$\therefore 3x+2=10-4x$$
$$\therefore 7x=8$$
$$\therefore x=\frac{8}{7}$$

Substitute $x=\frac{8}{7}$ into (2):

$$y=\frac{5}{2}-\frac{8}{7}$$
$$=\frac{35-16}{14}$$
$$=\frac{19}{14}$$

$x=\frac{8}{7}$ and $y=\frac{19}{14}$

Exercise 1.2

Solve for x and y in each of the following:

1. $3x+4yi=\sqrt{3}-\sqrt{3}i$
2. $(2x-y)+(x-2y)i=4-2i$
3. $\frac{-2+i}{2+i}=\frac{2i-3x}{y-1}$

The full solutions are at the end of the unit.

Summary

In this unit you have learnt the following:

- How to solve equations where some of the roots are complex.
- How to solve for unknowns in equivalent complex numbers using the technique of simultaneous equations.

Unit 1: Assessment

Suggested time to complete: 15 minutes

1. Solve for x, leaving any complex roots in standard form:
   a. $2x^2 + 4x = -11$
   b. $-x^2 + x - 23 = 0$
   c. $-3x^2 + 2x = 14$
   d. $\frac{1}{2}x^2 + 4x = -12$
   e. $4x^2 = -x^4$
   f. $\frac{4}{(x+1)} - \frac{3}{x} = 7$
2. Solve for x and y in the following equations:
   a. $(x+y) - (x-y)i = 3 + 2i$
   b. $\frac{3i-2}{3+2i} = \frac{i+x}{y-1}$
   c. $x + yi = \frac{(2-3i)^2}{1+i}$
3. Show that $(5+4i)$ and $(3+2i)$ are factors of $(7+22i)$. Hence, or otherwise, determine the prime factors of $(7^2 + 22^2)$.

The full solutions are at the end of the unit.

Unit 1: Solutions

Exercise 1.1

1. $x^2 + x + 1 = 0$
   $a = 1, b = 1, c = 1$
   $$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(1) \pm \sqrt{(1)^2 - 4(1)(1)}}{2(1)} \\ &= \frac{-1 \pm \sqrt{1-4}}{2} \\ &= \frac{-1 \pm \sqrt{-3}}{2} \\ &= -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i \end{aligned}$$
   $x = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ or $x = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$
2.

$$5x^2 - 8x = -6$$
$$\therefore 5x^2 - 8x + 6 = 0$$
$$a = 5, b = -8, c = 6$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-(-8) \pm \sqrt{(-8)^2 - 4(5)(6)}}{2(5)}$$
$$= \frac{8 \pm \sqrt{64 - 120}}{10}$$
$$= \frac{8 \pm \sqrt{-56}}{10}$$
$$= \frac{8 \pm \sqrt{56}i}{10}$$
$$= \frac{8 \pm 2\sqrt{14}i}{10}$$
$$= \frac{4}{5} \pm \frac{\sqrt{14}}{5}i$$
$x = \frac{4}{5} + \frac{\sqrt{14}}{5}i$ or $x = \frac{4}{5} - \frac{\sqrt{14}}{5}i$

3. $2x^2 - 12x + 19 = 0$
$$a = 2, b = -12, c = 19$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-(-12) \pm \sqrt{(-12)^2 - 4(2)(19)}}{2(2)}$$
$$= \frac{12 \pm \sqrt{144 - 152}}{4}$$
$$= \frac{12 \pm \sqrt{-8}}{4}$$
$$= \frac{12 \pm 2\sqrt{2}i}{4}$$
$$= 3 \pm \frac{\sqrt{2}}{2}i$$
$x = 3 + \frac{\sqrt{2}}{2}i$ or $x = 3 - \frac{\sqrt{2}}{2}i$

4.
$$\frac{x}{(x-2)} + \frac{3}{(x-3)} = 3 \qquad x \neq 2, x \neq 3 \quad \text{LCD: } (x-2)(x-3)$$
$$\therefore x(x-3) + 3(x-2) = 3(x-3)(x-2)$$
$$\therefore x^2 - 3x + 3x - 6 = 3x^2 - 15x + 18$$
$$\therefore 2x^2 - 15x + 24 = 0$$
$$a = 2, b = -15, c = 24$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-(-15) \pm \sqrt{(-15)^2 - 4(2)(24)}}{2(2)}$$
$$= \frac{15 \pm \sqrt{225 - 192}}{4}$$
$$= \frac{15 \pm \sqrt{33}}{4}$$
$x = \frac{15 + \sqrt{33}}{4}$ or $x = \frac{15 - \sqrt{33}}{4}$

5.
$$x^2 + (x+1)^2 + (x+2)^2 = -1$$
$$\therefore x^2 + x^2 + 2x + 1 + x^2 + 4x + 4 = -1$$
$$\therefore 3x^2 + 6x + 6 = 0$$
$a = 3, b = 6, c = 6$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-(6) \pm \sqrt{(6)^2 - 4(3)(6)}}{2(3)}$$
$$= \frac{-6 \pm \sqrt{36 - 72}}{6}$$
$$= \frac{-6 \pm \sqrt{-36}}{6}$$
$$= \frac{-6 \pm 6i}{6}$$
$$= -1 \pm i$$
$x = -1 + i$ or $x = -1 - i$

6.
$$x^4 = -8x^2$$
$$\therefore x^4 + 8x^2 = 0$$
$$\therefore x^2(x^2 + 8) = 0$$
$$\therefore x^2 = 0 \text{ or } x^2 = -8$$
$$\therefore x = 0 \text{ or } x = \pm\sqrt{8}i$$
$$\therefore x = 0 \text{ or } x = \pm 2\sqrt{2}i$$

/li>

Back to Exercise 1.1

Exercise 1.2

1. $3x + 4yi = \sqrt{3} - \sqrt{3}i$
   $3x = \sqrt{3}$
   $\therefore x = \frac{\sqrt{3}}{3}$
   $4y = -\sqrt{3}$
   $\therefore y = -\frac{\sqrt{3}}{4}$
2. $(2x - y) + (x - 2y)i = 4 - 2i$
   $2x - y = 4 \qquad (1)$
   $x - 2y = -2 \qquad (2)$

From (2):

$x = 2y - 2 \qquad (3)$

Substitute (3) into (1):

$$\begin{aligned} 2(2y-2) - y &= 4 \\ \therefore 4y - 4 - y &= 4 \\ \therefore 3y &= 8 \\ \therefore y &= \frac{8}{3} \end{aligned}$$

Substitute $y = \frac{8}{3}$ into (1):

$$\begin{aligned} 2x - \frac{8}{3} &= 4 \\ \therefore 2x &= 4 + \frac{8}{3} \\ &= \frac{20}{3} \end{aligned}$$

$x = \frac{20}{3}$ and $y = \frac{8}{3}$

3.

$$\begin{aligned} \frac{-2+i}{2+i} &= \frac{2i-3x}{y-1} && y \neq 1 \text{ LCD: } (2+i)(y-1) \\ \therefore (-2+i)(y-1) &= (2i-3x)(2+i) \\ \therefore -2y + 2 + yi - i &= 4i + 2i^2 - 6x - 3xi && \text{Collect real amd imaginary terms on both sides} \\ \therefore (2-2y) + (y-1)i &= (-2-6x) + (4-3x)i \end{aligned}$$

The real parts on both sides are equal and the imaginary parts on both sides are equal.

$$\begin{aligned} (2-2y) &= (-2-6x) \\ \therefore 2 - 2y &= -2 - 6x \\ \therefore 2y &= 6x + 4 \\ \therefore y &= 3x + 2 \qquad (1) \end{aligned}$$

$$\begin{aligned} (y-1) &= (4-3x) \\ \therefore y - 1 &= 4 - 3x \\ \therefore y &= 5 - 3x \qquad (2) \end{aligned}$$

Substitute (1) into (2):

$$\begin{aligned} 3x + 2 &= 5 - 3x \\ \therefore 6x &= 3 \\ \therefore x &= \frac{1}{2} \end{aligned}$$

Substitute $x = \frac{1}{2}$ into (1):

$$\begin{aligned} y &= 3\left(\frac{1}{2}\right) + 2 \\ &= \frac{3}{2} + 2 \\ &= \frac{7}{2} \end{aligned}$$

$x = \frac{1}{2}$ and $y = \frac{7}{2}$

Back to Exercise 1.2

Unit 1: Assessment

1.

a.

$$
\begin{aligned}
2x^2 + 4x &= -11 \\
\therefore 2x^2 + 4x + 11 &= 0
\end{aligned}
$$

$$
\begin{aligned}
&a = 2, b = 4, c = 11 \\
&x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&\quad = \frac{-(4) \pm \sqrt{(4)^2 - 4(2)(11)}}{2(2)} \\
&\quad = \frac{-4 \pm \sqrt{16 - 88}}{4} \\
&\quad = \frac{-4 \pm \sqrt{-72}}{4} \\
&\quad = \frac{-4 \pm \sqrt{72}i}{4} \\
&\quad = \frac{-4 \pm 6\sqrt{2}i}{4} \\
&\quad = -1 \pm \frac{3\sqrt{2}}{2}i
\end{aligned}
$$

$x = -1 + \frac{3\sqrt{2}}{2}i$ or $x = -1 - \frac{3\sqrt{2}}{2}i$

b.

$$
\begin{aligned}
&-x^2 + x - 23 = 0 \\
&\therefore x^2 - x + 23 = 0 \\
&a = 1, b = -1, c = 23 \\
&x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&\quad = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(23)}}{2(1)} \\
&\quad = \frac{1 \pm \sqrt{1 - 92}}{2} \\
&\quad = \frac{1 \pm \sqrt{-91}}{2} \\
&\quad = \frac{1 \pm \sqrt{91}i}{2} \\
&\quad = \frac{1}{2} \pm \frac{\sqrt{91}}{2}i
\end{aligned}
$$

$x = \frac{1}{2} + \frac{\sqrt{91}}{2}i$ or $x = \frac{1}{2} - \frac{\sqrt{91}}{2}i$

c.

$$
\begin{aligned}
-3x^2 + 2x &= 14 \\
\therefore 3x^2 - 2x + 14 &= 0
\end{aligned}
$$

$a = 3, b = -2, c = 14$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(3)(14)}}{2(3)}$$

$$= \frac{2 \pm \sqrt{4 - 168}}{6}$$

$$= \frac{2 \pm \sqrt{-164}}{6}$$

$$= \frac{2 \pm \sqrt{164}i}{6}$$

$$= \frac{2 \pm 2\sqrt{41}i}{6}$$

$$= \frac{1}{3} \pm \frac{\sqrt{41}}{3}i$$

$x = \frac{1}{3} + \frac{\sqrt{41}}{3}i$ or $x = \frac{1}{3} - \frac{\sqrt{41}}{3}i$

d.

$$\frac{1}{2}x^2 + 4x = -12$$

$$\therefore x^2 + 8x + 24 = 0$$

$$a = 1, b = 8, c = 24$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(8) \pm \sqrt{(8)^2 - 4(1)(24)}}{2(1)}$$

$$= \frac{-8 \pm \sqrt{64 - 96}}{2}$$

$$= \frac{-8 \pm \sqrt{-32}}{2}$$

$$= \frac{-8 \pm \sqrt{32}i}{2}$$

$$= \frac{-8 \pm 4\sqrt{2}i}{2}$$

$$= -4 \pm 2\sqrt{2}i$$

$x = -4 + 2\sqrt{2}i$ or $x = -4 - 2\sqrt{2}i$

e.

$$4x^2 = -x^4$$

$$\therefore x^4 + 4x^2 = 0$$

$$\therefore x^2(x^2 + 4) = 0$$

$$\therefore x^2 = 0 \text{ or } x^2 = -4$$

$$\therefore x = 0 \text{ or } x = \pm\sqrt{-4}$$

$$\therefore x = 0 \text{ or } x = \pm 2i$$

f.

$$\frac{4}{(x+1)} - \frac{3}{x} = 7 \qquad x \neq -1, x \neq 0 \text{ LCD: } x(x+1)$$

$$\therefore 4x - 3(x+1) = 7x(x+1)$$

$$\therefore 4x - 3x - 3 = 7x^2 + 7x$$

$$\therefore 7x^2 + 6x + 3 = 0$$

$$a = 7, b = 6, c = 3$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(6) \pm \sqrt{(6)^2 - 4(7)(3)}}{2(7)}$$

$$= \frac{-6 \pm \sqrt{36 - 84}}{14}$$

$$= \frac{-6 \pm \sqrt{-48}}{14}$$

$$= \frac{-6 \pm \sqrt{48}i}{14}$$

$$= \frac{-6 \pm 4\sqrt{3}i}{14}$$

$$= -\frac{3}{7} \pm \frac{2\sqrt{3}}{7}i$$

$x = -\frac{3}{7} + \frac{2\sqrt{3}}{7}i$ or $x = -\frac{3}{7} - \frac{2\sqrt{3}}{7}i$

2.

a. $(x + y) - (x - y)i = 3 + 2i$

$$x + y = 3$$
$$\therefore x = 3 - y \qquad (1)$$

$$-(x - y) = 2$$
$$\therefore -x + y = 2$$
$$\therefore x = y - 2 \qquad (2)$$

Substitute (1) into (2):

$$3 - y = y - 2$$
$$\therefore 2y = 5$$
$$\therefore y = \frac{5}{2}$$

Substitute $y = \frac{5}{2}$ into (1):

$$x = 3 - \frac{5}{2}$$
$$\therefore x = \frac{1}{2}$$

$x = \frac{1}{2}$ and $y = \frac{5}{2}$

b.

$$\frac{3i - 2}{3 + 2i} = \frac{i + x}{y - 1} \qquad y \neq 1 \text{ LCD: } (3 + 2i)(y - 1)$$

$$\therefore (3i - 2)(y - 1) = (i + x)(3 + 2i)$$
$$\therefore 3yi - 3i - 2y + 2 = 3i + 2i^2 + 3x + 2xi$$
$$\therefore (2 - 2y) + (3y - 3)i = (3x - 2) + (3 + 2x)i$$
$$(2 - 2y) = (3x - 2)$$
$$\therefore 2 - 2y = 3x - 2$$
$$\therefore 2y = -3x + 4$$
$$\therefore y = -\frac{3}{2}x + 2 \qquad (1)$$

$$(3y-3)=(3+2x)$$
$$\therefore 3y-3=3+2x$$
$$\therefore 3y=2x+6$$
$$\therefore y=\frac{2}{3}x+2 \qquad (2)$$

Substitute (1) into (2):

$$-\frac{3x}{2}+2=\frac{2x}{3}+2$$
$$\therefore -9x+12=4x+12$$
$$\therefore 13x=0$$
$$\therefore x=0$$

Substitute $x=0$ into (1):

$y=2$

$x=0$ and $y=2$

c.

$$x+yi=\frac{(2-3i)^2}{1+i}$$
$$\therefore (x+yi)(1+i)=(2-3i)(2-3i)$$
$$\therefore x+xi+yi+yi^2=4-6i-6i+9i^2$$
$$\therefore (x-y)+(x+y)i=-5-12i$$

$$x-y=-5 \qquad x+y=-12$$
$$\therefore x=y-5 \quad (1) \qquad \therefore x=-y-12 \quad (2)$$

Substitute (1) into (2):

$$y-5=-y-12$$
$$\therefore 2y=-7$$
$$\therefore y=-\frac{7}{2}$$

Substitute $y=-\frac{7}{2}$ into (1):

$$x=-\frac{7}{2}-7$$
$$=\frac{-7-14}{2}$$
$$=-\frac{21}{2}$$

$x=-\frac{21}{2}$ and $y=-\frac{7}{2}$

3.

$$(5+4i)\times(3+2i)=15+10i+12i+8i^2$$
$$=15+22i-8$$
$$=7+22i$$

Therefore $(5+4i)$ and $(3+2i)$ are factors of $(7+22i)$.

$$(7^2+22^2)=7^2-22^2i^2$$
$$=7^2-(22i)^2$$
$$=(7+22i)(7-22i)$$

Now:

$$(5-4i)\times(3-2i)=15-10i-12i+8i^2$$
$$=15-22i-8$$
$$=7-22i$$

And $(7+22i)=(5+4i)(3+2i)$ (from above)

Therefore:

$$\begin{aligned}(7^2+22^2) &= (7+22i)(7-22i)\\ &= (5+4i)(3+2i)(5-4i)(3-2i)\\ &= ((5+4i)(5-4i))\,((3+2i)(3-2i))\\ &= (25-16i^2)(9-4i^2)\\ &= (25+16)(9+4)\\ &= (41)(13)\end{aligned}$$

Therefore, the prime factors of (7^2+22^2) are 41 and 13.

Back to Unit 1: Assessment

Media Attributions

SUBJECT OUTCOME III

FUNCTIONS AND ALGEBRA: WORK WITH ALGEBRAIC EXPRESSIONS USING THE REMAINDER AND THE FACTOR THEOREMS

Subject outcome

Subject outcome 2.1: Work with algebraic expressions using the remainder and the factor theorems

Learning outcomes

- Use and apply the remainder and the factor theorem.
  - Find the remainder.
  - Prove that an expression is a factor.
  - Find an unknown variable in order to make an expression, a factor or to leave a remainder.
- Factorise third degree polynomials including examples that require the factor theorem (long division or any other method may be used).

Unit 1 outcomes

By the end of this unit you will be able to:

- Find the factors of a cubic polynomial.
- Find the remainder of cubic polynomial.
- Use division by inspection, synthetic division or long division to factorise and solve cubic polynomials.

Unit 1: Use the remainder and factor theorem to factorise third degree polynomials

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Unit 1 outcomes

By the end of this unit you will be able to:

- Find the factors of a cubic polynomial.
- Find the remainder of cubic polynomial.
- Use division by inspection, synthetic division or long division to factorise and solve cubic polynomials.

What you should know

Before you start this unit, make sure you can:

- Manipulate and simplify algebraic expressions. To revise this topic, view level 3 subject outcome 2.2.
- Solve algebraic equations and inequalities. To revise this topic, view level 3 subject outcome 2.3.

Introduction

You have already seen examples of polynomial expressions in level 2 subject outcome 2.1, unit 1. Remember that a polynomial is an expression with one or more variables with different coefficients and non-negative integer powers.

We define a polynomial as: $a_nx^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0x^0$ where $n \in N_0$. a_n, a_{n-1}, a_2, a_1 and a_0 are the coefficients of each term and are usually constants.

There is an unlimited variety in the number of terms and the powers of the variable. While the order of the terms in a polynomial is not important for performing operations, we generally arrange the terms in decreasing powers of the variable. This is called the **general form**.

Notice that the definition of a polynomial states that all exponents of the variables must be elements of the set of whole numbers. If an expression contains terms with exponents that are not whole numbers, then it is not a polynomial.

For example, $\frac{2}{x} + 2x^2 + 1$, $3\sqrt{y} + y$ and $k^2 - 6k + 3k^{-\frac{1}{2}}$ are not polynomials. Look at each expression and make sure you understand why these are not polynomials. Can you see that in each case the exponents are not whole numbers?

The **degree** of a polynomial is the **highest power** of the variable. If the expression has been written in general form then it is the power of the first variable. The **leading term** is the term with the highest power of the variable, or the term with the highest degree. The **leading coefficient** is the coefficient of the leading term.

The polynomial $2x^3 - 3x^2 + x - 1$ has a degree of 3, the leading term is $2x^3$ and the leading coefficient is 2.

Exercise 1.1

State whether the following statements are true or false:

1. $2x^{-1}$ is a monomial because it only has one term.
2. 45 is a constant polynomial of degree 0.
3. $2x^2 - 3x + 1$ is a quadratic polynomial of degree 2 with a leading coefficient of 2.
4. A cubic polynomial always has three terms and all the exponents are natural numbers.

The full solutions are at the end of the unit.

Dividing cubic polynomials

In level 3 subject outcome 2.3 we looked at various methods to factorise and solve quadratic equations. In this unit we will factorise cubic polynomials with one variable. Remember that a cubic polynomial is an algebraic expression with a highest power of 3. The standard form of a cubic polynomial is $ax^3 + bx^2 + cx + d$.

The following activity is useful to remind ourselves of the basics of long division, which can be applied to polynomials too and is needed to factorise cubic polynomials.

Activity 1.1: Investigate simple division

Time required: 10 minutes

What you need:

- a pen and paper

What to do

Consider the following situation and answer the questions that follow.

Six learners are at a product promotion and there are 15 free gifts to be given away. Each learner must receive the same number of gifts.

1. How many gifts does each learner get?
2. How many gifts will be left over?
3. Use the following variables to express the above situation as a mathematical equation:
   $a =$ total number of gifts
   $b =$ total number of learners

$q =$ number of gifts for each learner
$r =$ number of gifts remaining

4. What is each part of the division expression called?
5. What does your equation say in words?

What did you find?

1. Since each learner has to a get the same whole number of gifts, each learner will get 2 gifts.
2. In total, the learners will get 12 gifts. Therefore, there will be 3 left over.
3. Using numerical values we can write this as $\frac{15}{6} = 2$ remainder 3. This is the same as$15 = 6 \times 2 + 3$.
   Using the variables this becomes $a = b \times q + r, \ b \neq 0$.
4. 15 is the dividend
   6 is the divisor
   2 is the quotient
   3 is the remainder
5. The dividend is equal to the divisor multiplied by the quotient, plus the remainder.

The activity reminded us that if an integer a is divided by an integer b, then the answer is q with a remainder of r. Sometimes $r = 0$.

For example, 8 divided by 3 gives a whole number answer of 2 with a remainder of 2.

We can write this as $\frac{8}{3} = 2 + \frac{2}{3}$ which is the same as saying that $8 = 3 \times 2 + 2$.

Take note!

This rule can be extended to include the division of polynomials; if a polynomial $f(x)$ is divided by a polynomial $g(x)$, then the answer is $Q(x)$ with a remainder $R(x)$.

$f(x) = g(x) \cdot Q(x) + R(x)$ where $g(x) \neq 0$.

We are familiar with the process of long division in ordinary arithmetic. Let's have a look at the process again to remind us how this is done.

$$\begin{array}{rl} \overline{59} & \\ 3)178 & \\ \underline{-15}\downarrow & \text{Step 1: } 5 \times 3 = 15 \text{ and } 17 - 15 = 2 \\ 28 & \text{Step 2: Bring down the } 8 \\ \underline{-27} & \text{Step 3: } 9 \times 3 = 27 \text{ and } 28 - 27 = 1 \\ 1 & \end{array}$$

$\frac{178}{3} = 59$ remainder 1 or $59\frac{1}{3}$.

Another way to look at the solution is as a sum of parts. This should look familiar, since it is the same method used to check division in elementary arithmetic.

$$\begin{aligned} \text{Dividend} &= \text{divisor} \cdot \text{quotient} + \text{remainder} \\ 178 &= 3 \cdot 59 + 1 \\ &= 177 + 1 \\ &= 178 \end{aligned}$$

You can use long division and synthetic division, which is explained in detail shortly, to find the quotient and remainder when a cubic polynomial is divided by another polynomial.

Example 1.1

Use the method of long division to find the quotient and remainder when $f(x) = 2x^3 + 3x^2 + x - 5$ is divided by $x - 1$.

Solution

Write down the known and unknown expressions.

$f(x) = g(x) \cdot Q(x) + R(x)$

$2x^3 + 3x^2 + x - 5 = (x - 1) \cdot Q(x) + R(x)$

Use long division to find $Q(x)$ and $R(x)$.

Make sure that $f(x)$ and $g(x)$ are written in descending order of the exponents. If a term of a certain degree is missing from $f(x)$, then write the term with a coefficient of 0.

$$\begin{array}{l} \qquad\quad \overline{2x^2+5x+4} \\ x-1)\,2x^3+3x^2+x-5 \\ \quad -\underline{(2x^3-2x^2)} \\ \qquad 5x^2+x \\ \quad -\underline{(5x^2-5x)} \\ \qquad 6x-5 \\ \quad -\underline{(6x-6)} \\ \qquad 1 \end{array}$$

Divide $2x^3$ by x and put the answer in the quotient

Multiply the quotient by the divisor and subtract from $f(x)$

Bring down the next term from $f(x)$ and repeat the process

Write the final answer.

$$\begin{aligned} Q(x) &= 2x^2 + 5x + 6 \\ R(x) &= 1 \\ \therefore 2x^3 + 3x^2 + x - 5 &= (x - 1) \cdot (2x^2 + 5x + 6) + 1 \end{aligned}$$

You can multiply the brackets and simplify to check that the answer is correct.

Exercise 1.2

1. Find the quotient and remainder of the following cubic polynomials by using long division:
   a. $6x^3 + x^2 - 4x + 5$ divided by $2x - 1$

b. x^3+3x^2-x-3 divided by $x-1$

2. Hence or otherwise, factorise 1b) completely.

The full solutions are at the end of the unit.

As you have seen, long division of polynomials involves many steps and can be quite cumbersome. Synthetic division is a shorthand method of dividing polynomials for the special case of dividing by a linear factor whose leading coefficient is 1.

Example 1.2

Use the method of synthetic division to find the quotient and remainder when:

1. $f(x)=2x^3+3x^2+x-5$ is divided by $x-1$.
2. $g(x)=6x^3+x^2-4x+5$ is divided by $2x-1$.

Solutions

1. Recall the long division solution from Example 1.1 looked this this:

$$\begin{array}{r}
2x^2+5x+6 \\
x-1\overline{)2x^3+3x^2+x-5} \\
-(2x^3-2x^2) \\
\hline
5x^2+x \\
-(5x^2-5x) \\
\hline
6x-5 \\
-(6x-6) \\
1
\end{array}$$

There is a lot of repetition in the long division. Let's see how we can simplify the process.

Synthetic division allows us to collapse the 'table' of long division by moving each of the rows up to fill any vacant spots. Also, instead of dividing by -1, as we would in division of whole numbers, then multiplying and subtracting the middle product, we change the sign of the 'divisor' to $+1$, multiply and add.

This will become clearer by completing the example $f(x)=2x^3+3x^2+x-5$ divided by $x-1$ using synthetic division.

The process starts by writing the constant term of the divisor with the opposite sign and the coefficients of the polynomial. So the -1 constant term in the divisor becomes $+1$.

$$1\,|\underline{2\ \ 3\ \ 1\ \ -5}$$

Bring down the leading coefficient then multiply this leading coefficient by the divisor and write that answer in column 2.

$$\begin{array}{c|cccc} 1 & 2 & 3 & 1 & -5 \\ & & 2 & & \\ \hline & 2 & & & \end{array}$$

Next, add the numbers in the second column and write this answer down. Next, multiply this answer by the divisor and write the result in the third column and add the numbers in the third column.

$$\begin{array}{c|cccc} 1 & 2 & 3 & 1 & -5 \\ & & 2 & 5 & 6 \\ \hline & 2 & 5 & 6 & 1 \end{array}$$

Once again, multiply this result of adding the numbers in the third column by the divisor and write this number in the fourth column. Add the numbers in the fourth column. This final entry in the last column is the remainder.

Using synthetic division the quotient is $2x^2 + 5x + 6$ and the remainder is 1 just as it was in Example 1.1.

2. In this case we see that the coefficient of the divisor is not 1. To use synthetic division, the coefficient of the divisor must be 1. So, we make the leading coefficient of the divisor equal to 1 by dividing by a factor of 2 to get $2(x - \frac{1}{2})$.

   Now the $-\frac{1}{2}$ constant term in the divisor becomes $+\frac{1}{2}$. The process for synthetic division will follow the same steps as before.

$$\begin{array}{c|cccc} \frac{1}{2} & 6 & 1 & -4 & 5 \\ \hline \end{array}$$

$$\begin{array}{c|cccc} \frac{1}{2} & 6 & 1 & -4 & 5 \\ & & 3 & 2 & -1 \\ \hline & 6 & 4 & -2 & 4 \end{array}$$

Using synthetic division the quotient is $6x^2 + 4x - 2 = 2(3x^2 + 2x - 1)$ and the remainder is 4. But, we are not done yet. Remember, we wrote the divisor as $2(x - \frac{1}{2})$. Now we need to rewrite the expression to take it back to its original form.

$$\begin{aligned} 6x^3 + x^2 - 4x + 5 &= \frac{1}{2} \cdot 2(x - \frac{1}{2}) \cdot 2(3x^2 + 2x - 1) + 4 \\ &= 2(x - \frac{1}{2}) \cdot (3x^2 + 2x - 1) + 4 \\ &= (2x - 1)(3x^2 + 2x - 1) + 4 \end{aligned}$$

Note

For a demonstration of the synthetic division method watch the video "Synthetic Division".

Synthetic Division (Duration: 05.20)

Synthetic division is an alternative that can only be used when the divisor is a binomial in the form $x-k$ where k is a real number. To use synthetic division, the coefficient of the divisor must be 1. Say, for example, the divisor was $2x-1$. In this case, we make the leading coefficient of the divisor equal to 1 by rewriting it as $2(x-\frac{1}{2})$. In synthetic division, only the coefficients are used in the division process.

Take note!

To divide two polynomials using synthetic division do the following:

1. Write k for the divisor.
2. Write the coefficients of the dividend.
3. Bring the lead coefficient down.
4. Multiply the lead coefficient by k. Write the product in the next column.
5. Add the terms of the second column.
6. Multiply the result by k. Write the product in the next column.
7. Repeat steps 5 and 6 for the remaining columns.
8. Use the last row of numbers to write the quotient. The number in the last column is the remainder and has degree 0. The next number from the right has degree 1, the next number from the right has degree 2 and so on.

Exercise 1.3

Find the quotient and remainder of the following by using synthetic division:

1. $2x^3-3x^2+4x+5$ divided by $x+2$
2. $4x^3+10x^2-6x-20$ divided by $x-2$
3. $2x^3+5x-4$ divided by $x-1$

The full solutions are at the end of the unit.

Remainder theorem

Now that we know how to divide polynomials, we can use polynomial division to find the value of polynomials using the remainder theorem.

Activity 1.2: Evaluate polynomials using the remainder theorem

Time required: 15 minutes

What you need:

- a pen and paper

What to do:

Given the following functions:

$$f(x) = x^3 + 3x^2 + 4x + 12$$
$$d(x) = x - 1$$
$$g(x) = 4x^3 - 2x^2 + 2x - 4$$
$$h(x) = 2x + 1$$

Determine $\frac{f(x)}{d(x)}$ and $\frac{g(x)}{h(x)}$.

Write your answers in the form $a(x) = b(x) \cdot Q(x) + R(x)$.

Calculate $f(1)$ and $g(-\frac{1}{2})$.

Compare your answers to question 1 and 3. What do you notice?

Write a mathematical equation to describe your conclusions.

Complete the following sentence: a cubic function divided by a linear expression gives a quotient with a degree of ________ and a remainder with a degree of ______, which is called a constant.

What did you find?

1. Using synthetic division or long division we get:

$$\frac{f(x)}{d(x)} = \frac{x^3 + 3x^2 + 4x + 12}{x - 1}$$

$$\begin{array}{c|cccc} 1 & 1 & 3 & 4 & 12 \\ & & 1 & 4 & 8 \\ \hline & 1 & 4 & 8 & 20 \end{array}$$

$$\text{Quotient} = x^2 + 4x + 8$$
$$\text{Remainder} = 20$$

$$\frac{g(x)}{h(x)} = \frac{4x^3 - 2x^2 + 2x - 4}{2x+1}$$

$$\begin{array}{r|l} & \quad\quad 2x^2-2x+2 \\ \hline 2x+1 & 4x^3 - 2x^2 + 2x - 4 \\ & -(4x^3 + 2x^2) \\ \hline & -4x^2 + 2x \\ & -(-4x^2 - 2x) \\ \hline & 4x - 4 \\ & -(4x+2) \\ \hline & -6 \end{array}$$

$$\begin{aligned} \text{Quotient} &= 2x^2 - 2x + 2 \\ \text{Remainder} &= -6 \end{aligned}$$

2. $x^3 + 3x^2 + 4x + 12 = (x-1)(x^2+4x+8) + 20$
   $4x^3 - 2x^2 + 2x - 4 = (2x+1)(2x^2-2x+2) - 6$

3.
$$\begin{aligned} f(1) &= (1)^3 + 3(1)^2 + 4(1) + 12 \\ &= 20 \end{aligned}$$

$$\begin{aligned} g(-\frac{1}{2}) &= 4(-\frac{1}{2})^3 - 2(-\frac{1}{2})^2 + 2(-\frac{1}{2}) - 4 \\ &= -6 \end{aligned}$$

4. The remainders using division of the polynomials are the same as finding the function values using the divisor.
5. If a polynomial is divided by $x - a$, the remainder may be found quickly by evaluating the polynomial function at a.
   So we can say $R = p(a)$.
6. A cubic function divided by a linear expression gives a quotient with a degree of $\underline{2}$ and a remainder with a degree of $\underline{0}$, which is called a constant.

The remainder theorem:

If a polynomial $f(x)$ is divided by $ax - k$, then the remainder is the value $f\left(\frac{k}{a}\right)$.

You can also use the remainder theorem to solve for an unknown variable. This is shown in the next example.

Example 1.3

Determine the value of t if $f(x) = x^3 + tx^2 + 4x + 2$ gives a remainder of 16 when divided by $2x+1$.

Solution

The remainder theorem tells us that the remainder when $f(x)$ is divided by $2x+1$ can be found using $f(-\frac{1}{2})$. We are also told that $f(-\frac{1}{2})=16$.

$$f(x)=x^3+tx^2+4x+2$$
$$\begin{aligned} f(-\frac{1}{2}) &= (-\frac{1}{2})^3+t(-\frac{1}{2})^2+4(-\frac{1}{2})+2 \\ &= -\frac{1}{8}+\frac{1}{4}t-2+2 \\ &= \frac{1}{4}t-\frac{1}{8} \end{aligned}$$

But $f(-\frac{1}{2})=16$

$$\frac{1}{4}t-\frac{1}{8}=16$$
$$\therefore 2t-1=128$$
$$\therefore 2t=129$$
$$\therefore t=\frac{129}{2}$$

Exercise 1.4

1. Use the remainder theorem to determine the remainder when $f(x)=3x^3+5x^2-x+1$ is divided by:
   a. $x+2$
   b. $2x-1$
   c. $3x+1$
2. Calculate the value of m if $2x^3-7x^2+mx-26$ is divided by $x-2$ and gives a remainder of -24.

The full solutions are at the end of the unit.

Using the factor theorem to solve cubic polynomials

You can use the factor theorem to solve for the zeros or roots (x-intercepts) of a cubic function. The factor theorem describes the relationship between the root of a polynomial and a factor of the polynomial.

You have seen in arithmetic division that if an integer a is divided by an integer b and the answer is q with remainder $r=0$ then b is a factor of a. For example, 27 divided by 9 is 3 with remainder 0. Therefore, 9 is a factor of 27. This is also true of polynomials.

Recall that if $f(x)\div(x-k)$ then $f(x)=(x-k)\cdot Q(x)+R(x)$.

If k is a root of the function, then the remainder R is zero and $f\,(k)=0$. Therefore, $f(x)=(x-k)\cdot Q(x)+0$ or $f(x)=(x-k)\cdot Q(x)$.

Written in this form, where the remainder is zero, we can say that $x-k$ is a factor of $f(x)$. We can conclude if k is a zero of $f(x)$, then $x-k$ is a factor of $f(x)$.

Conversely, if $x-k$ is a factor of $f(x)$ then the remainder is 0.

Factor theorem:

If $f(x)$ is divided by $ax-k$ and the remainder, given by $f\left(\frac{k}{a}\right)$ is equal to 0, then $ax-k$ is a factor of $f(x)$

.

Converse: If $ax-k$ is a factor of $f(x)$ then $f\left(\frac{k}{a}\right)=0$.

Cubic polynomials are factorised by using long division, division by inspection or synthetic division. You may use whichever method you are most comfortable with to factorise cubic polynomials. Remember that the degree of the polynomial tells you how many roots the function will have at most. So, a cubic polynomial will have at most three roots, a quadratic function at most two roots and so on.

Example 1.4

Show that $x+2$ is a factor of $g(x)=x^3-6x^2-x+30$. Find the remaining factors of $g(x)$. Use the factors to determine the zeros of the cubic polynomial.

Solution

Use the remainder theorem to show that $g(-2)=0$. If $g(-2)=0$ then $x+2$ is a factor of $g(x)$.

$$\begin{aligned} g(-2) &= (-2)^3-6(-2)^2-(-2)+30 \\ &= -8-24+2+30 \\ &= 0 \end{aligned}$$

Since $g(-2)=0$ then $x+2$ is a factor of x^3-6x^2-x+30.

Now, that we have a linear factor we can divide $g(x)$ by $x+2$ to find the quadratic factor.

You can use long division or synthetic division to factorise further but one of the quickest methods to factorise a cubic polynomial is **division by inspection**.

We know that $x^3-6x^2-x+30=(x+2)(\dots)$.

The first term in the second bracket must be x^2 to give x^3 when we multiply the brackets to make the polynomial a cubic.

The last term in the second bracket must be 15 because $15\times 2=30$. The middle term must be some ax term.

So far we have $x^3-6x^2-x+30=(x+2)(x^2+ax+15)$.

Now, we must find the coefficient of the middle term in the second bracket. When you multiply the brackets out, the terms that you will multiply together and collect to get the $-6x^2$ term are: $2\times x^2$ and $x\times ax^2$.

$(x+2)(x^2+ax+15)$

$2\times x^2+a\times x^2=-6x^2$

Solve this simple equation (which can be done in your head without showing working) to find a.

$$\begin{aligned} 2x^2 + ax^2 &= -6x^2 \\ ax^2 &= -8x^2 \\ \therefore a &= -8 \end{aligned}$$

So the coefficient of the x-term is -8.

$\therefore x^3 - 6x^2 - x + 30 = (x+2)(x^2 - 8x + 15)$

We must factorise the quadratic factor further to fully factorise the cubic polynomial.

$$\begin{aligned} x^3 - 6x^2 - x + 30 &= (x+2)(x^2 - 8x + 15) \\ &= (x+2)(x-3)(x-5) \end{aligned}$$

The zeros of the function are found by solving the previous equation.

$$\begin{aligned} &(x+2)(x-3)(x-5) = 0 \\ &\therefore x = -2, 3 \text{ or } 5 \end{aligned}$$

Take note!

To factorise a cubic polynomial we do the following:

1. Find one linear factor by trial and error. Note: generally only try factors that would divide into the constant term. For example, if the constant term is 3 then the only factors you need to try are numbers that divide into $3: -3, -1, 1 \text{ and } 3$.
2. Use the factor theorem to confirm that your answer in question 1 is a factor by showing that $f(k) = 0$.
3. Divide the given cubic polynomial by your linear factor to get the quadratic factor.
4. Factorise the quadratic trinomial to find the other two factors of the cubic polynomial.

Exercise 1.5

1. Factorise:
   a. $f(x) = x^3 + x^2 - 9x - 9$
   b. $g(x) = x^3 - 3x^2 + 4$
2. $f(x) = 2x^3 + x^2 - 5x + 2$
   a. Find $f(1)$.
   b. Factorise $f(x)$ and list the zeros of the function.

The full solutions are at the end of the unit.

Sometimes it is not possible to factorise a quadratic expression using inspection, in which case we use the quadratic formula to fully factorise and solve the cubic equation.

Summary

In this unit you have learnt the following:

- How to divide polynomials using long division to find the quotient and remainder.
- How to divide polynomials using synthetic division to find the quotient and remainder.
- How to use the remainder theorem to find the remainder of a polynomial.
- How to use the remainder theorem to find an unknown value.
- How to find a linear factor of a cubic polynomial using the factor theorem.
- How to fully factorise a cubic polynomial using division by inspection.
- How to solve equations with cubic polynomials.

Unit 1: Assessment

Suggested time to complete: 45 minutes

1. The volume of a rectangular solid is given by the polynomial $3x^4 - 3x^3 - 33x^2 + 54x$. The length is $3x$ and the width is $x - 2$. Find the height, h.
2. When $f(x) = 3x^5 + px^4 + 10x^2 - 21x + 12$ is divided by $x - 2$ it leaves a remainder of 10. Find the value of p.
3. Solve $x^3 + x^2 - 16x = 16$.
4. Let $g(x)$ be a polynomial in x.
   a. If $g(x)$ is divided by $x - k$, what does $g(k)$ represent?
   b. If $g(k) = 0$ what can be said about $x - k$?
5. The polynomial $f(x) = x^3 + px^2 - qx - 6$ is exactly divisible by $x^2 + x - 2$.
   a. Factorise $x^2 + x - 2$.
   b. Calculate the values of p and q.

The full solutions are at the end of the unit.

Unit 1: Solutions

Exercise 1.1

1. False. It is not a polynomial since the exponent is not a whole number.
2. True.
3. True.

4. False. A cubic polynomial is an expression with a degree of 3 and does not always have three terms.

Back to Exercise 1.1

Exercise 1.2

1.

a.

$$
\begin{array}{r}
3x^2+2x-1 \\
2x-1\overline{\big)\,6x^3+x^2-4x+5} \\
-(6x^3-3x^2) \\
\hline
4x^2-4x \\
-(\ 4x^2-2x) \\
\hline
-2x+5 \\
-(2x+1) \\
\hline
4
\end{array}
$$

$$Q(x)=3x^2+2x-1$$
$$R(x)=4$$
$$\therefore 6x^3+x^2-4x+5=(2x-1)\cdot(3x^2+2x-1)+4$$

b.

$$
\begin{array}{r}
x^2+4x+3 \\
x-1\overline{\big)\,x^3+3x^2-x-3} \\
-(x^3-x^2) \\
\hline
4x^2-x \\
-(4x^2-4x) \\
\hline
3x-3 \\
-(3x-3) \\
\hline
0
\end{array}
$$

$$Q(x)=x^2+4x+3$$
$$R(x)=0$$
$$\begin{aligned}\therefore x^3+3x^2-x-3&=(x-1)\cdot(x^2+4x+3)+0\\&=(x-1)\cdot(x^2+4x+3)\end{aligned}$$

2. Since there is no remainder when x^3+3x^2-x-3 is divided by $(x-1)$, we can factorise the divisor multiplied by the quotient completely and we will get the dividend.
$$\begin{aligned}x^3+3x^2-x-3&=(x-1)\cdot(x^2+4x+3)\\&=(x-1)(x+1)(x+3)\end{aligned}$$

Back to Exercise 1.2

Exercise 1.3

1.

$$
\begin{array}{r|rrrr}
-2 & 2 & -3 & 4 & 5 \\
 & & -4 & 14 & -36 \\
\hline
 & 2 & -7 & 18 & -31
\end{array}
$$

$$\text{Quotient}=2x^2-7x+18$$
$$\text{Remainder}=-31$$

2.

$$\begin{array}{r|rrrr} 2 & 4 & 10 & -6 & -20 \\ & & 8 & 36 & 60 \\ \hline & 4 & 18 & 30 & 40 \end{array}$$

$\text{Quotient} = 4x^2 + 18x + 30$

$\text{Remainder} = 40$

3\.

Remember if a term is missing in the dividend we write the coefficient as 0.

$$\begin{array}{r|rrrr} 1 & 2 & 0 & 5 & -4 \\ & & 2 & 2 & 7 \\ \hline & 2 & 2 & 7 & 3 \end{array}$$

$\text{Quotient} = 2x^2 + 2x + 7$

$\text{Remainder} = 3$

Back to Exercise 1.3

Exercise 1.4

1.
   a.
   $$\begin{aligned} f(x) &= 3x^3 + 5x^2 - x + 1 \\ f(-2) &= 3(-2)^3 + 5(-2)^2 - (-2) + 1 \\ &= -24 + 20 + 2 + 1 \\ &= -1 \end{aligned}$$
   b.
   $$\begin{aligned} f(x) &= 3x^3 + 5x^2 - x + 1 \\ f(\frac{1}{2}) &= 3\left(\frac{1}{2}\right)^3 + 5\left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right) + 1 \\ &= \frac{3}{8} + \frac{5}{4} - \frac{1}{2} + 1 \\ &= \frac{17}{8} \end{aligned}$$
   c.
   $$\begin{aligned} f(x) &= 3x^3 + 5x^2 - x + 1 \\ f(-\frac{1}{3}) &= 3(-\frac{1}{3})^3 + 5(-\frac{1}{3})^2 - (-\frac{1}{3}) + 1 \\ &= -\frac{1}{9} + \frac{5}{9} + \frac{1}{3} + 1 \\ &= \frac{16}{9} \end{aligned}$$
2.
   Let $g(x) = 2x^3 - 7x^2 + mx - 26$
   Then $g(2) = -24$
   $g(2) = 2(2)^3 - 7(2)^2 + m(2) - 2$
   $\therefore -24 = 16 - 28 + 2m - 2$
   $\therefore -2m = 10$
   $\therefore m = -5$

Back to Exercise 1.4

Exercise 1.5

1.

a. Find a factor by trial and error.

$$f(3) = (3)^3 + (3)^2 - 9(3) - 9$$
$$= 0$$
$$f(3) = 0$$
$$\therefore x - 3 \text{ is a factor of } f(x)$$

Divide to find the other factors. Use division by inspection, long division or synthetic division.

$$x^3 + x^2 - 9x - 9 = (x-3)(x^2 + ax + 3)$$
$$x^2 = -3x^2 + ax^2$$
$$4x^2 = ax^2$$
$$4 = a$$

$$\therefore x^3 + x^2 - 9x - 9 = (x-3)(x^2 + 4x + 3)$$
$$= (x-3)(x+1)(x+3)$$

To solve, let $(x-3)(x^2+1)(x+3) = 0$

$\therefore x = 3, -1$ or -3

b.

$$g(x) = x^3 - 3x^2 + 4$$
$$g(-1) = -1 - 3 + 4$$
$$= 0$$
$$\therefore (x+1) \text{ is a factor of } g(x)$$

Remember to include the $0x$ term especially if using long division or synthetic division.

$$g(x) = x^3 - 3x^2 + 0x + 4$$
$$= (x+1)(x^2 + ax + 4)$$

To find coefficient of the middle term:

$$x^2 + ax^2 = -3x^2$$
$$ax^2 = -4x^2$$
$$a = -4$$
$$x^3 - 3x^2 + 4 = (x+1)(x^2 - 4x + 4)$$
$$= (x+1)(x-2)(x-2)$$

Solve:

$$(x+1)(x-2)(x-2) = 0$$
$$\therefore x = -1 \text{ or } 2$$

2.

a.

$$f(1) = 2(1)^3 + (1)^2 - 5(1) + 2$$
$$= 2 + 1 - 5 + 2$$
$$= 0$$

b.

$x - 1$ is a factor of $f(x)$

$$2x^3 + x^2 - 5x + 2 = (x-1)(2x^2 + ax - 2)$$
$$-2x^2 + ax^2 = x^2$$
$$ax^2 = 3x^2$$
$$\therefore a = 3$$
$$2x^3 + x^2 - 5x + 2 = (x-1)(2x^2 + 3x - 2)$$
$$= (x-1)(2x-1)(x+2)$$

The zeros of the function are $1, \frac{1}{2} \text{ or } -2$

Back to Exercise 1.5

Unit 1: Assessment

1.
$v = l \cdot w \cdot h$
$\therefore 3x^4 - 3x^3 - 33x^2 + 54x = 3x(x-2)h$
$\therefore (x-2)h = \dfrac{3x^4 - 3x^3 - 33x^2 + 54x}{3x} = \dfrac{3x(x^3 - x^2 - 11x + 18)}{3x} = x^3 - x^2 - 11x + 18$
$\therefore h = \dfrac{x^3 - x^2 - 11x + 18}{x-2}$
$(x-2)$ is a factor of $x^3 - x^2 - 11x + 18$

$x^3 - x^2 - 11x + 18 = (x-2)(x^2 + ax - 9)$
$\therefore -2x^2 + ax^2 = -x^2$
$\therefore ax^2 = x^2$
$\therefore a = 1$
$\therefore x^3 - x^2 - 11x + 18 = (x-2)(x^2 + x - 9)$
$h = \dfrac{(x-2)(x^2 + x - 9)}{(x-2)}$
$\therefore h = x^2 + x - 9$

2.
$f(x) = 3x^5 + px^4 + 10x^2 - 21x + 12$
But $f(2) = 10$

$\therefore 3(2)^5 + p(2)^4 + 10(2)^2 - 21(2) + 12 = 10$
$\therefore 96 + 16p + 40 - 42 + 12 = 10$
$\therefore 16p = 10 - 106 = -96$
$\therefore p = -6$

3.
$x^3 + x^2 - 16x - 16 = 0$
Find a factor.
$(4)^3 + (4)^2 - 16(4) - 16 = 0$
$\therefore x - 4 \text{ is factor}$

$$\begin{aligned} x^3 + x^2 - 16x - 16 &= (x-4)(x^2 + ax + 4) \\ &= (x-4)(x^2 + 5x + 4) \\ &= (x-4)(x+1)(x+4) \end{aligned}$$
$(x-4)(x+1)(x+4) = 0$
$\therefore x = 4; -1 \text{ and } -4$

4. Let $g(x)$ be a polynomial in x.
   a. $g(k)$ represents the remainder.
   b. If then $x - k$ is a factor of $g(x)$.
5. The polynomial $f(x) = x^3 + px^2 - qx - 6$ is exactly divisible by $x^2 + x - 2$.
   a.
      $x^2 + x - 2 = (x+2)(x-1)$
   b.
      $f(-2) = 0$

$\therefore (-2)^3 + p(-2)^2 - q(-2) - 6 = 0$
$\therefore -8 + 4p + 2q - 6 = 0$
$\therefore 4p + 2q - 14 = 0$
$\therefore 2p + q - 7 = 0$

$f(1) = 0$
$\therefore (1)^3 + p(1)^2 - q(1) - 6 = 0$
$\therefore 1 + p - q - 6 = 0$
$\therefore p - q - 5 = 0$
Solve simultaneously
$2p + q - 7 = 0 \ (1)$
$p - q - 5 = 0 \ (2)$

From (2):$p = q + 5(3)$
Substitute into (1):
$\therefore 2(q + 5) + q - 7 = 0$
$\therefore 2q + 10 + q - 7 = 0$
$\therefore 3q + 3 = 0$
$\therefore q = -1$
Substitute into (3):
$p = -1 + 5 = 4$

Back to Unit 1: Assessment

SUBJECT OUTCOME IV

FUNCTIONS AND ALGEBRA: USE A VARIETY OF TECHNIQUES TO SKETCH AND INTERPRET INFORMATION FOR GRAPHS OF THE INVERSE OF A FUNCTION

Subject outcome

Subject outcome 2.2: Use a variety of techniques to sketch and interpret information for graphs of the inverse of a function theorems

Learning outcomes

- Determine the equations of the inverses of the functions:
  - $y = ax + q$
  - $y = ax^2$
  - $y = a^x, a > 0$ ($y = a^x$ may be left with x as the subject of the formula. Note: No logarithms required.)
- Sketch the graphs of the inverse of the functions:
  - $y = ax + q$
  - $y = ax^2$
  - $y = a^x, a > 0$

  Note: Sketching the graphs using point by point plotting is an option.
- Obtain the equation of any of the following inverse graphs given as a sketch:
  - $y = ax + q$
  - $y = ax^2$
  - $y = a^x, a > 0$
- Identify characteristics as listed below with respect to the following functions and their inverses:
  $y = ax + q$
  $y = ax^2$
  $y = a^x, a > 0$
  - Domain and range
  - Intercepts with axes
  - Turning points, minima and maxima
  - Asymptotes
  - Shape and symmetry
  - Functions or non-functions
  - Continuous or discontinuous
  - Intervals in which a function increases/decreases.

Unit 1 outcomes

By the end of this unit you will be able to:

- Define an inverse function.
- Find the inverse of $y = ax + q$.
- Sketch the inverse of $y = ax + q$.
- Answer questions about the domain, range, shape, continuity and other characteristics of the inverse graph.

Unit 2 outcomes

By the end of this unit you will be able to:

- Find the inverse of $y = ax^2$.
- Sketch the inverse of $y = ax^2$.
- Answer questions about the domain, range, shape, continuity and other characteristics of the inverse graph.

Unit 3 outcomes

By the end of this unit you will be able to:

- Find the inverse of $y = a^x$ where $a > 0$.
- Sketch the inverse of $y = a^x$ where $a > 0$.
- Answer questions about the domain, range, shape, continuity and other characteristics of the inverse graph.

Unit 1: Determine and sketch the inverse of a linear function

DYLAN BUSA

Unit outcomes

By the end of this unit you will be able to:

- Define an inverse function.
- Find the inverse of $y = ax + q$.
- Sketch the inverse of $y = ax + q$.
- Answer questions about the domain, range, shape, continuity and other characteristics of the inverse graph.

What you should know

Before you start this unit, make sure you can:

- Define a relation and a function. Refer to level 2 subject outcome 2.1 unit 1 if you need help with this.
- Sketch linear functions using the dual-intercept method. Refer to level 2 subject outcome 2.1 unit 1 if you need help with this.

Introduction

In levels 2 and 3 we learnt about the following functions:

- Linear functions
  - $y = ax + q$
  - $y = mx + c$
- Quadratic functions
  - $y = ax^2 + q$
  - $y = a(x + p)^2 + q$
  - $y = ax^2 + bx + c$
- Exponential function
  - $y = a \cdot b^{x+p} + q, b > 0, b \neq 1$
- Hyperbolic function
  - $y = \dfrac{a}{x + p} + q$
- Trigonometric functions
  - $y = a\sin(x + p) + q$
  - $y = a\sin kx$
  - $y = a\cos(x + p) + q$
  - $y = a\cos kx$

- $y = a\tan(x+p)+q$
- $y = a\tan kx$

We learnt about the characteristics of each function, how to sketch their graphs and how the values of a, p, q, b and k affect the shape and position of these graphs. We also learnt how to find the equations of these functions based on their graphs.

The important thing to note is that they were all **functions**.

Function revision

Relations are mathematical rules (equations) that associate each element of one set of numbers with **at least one** element of another set of numbers. We can think of the first set of numbers as the **input** (or independent variables) and the second set of numbers the **output** (or the dependent variables). We put an input number into the relation rule or equation and get **at least one** output number. The value of the output **depends** on the value of the input.

Functions are special types of relations. They are rules that associate each element of the input set with **only one** element of the output set. Therefore, when we put an input number into the function rule or equation, we only ever get **one** output number. Every function is a relation but not every relation is a function (see Figure 1).

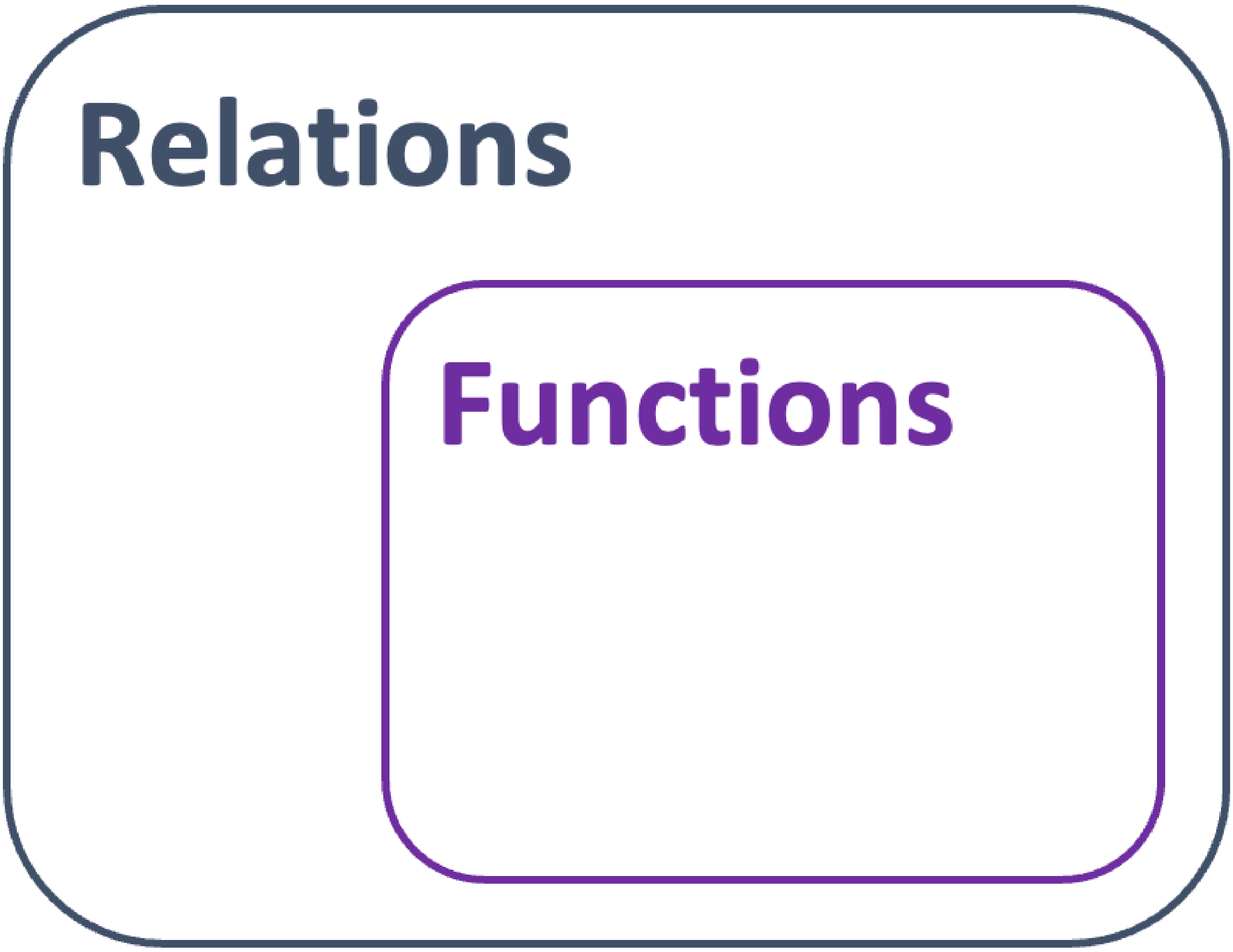


Figure 1: Relationship between relations and functions

We call the set of input numbers or independent variables the **domain**. We call the set of output numbers or dependent variables the **range**.

Relation:

A mathematical rule that associates each element of a set (A) with at least one element in another set (B).

Function:

A mathematical rule that uniquely associates elements of one set (A) with the elements of another set (B) such that each element in a set (A) maps to only one element in the other set (B).

Some functions are **one-to-one functions** (see Figure 2). In one-to-one functions, each element of the input set is uniquely associated with an output. In other words, each output is produced by only one input.

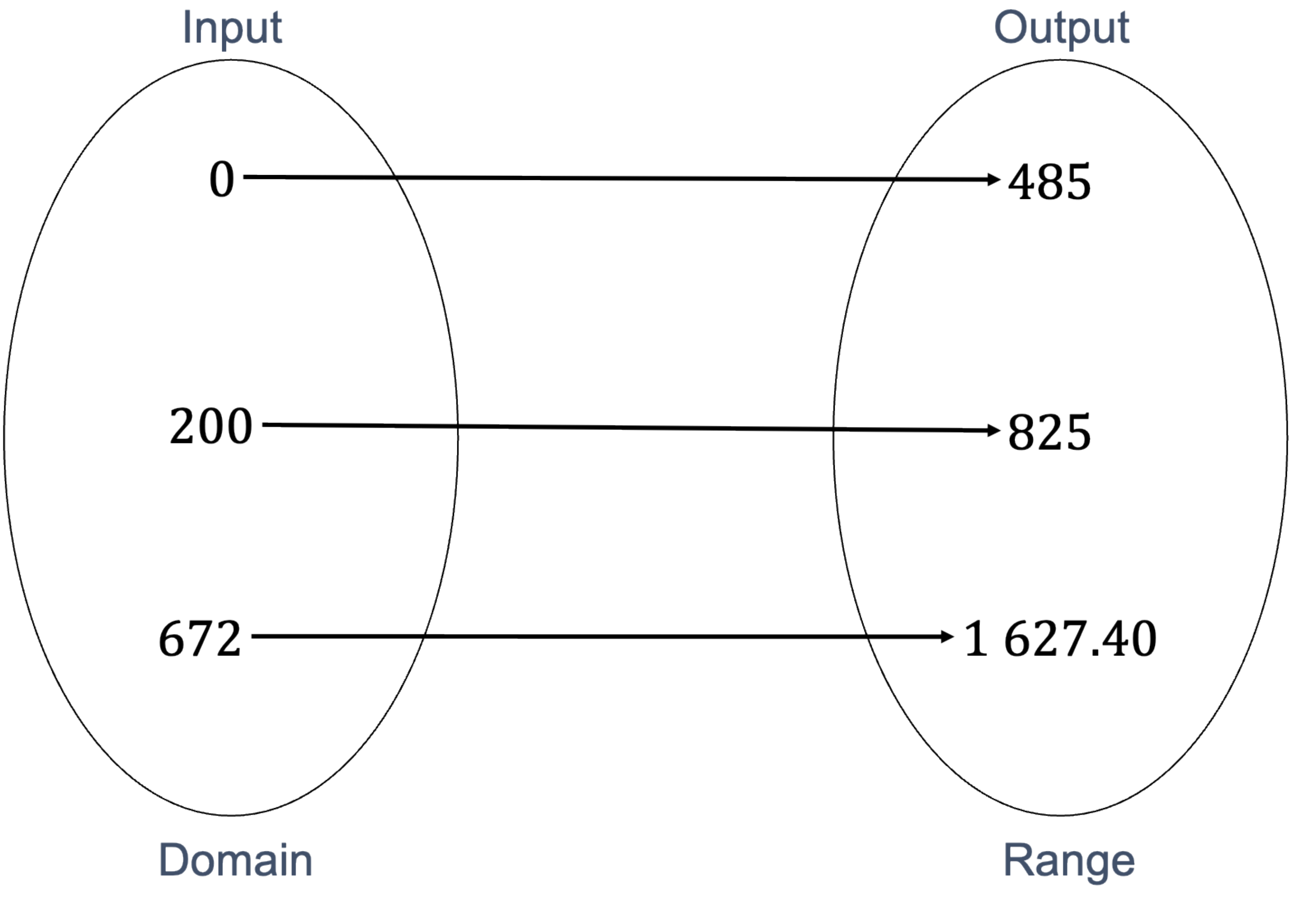


Figure 2: A one-to-one function mapping

The linear function is an example of a one-to-one function. Each output value or y-value is produced by only one input value or x-value (see Figure 3).

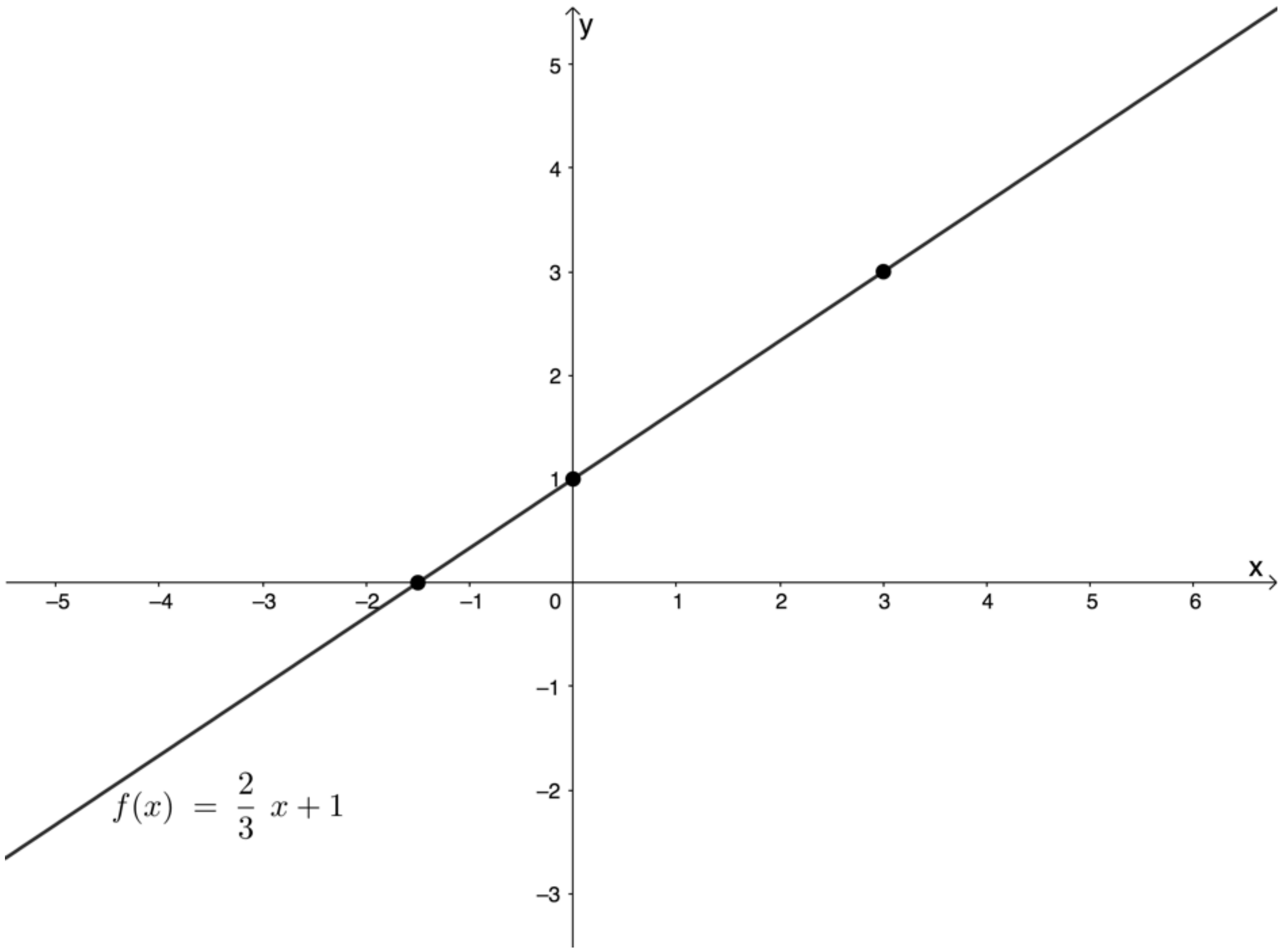


Figure 3: Graph of the function $f(x) = \frac{2}{3}x + 1$

Some functions are many-to-one functions (see Figure 4). In many-to-one functions, each element of the input set is associated with an output but not uniquely. In other words, an output can be produced by more than one input.

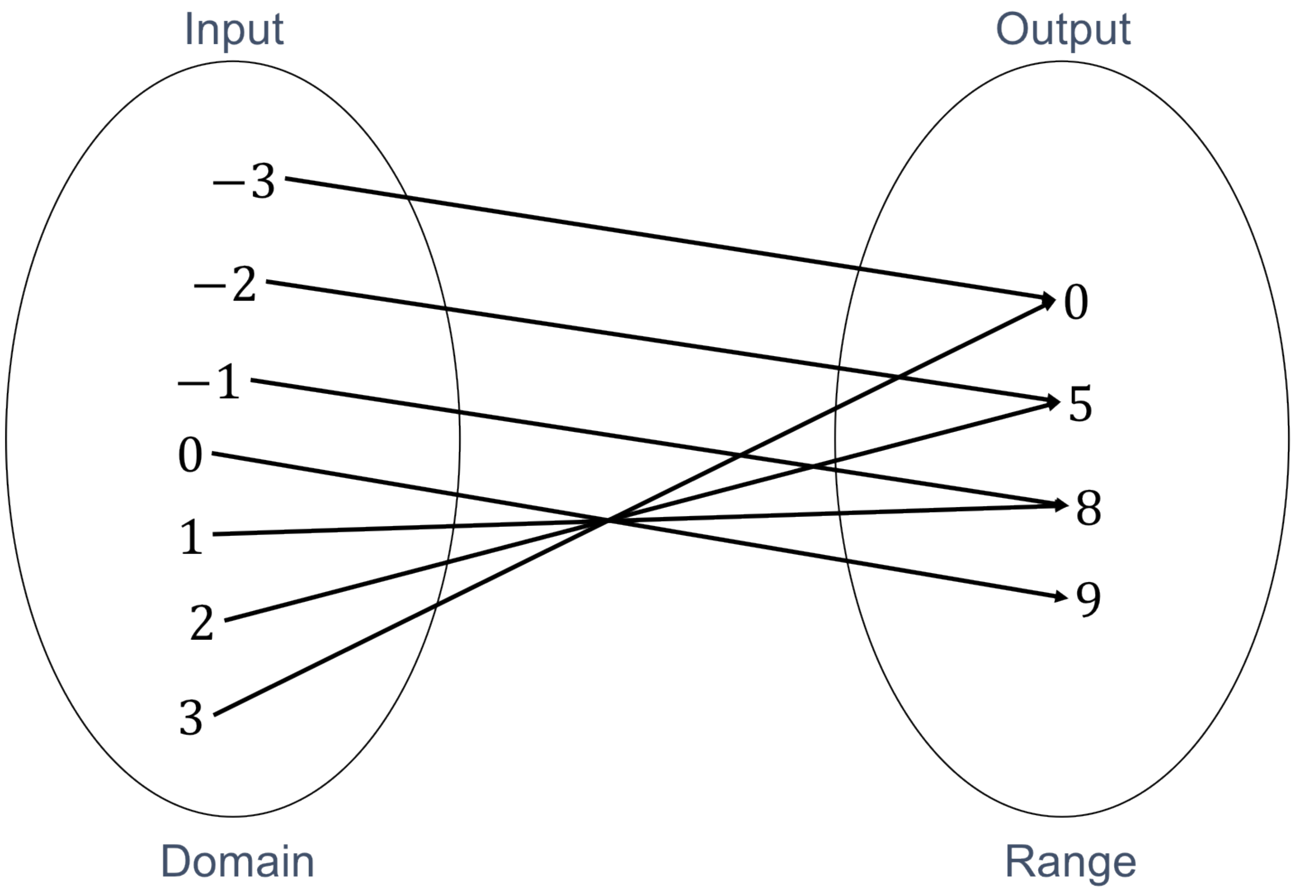


Figure 4: A many-to-one function mapping

The quadratic function is an example of a many-to-one function. Each output value or y-value is produced by two input values or x-values (see Figure 5).

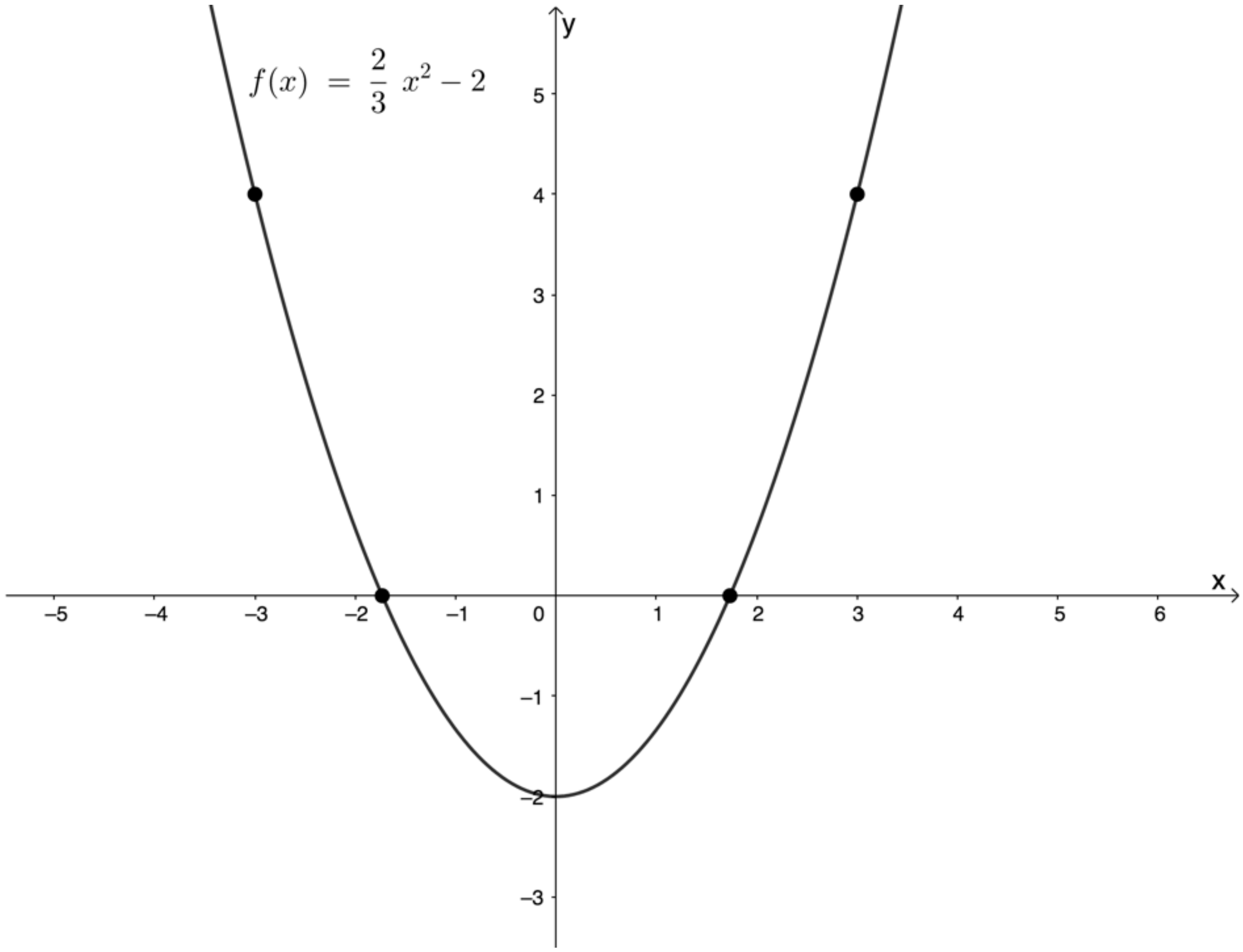


Figure 5: Graph of the function $f(x) = \frac{2}{3}x^2 - 2$

The vertical line test

Because we know that for a relation to be a function each element of the input set must be associated with **only one** element of the output set, we can use a simple visual test to see if a graph is the graph of a function or not. This is the **vertical line test**.

If we can place a vertical line over the graph and have it cut the graph more than once, then we know that there is an input value which produces more than one output value and so the graph is not the graph of a function.

Figure 6 uses the vertical line test to confirm that $y = \frac{2}{3}x^2 - 2$ is a function. At no point is any input value associated with more than one output value.

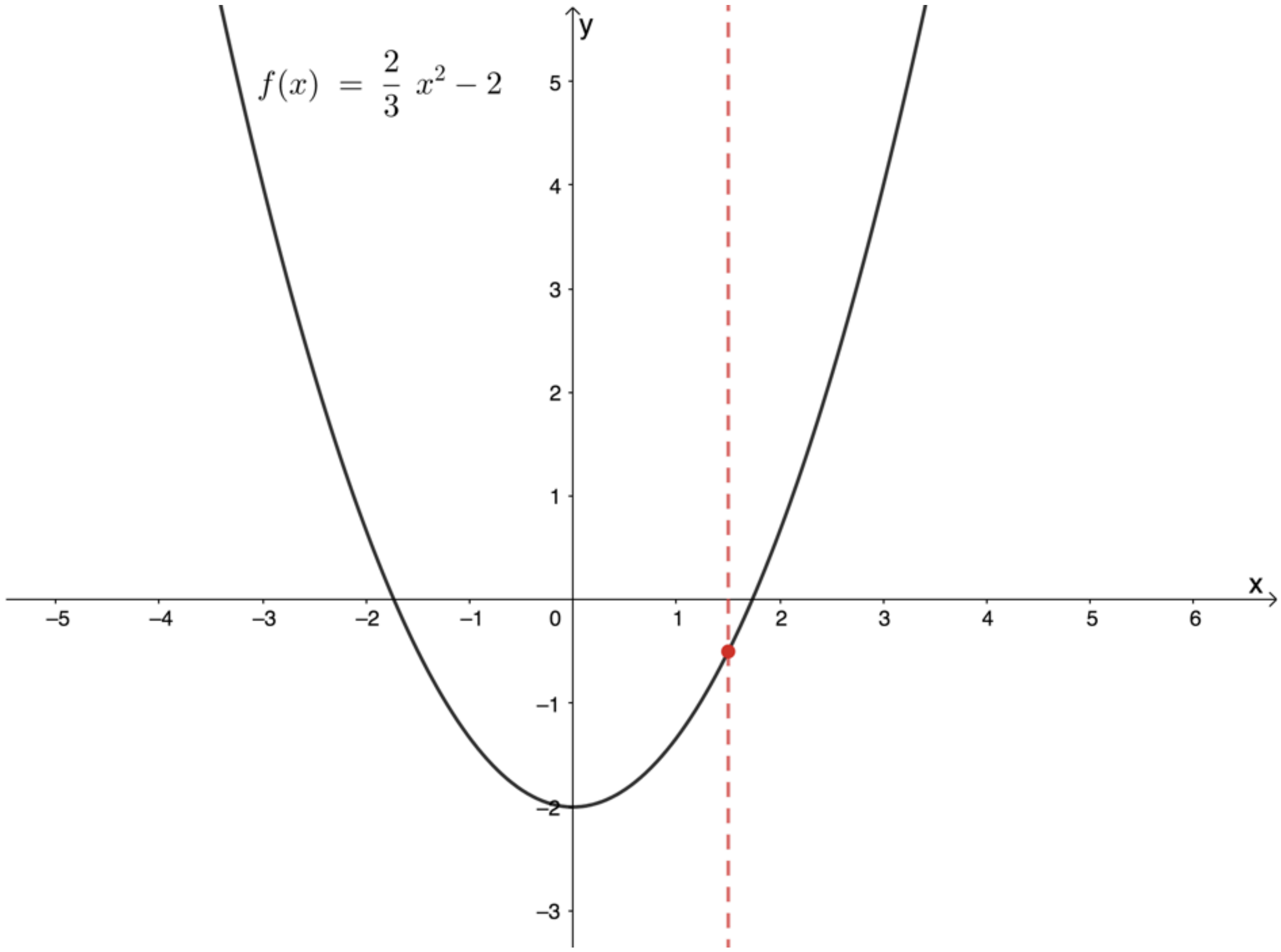


Figure 6: Vertical line test confirming that $y = \frac{2}{3}x^2 - 2$ *is a function*

Figure 7 shows that $y = \pm\sqrt{25 - x^2}$ is not a function. We can see that at least one input value is associated with more than one output value. In fact, almost all the input values are associated with more than one output value.

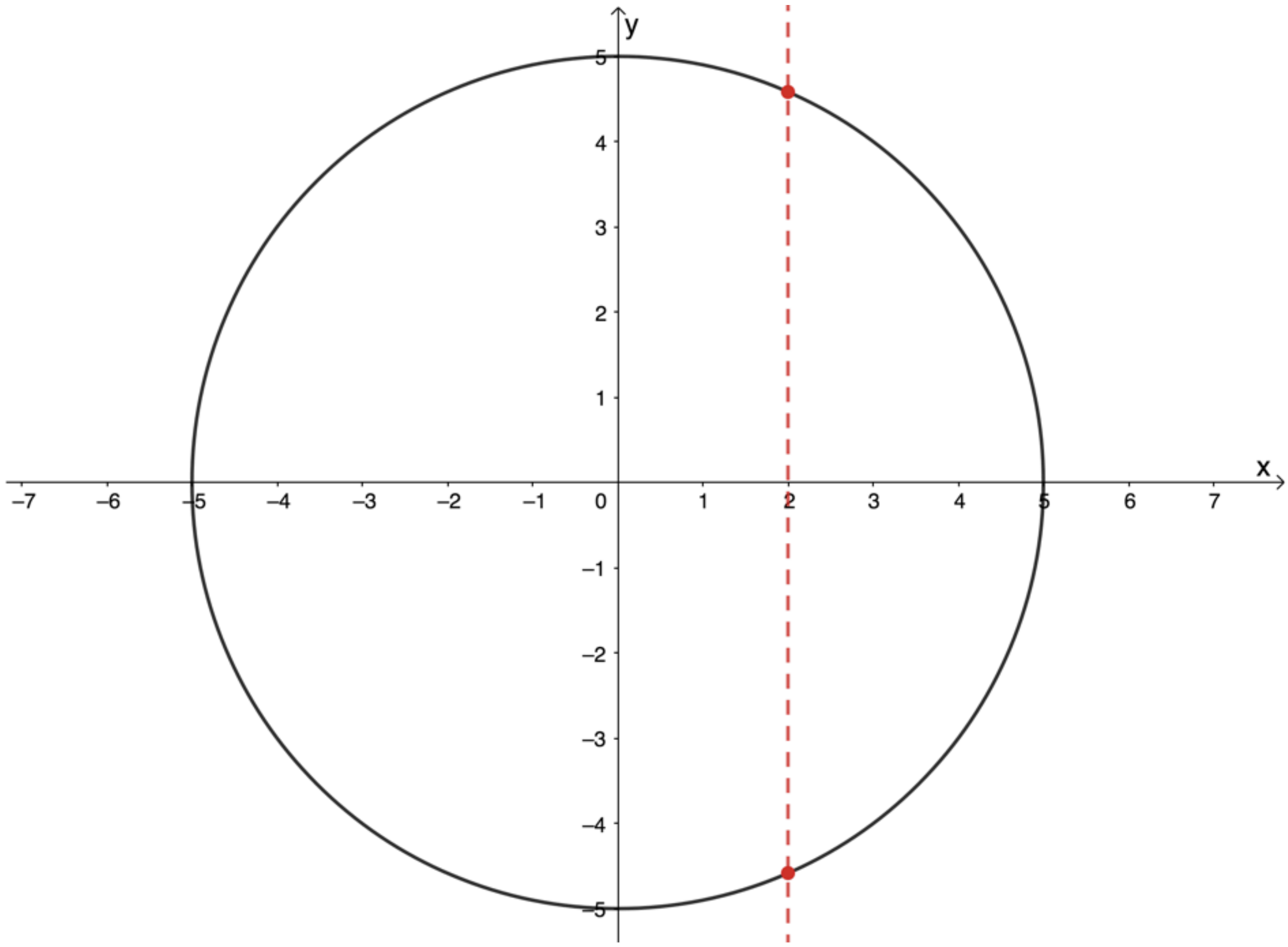
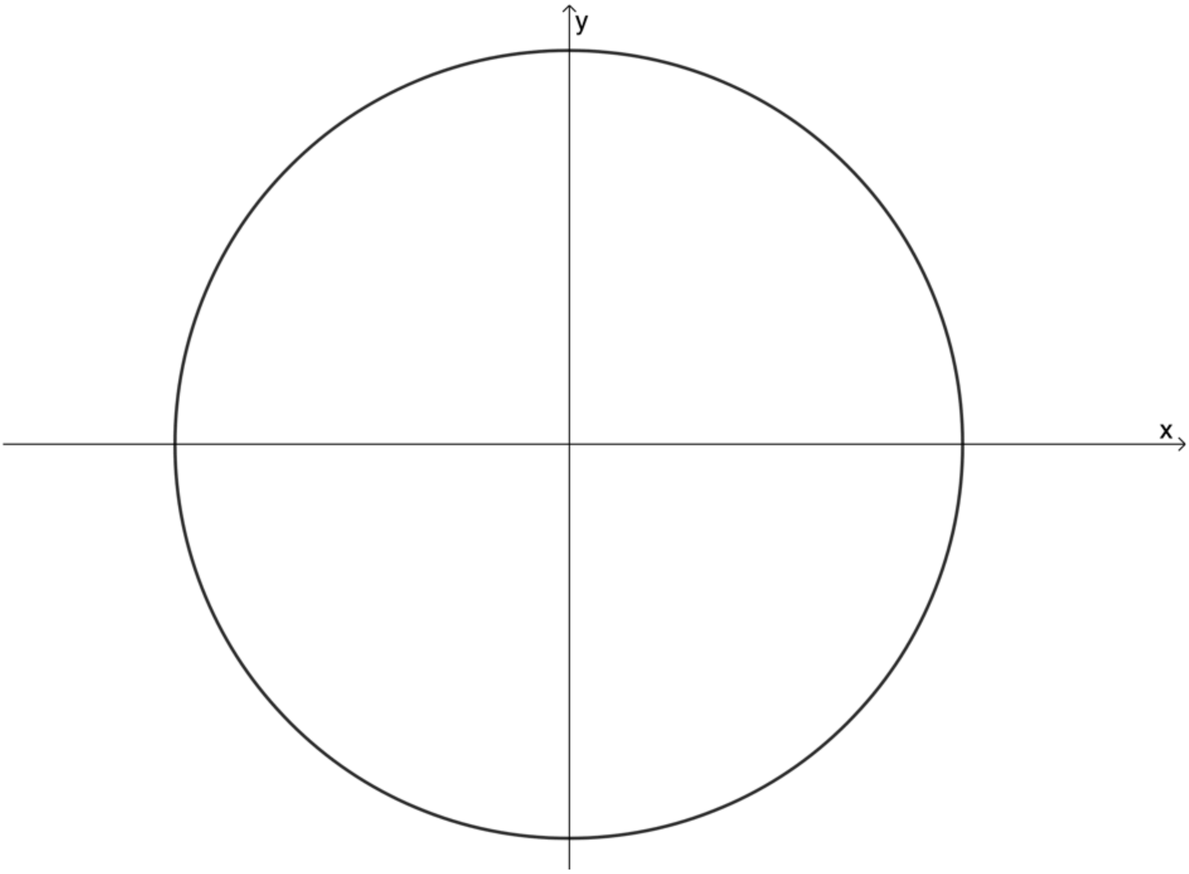


Figure 7: Vertical line test confirming that $y = \pm\sqrt{25 - x^2}$ *is not a function*

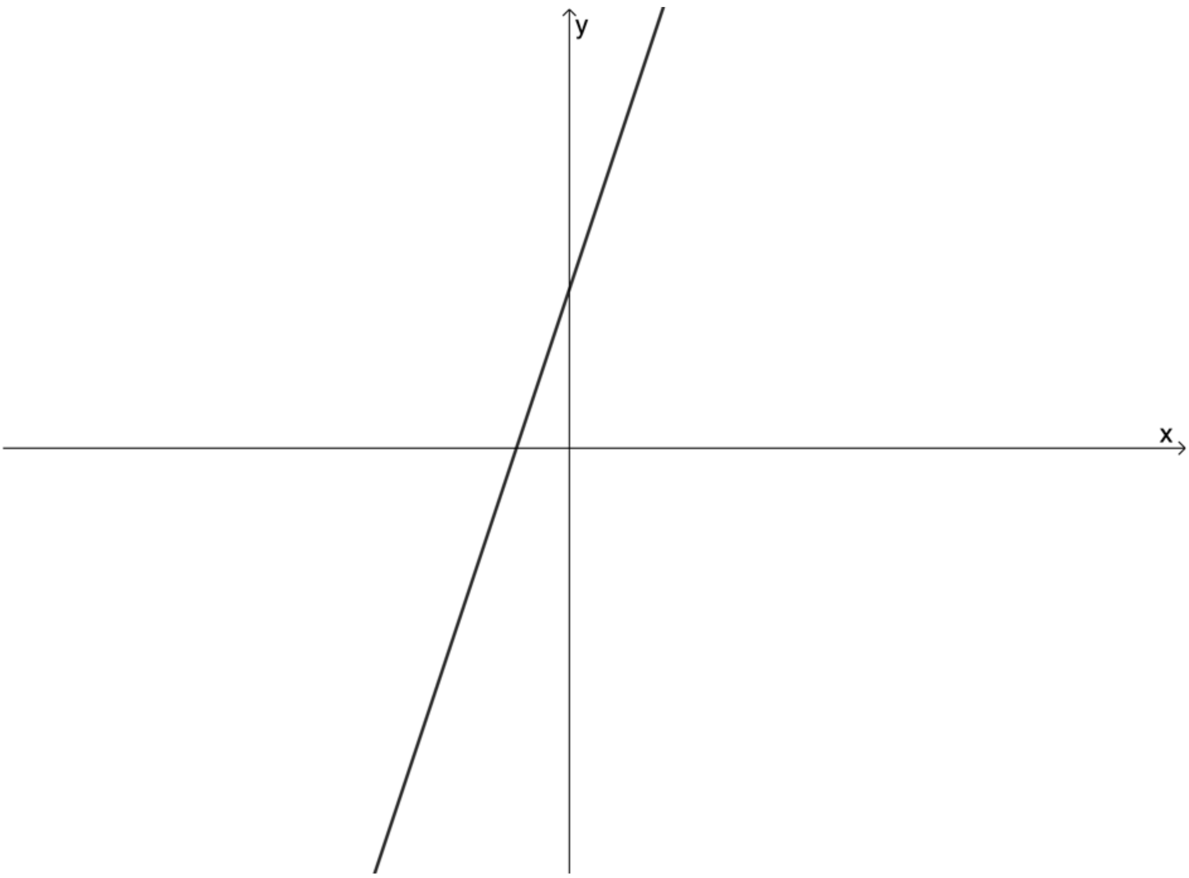
Exercise 1.1

Use the vertical line test to determine whether the following graphs are graphs of functions or not:

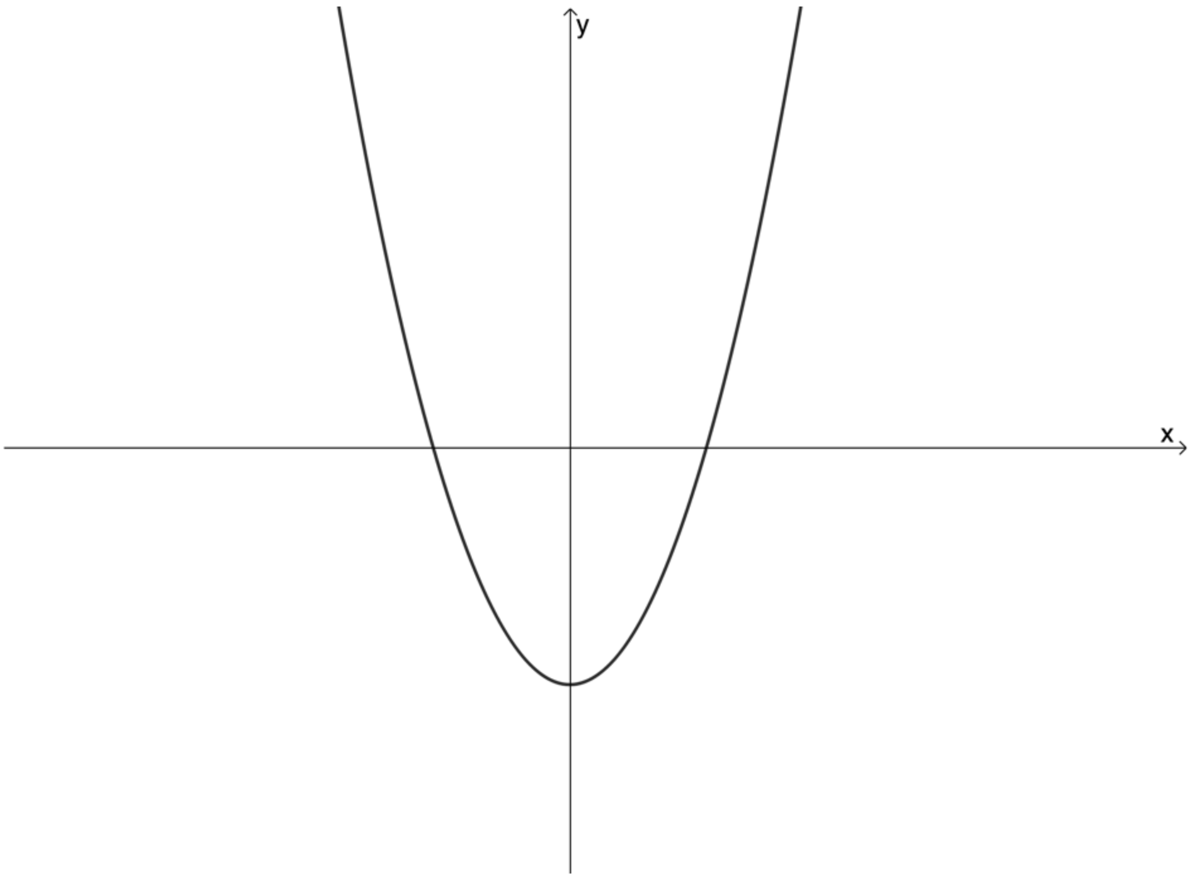
1.



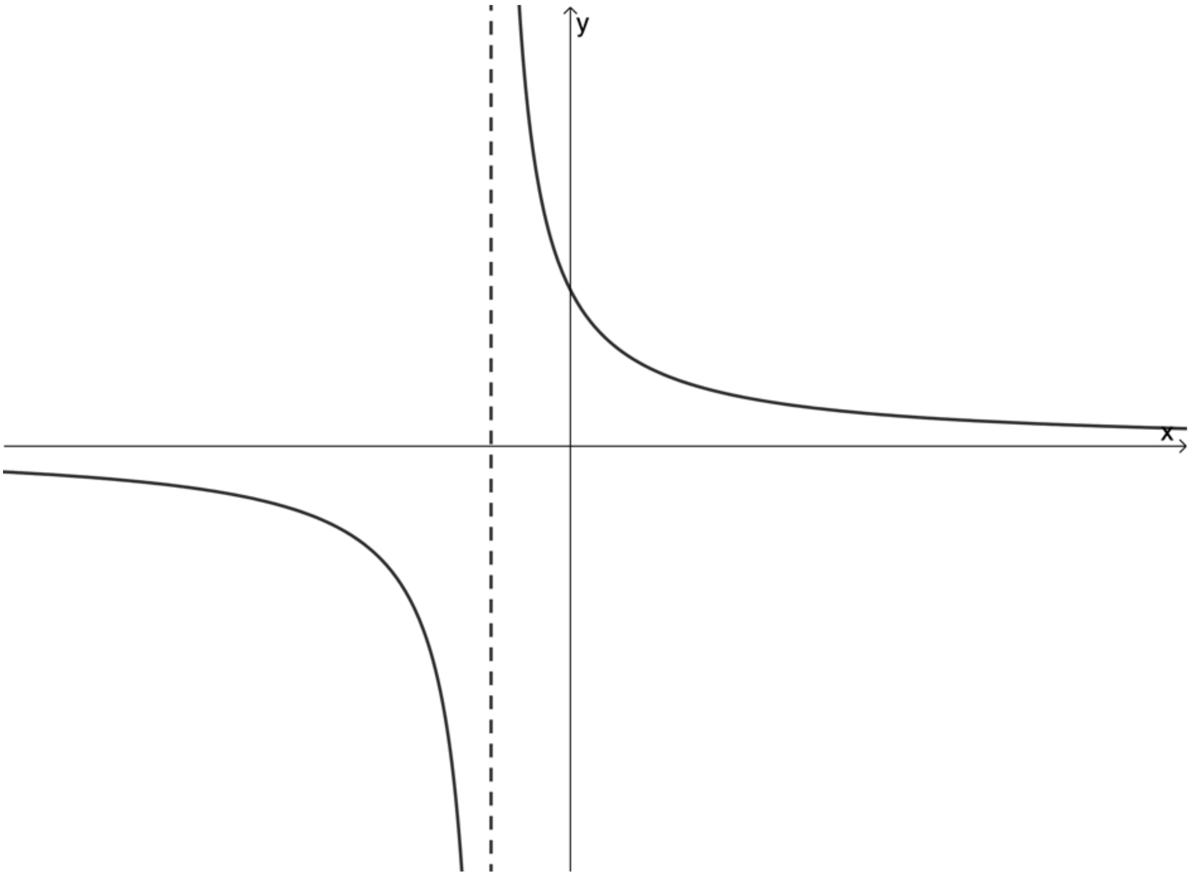
2.



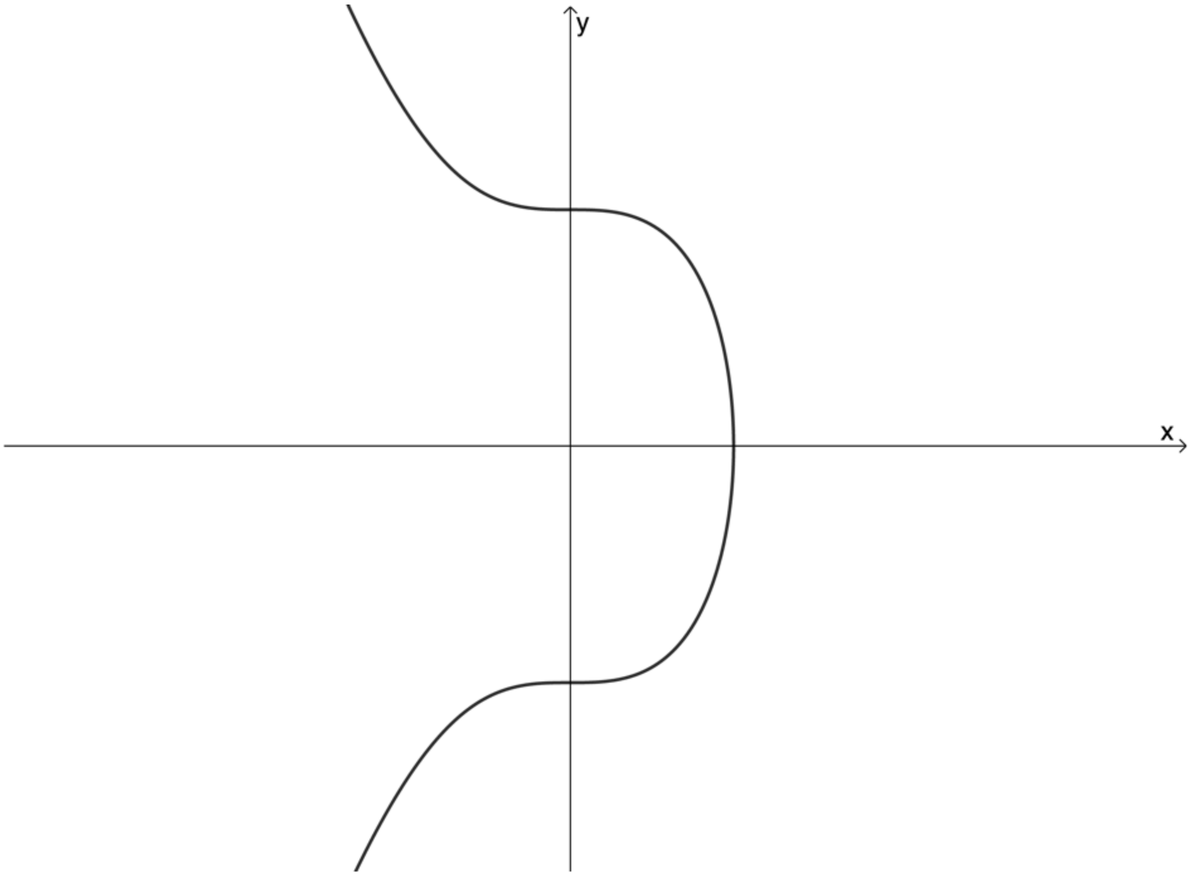
3.



4.



5.



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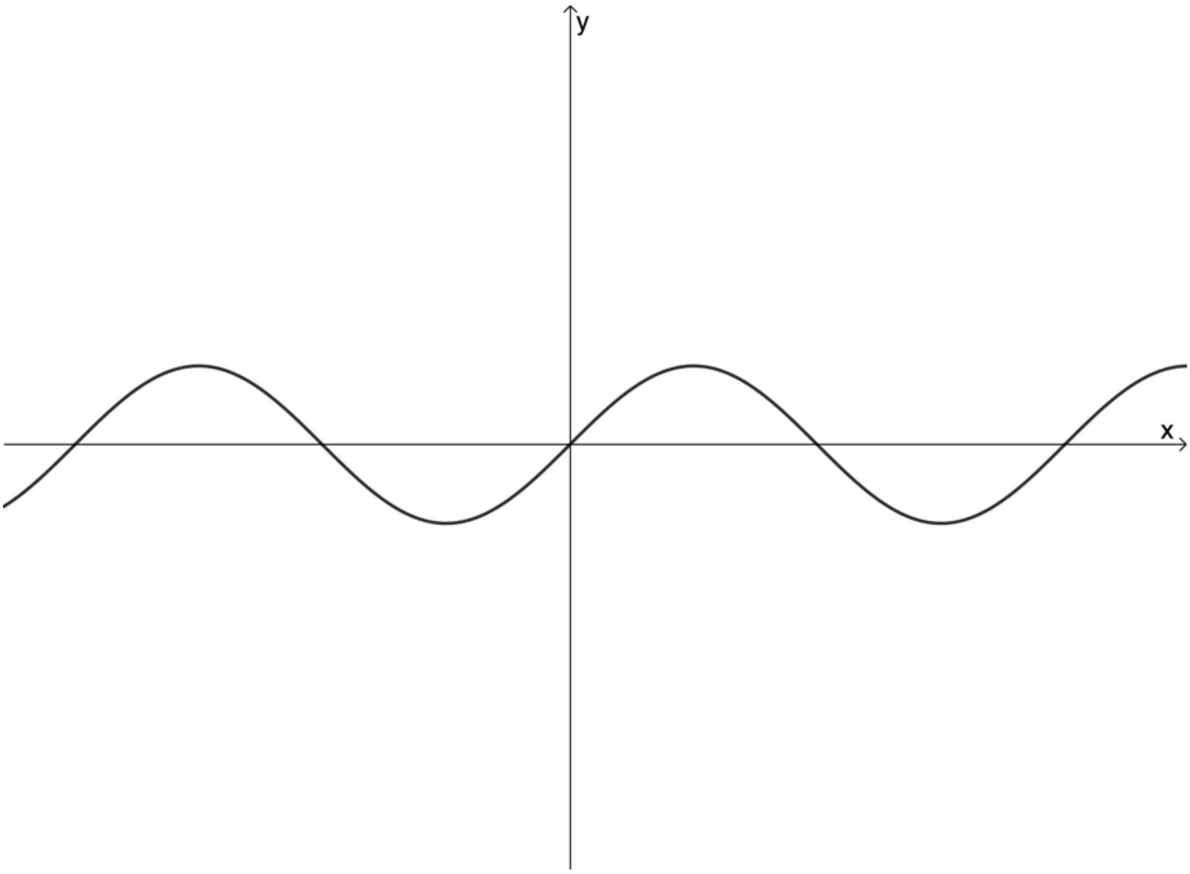

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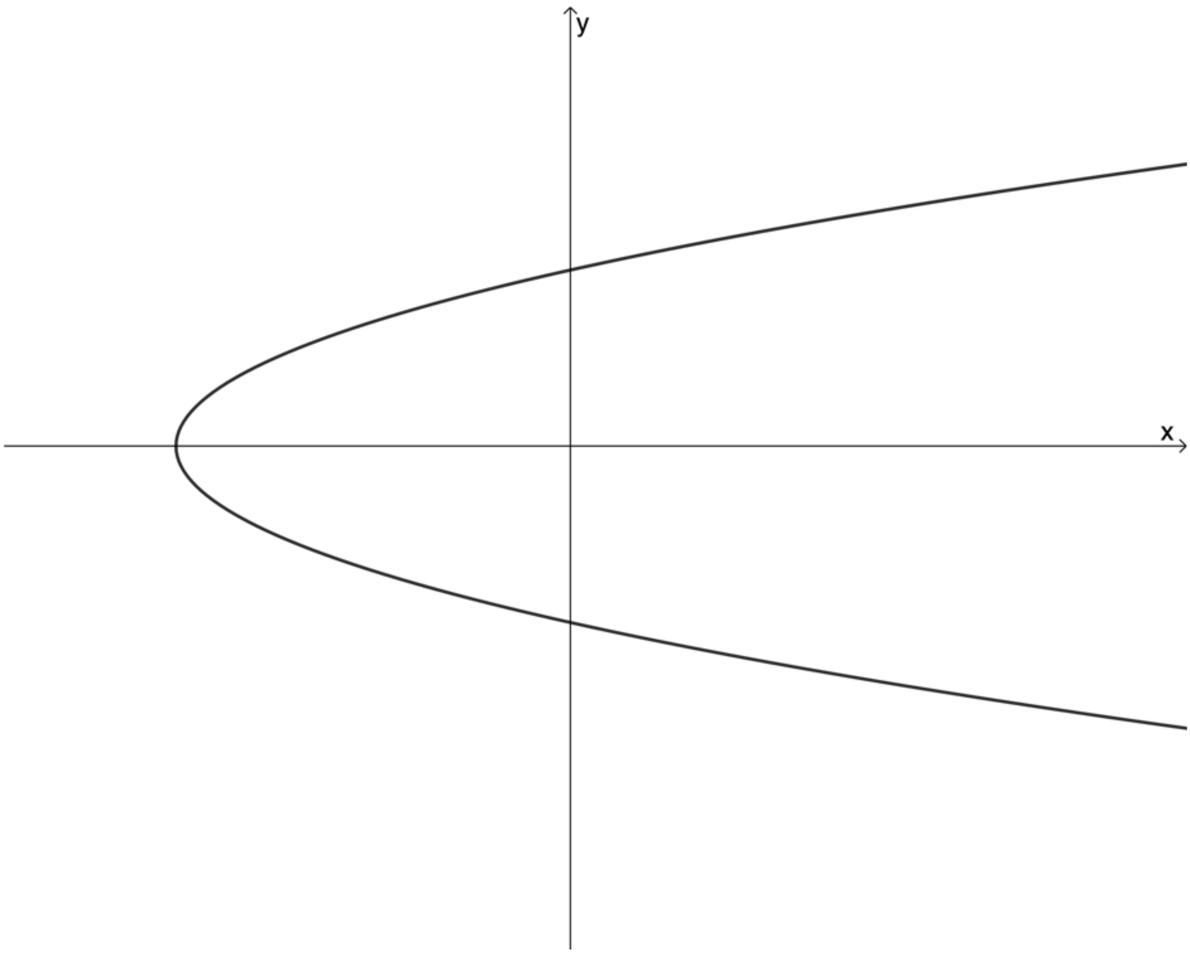

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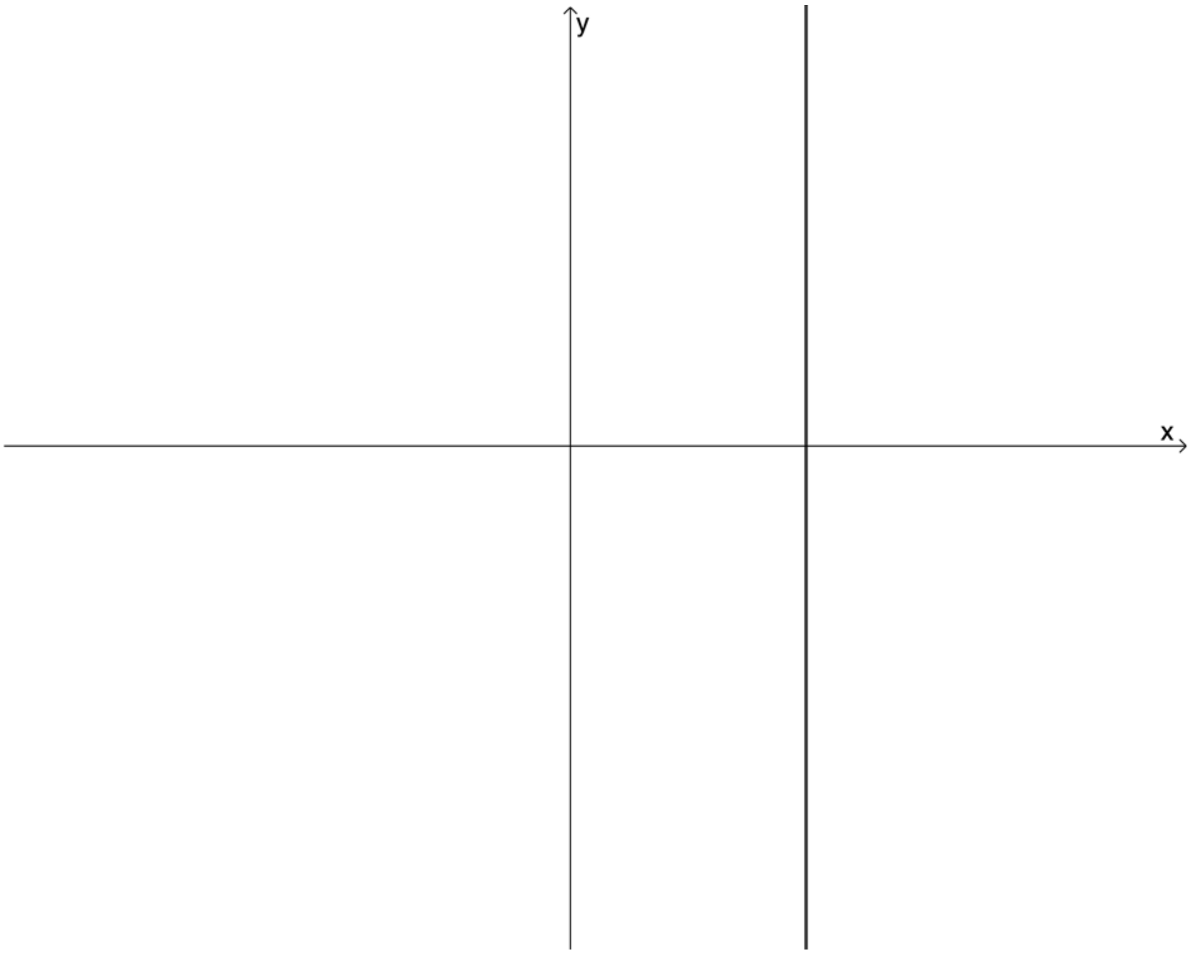
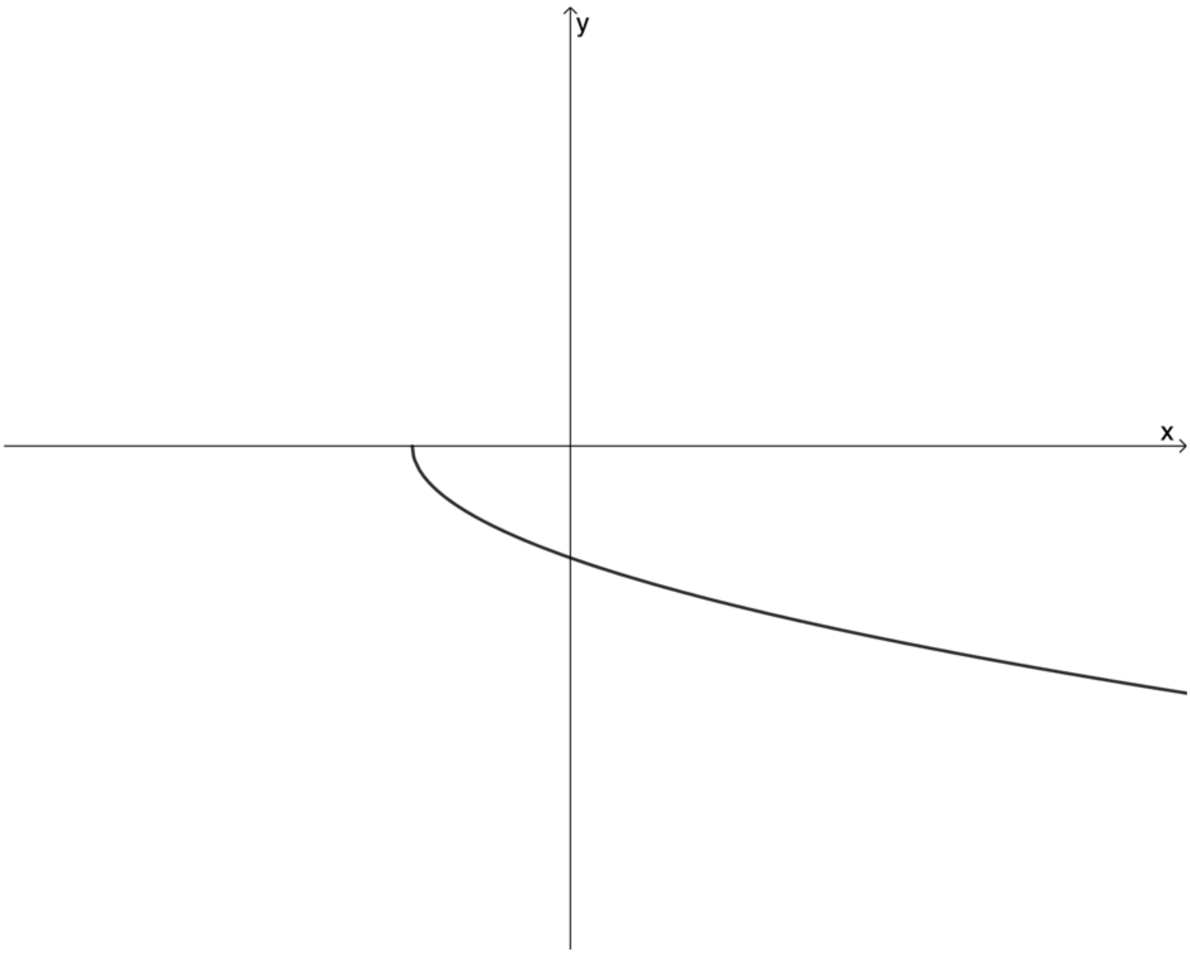
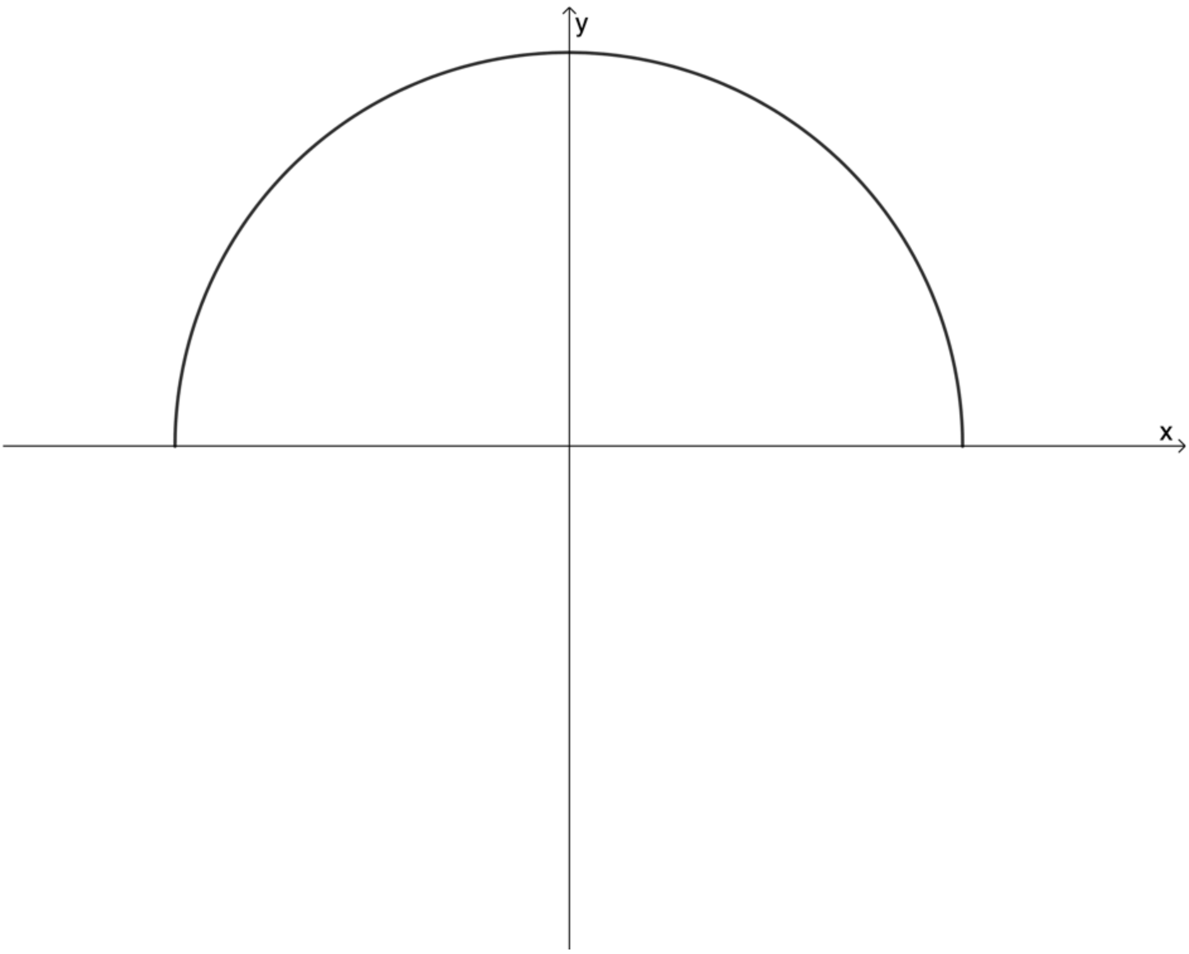


9.



10.



The full solutions are at the end of the unit.

Function notation

We know that we can use a special notation to represent functions, called **function notation**. For example, in $h(s) = 3s + 2$ the function is called h and the independent variable is s. We say that 'h of s is equal to three s plus two'.

$h(2)$ (h of two) means the function value (or the output value) when the input value is two. Therefore $h(2) = 3(2) + 2 = 8$.

If $h(s) = 1$ then $3s + 2 = 1$. Therefore, $s = -\frac{1}{3}$.

Unless a relation is a function, we may not represent it using function notation. From Figure 7, we know that $y = \pm\sqrt{25 - x^2}$ is not a function. Therefore, we cannot represent this relation using function notation as $f(x) = \pm\sqrt{25 - x^2}$.

However, we can say that $f(x) = +\sqrt{25 - x^2}$ (see Figure 8) or that $f(x) = -\sqrt{25 - x^2}$ (see Figure 9) because, once we restrict the output to being either the positive or negative root, the relation is a function.

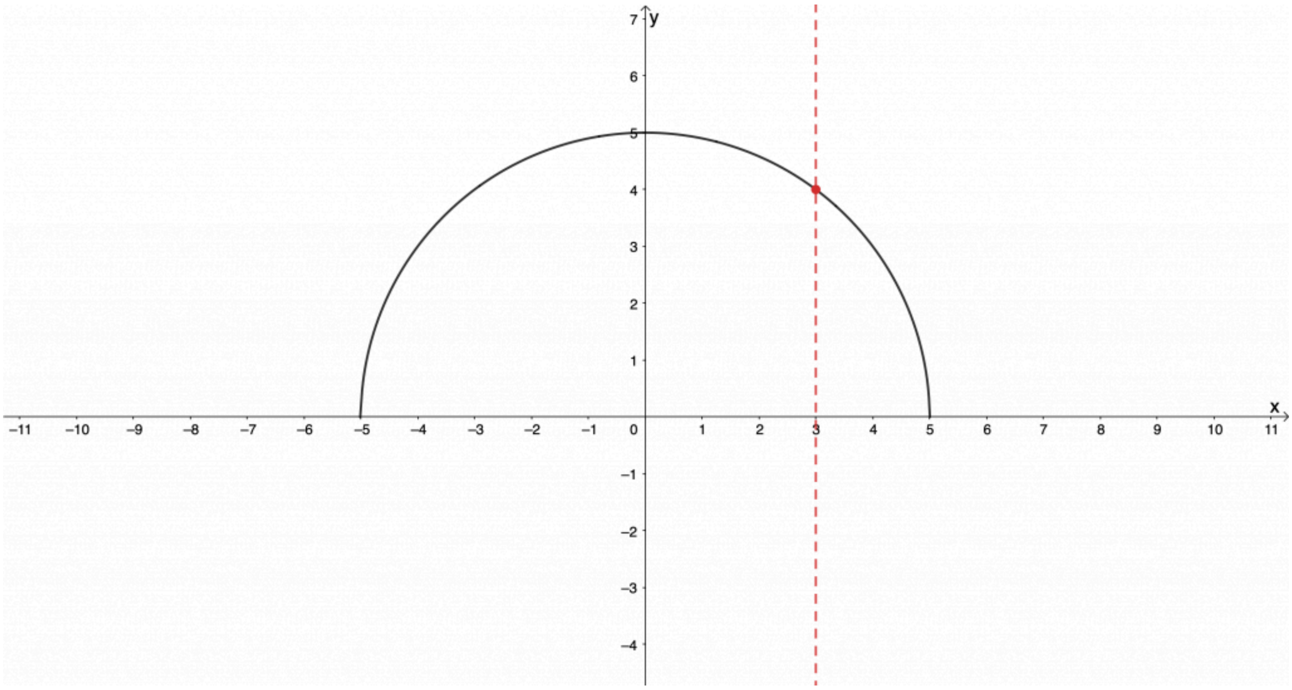


Figure 8: Graph of $f(x) = \sqrt{25 - x^2}$

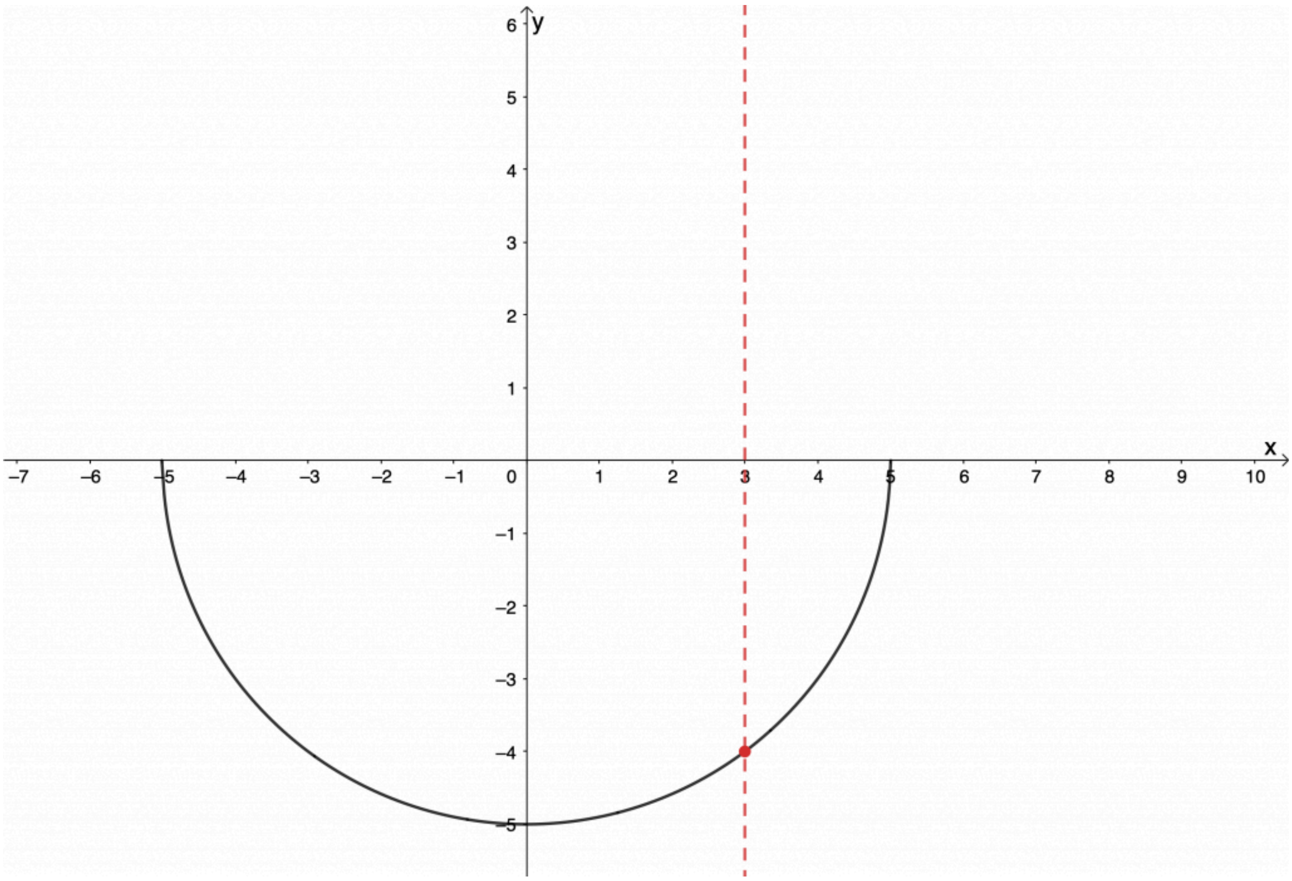


Figure 9: Graph of $f(x) = -\sqrt{25 - x^2}$

Inverse functions

If we have the function $f(x) = 4x - 5$, then we know that if we put $x = 2$ into the function, we will get the output of $y = 3$.

The **inverse** of a function does the 'reverse' of a given function. The inverse of the function f will, therefore, take the input of $x = 3$ and produce the output of $y = 2$.

If a function is a **one-to-one function**, then its inverse will also be a function and we can represent the inverse using **inverse function notation**. If the function is $f(x)$, then the inverse function is denoted as $f^{-1}(x)$. Note that the -1 does not represent an exponent.

We say that functions whose inverses are also functions are **invertible**.

To find the inverse of a function, we simply need to swop the x and y in the original function equation. If $f(x) = 4x - 5$ then we know that $y = 4x - 5$. Therefore, the inverse of f is $x = 4y - 5$.

We then need to rearrange the inverse equation into the $y =$ form.

$$\begin{aligned} x &= 4y - 5 \\ \therefore 4y &= x + 5 \\ \therefore y &= \frac{1}{4}x + \frac{5}{4} \end{aligned}$$

So, if $f(x) = 4x - 5$ then we can say that $f^{-1}(x) = \frac{1}{4}x + \frac{5}{4}$. Remember that we can only use the inverse function notation because f is a one-to-one function and, therefore, $f^{-1}(x)$ is also a function.

If we saw that $f(2) = 3$, check to make sure that $f^{-1}(3) = 2$.

Take note!

If the inverse of a function is also a function, that function is said to be invertible. Only one-to-one functions are invertible.

Often, however, the inverse of a function is not a function. Let's look at the following example.

Example 1.1

Find the inverse of $g(x) = 2x^2 - 8$ and state whether it is a function or not.

Solution

We start by writing the given function in $y =$ form.

$$\begin{aligned} g(x) &= 2x^2 - 8 \\ \therefore y &= 2x^2 - 8 \end{aligned}$$

To find the inverse, we interchange the x and y in the original function equation and get the new equation into $y =$ form.

$$x = 2y^2 - 8$$
$$\therefore 2y^2 = x + 8$$
$$\therefore y^2 = \frac{x}{2} + 4$$
$$\therefore y = \pm\sqrt{\frac{x}{2} + 4}$$

We can see that for every input we put into the inverse of g we will get two outputs. Therefore, the inverse of g is not a function and we cannot write it using inverse function notation.

Take note!

Do not confuse inverse function notation with the reciprocal of a function.
If $f(x) = 2x$ then:

- the **inverse** of f is $f^{-1}(x) = \frac{1}{2}x$
- the **reciprocal** of f is $[f(x)]^{-1} = \frac{1}{f(x)} = \frac{1}{2x}$

The inverse of the linear function

Because linear functions are one-to-one functions, we know that their inverses will also be functions. Therefore, linear functions are invertible.

Example 1.2

Given $h(x) = -2x + 3$:

1. Determine $h^{-1}(x)$.
2. Sketch $h(x)$ and $h^{-1}(x)$ on the same set of axes, showing the intercepts with the axes.
3. State the domain and range of $h(x)$ and $h^{-1}(x)$.
4. About what line are the graphs of $h(x)$ and $h^{-1}(x)$ symmetrical?

Solutions

1. **Step 1**: Start by writing the original function in $y =$ form.
   $y = -2x + 3$

   Step 2: Interchange x and y and write the inverse equation in $y =$ form.

$x = -2y + 3$

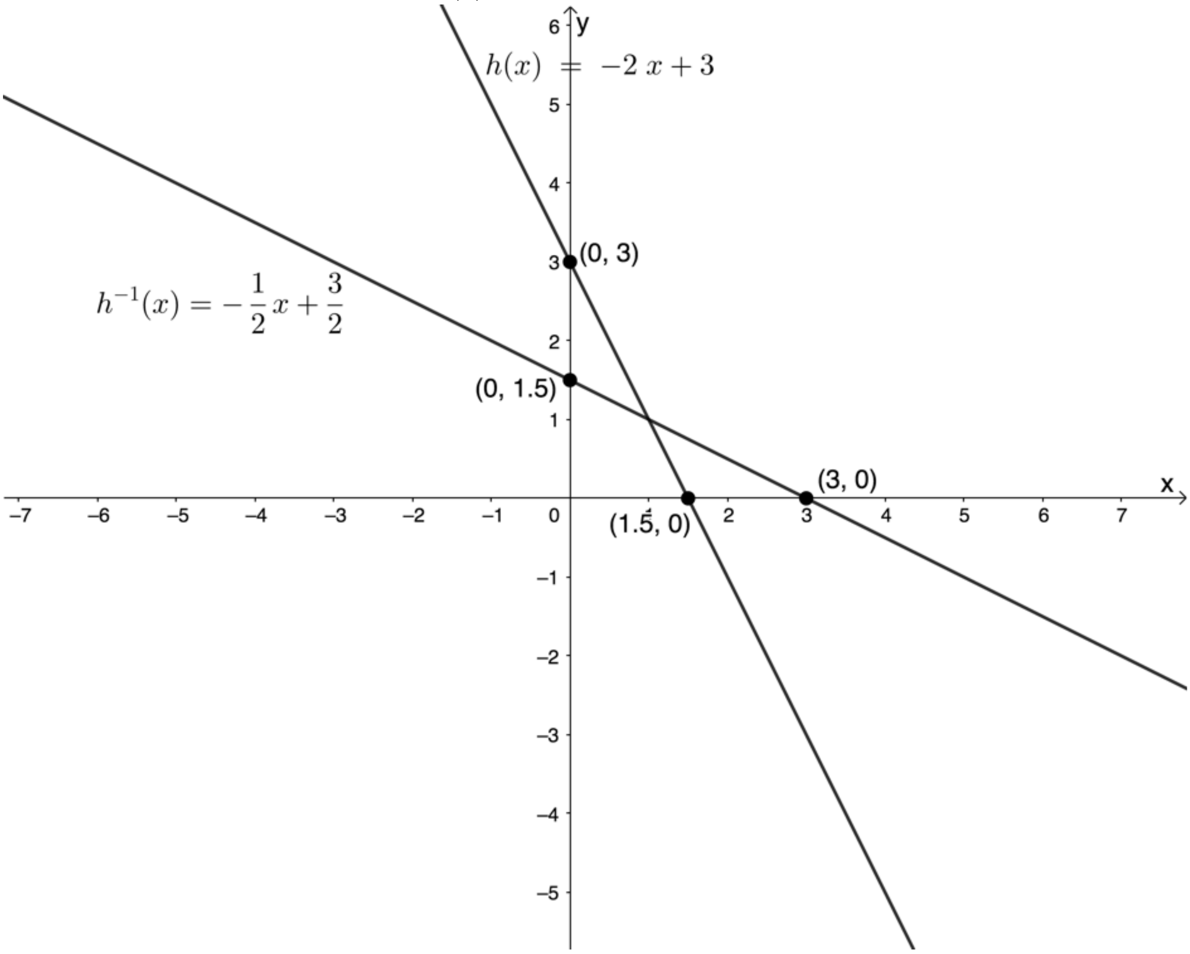
$\therefore 2y = -x + 3$

$\therefore y = -\frac{1}{2}x + \frac{3}{2}$

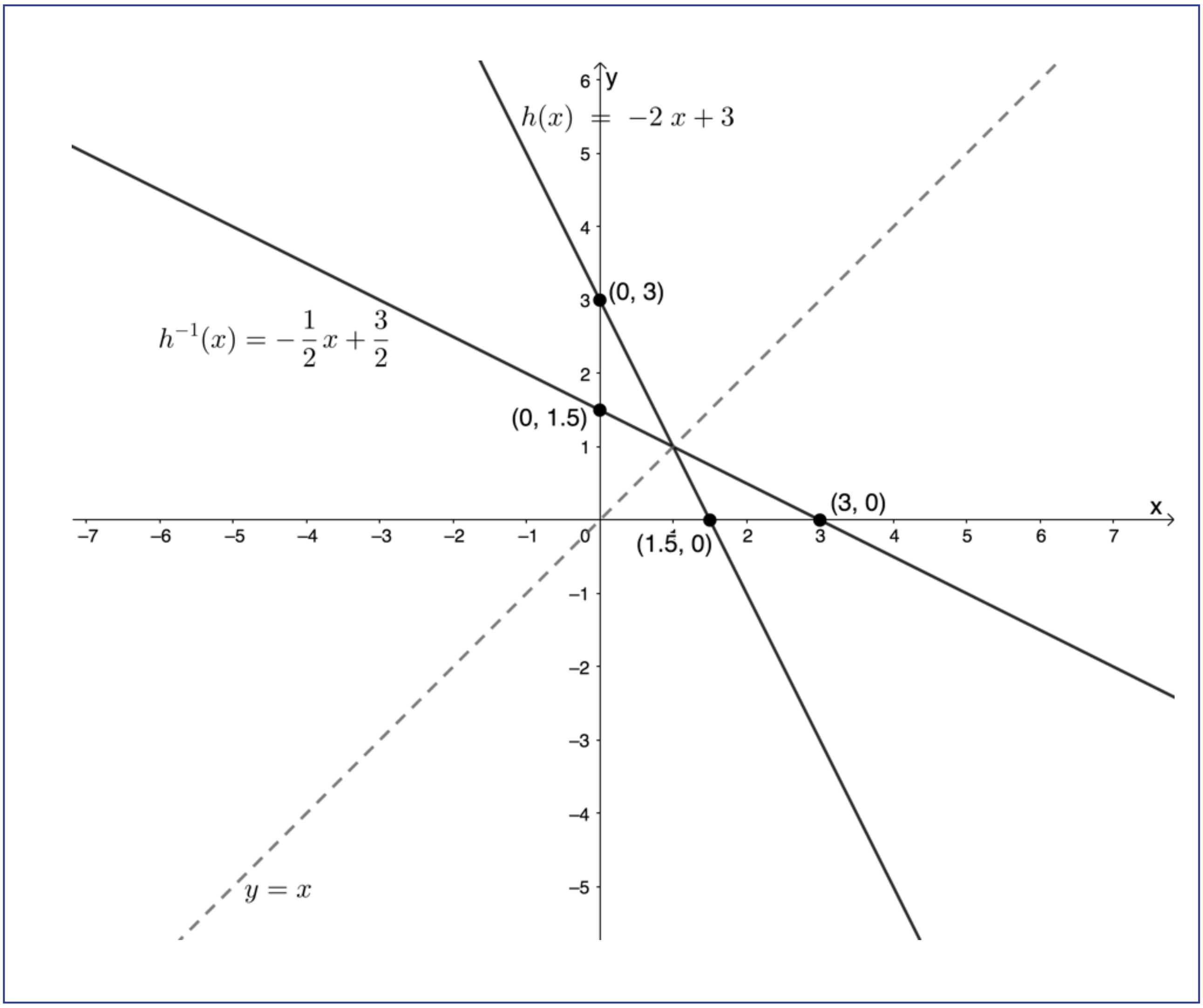
Step 3: Write the inverse using inverse function notation if it is also a function.

$h^{-1}(x) = -\frac{1}{2}x + \frac{3}{2}$

2. Here are the sketches of $h(x)$ and $h^{-1}(x)$.



3. Domain of $h(x)$: $\{x|x \in \mathbb{R}\}$
   Range of $h(x)$: $\{x|x \in \mathbb{R}\}$
   Domain of $h^{-1}(x)$: $\{x|x \in \mathbb{R}\}$
   Range of $h^{-1}(x)$: $\{x|x \in \mathbb{R}\}$
4. The graphs of $h(x)$ and $h^{-1}(x)$ are symmetrical about the line $y = x$.



In Example 1.2, we saw that the graphs of $h(x)$ and its inverse $h^{-1}(x)$ are symmetrical about the line $y = x$. This makes sense given the fact that we found the equation for $h^{-1}(x)$ by interchanging x and y.

This symmetry was also visible in the intercepts of each graph with the axes. The y-intercept of $h(x)$ became the x-intercept of $h^{-1}(x)$ and the x-intercept of $h(x)$ became the y-intercept of $h^{-1}(x)$.

Exercise 1.2

1. Given $f(x) = -2x + 3$:
   a. Find $f^{-1}(x)$
   b. Find $[f(x)]^{-1}$
2. Consider the relation $y = \frac{3}{4}x + \frac{1}{2}$:
   a. Is the relation a function?
   b. Determine the inverse of this relation.
   c. Is this relation an invertible function?

3. Given $q(x) = 2x + 1$:
    a. Determine $q^{-1}(x)$.
    b. Sketch the graphs of $q(x)$ and $q^{-1}(x)$ on the same system of axes.
    c. If $S(2, 5)$ is a point on $q(x)$, determine the coordinates of T if T is a point on $q^{-1}(x)$ and if S and T are symmetrical.
4. Given $t^{-1}(x) = -3x - 3$:
    a. Determine $t(x)$.
    b. Determine the intercepts with the axes of $t(x)$ and $t^{-1}(x)$.
    c. Determine the coordinates of A, the point of intersection of $t(x)$ and $t^{-1}(x)$.
5. If $f(x) = 3x - 2$, is $f^{-1}(x)$ an increasing or decreasing function? Explain your answer.

The full solutions are at the end of the unit.

Summary

In this unit you have learnt the following:

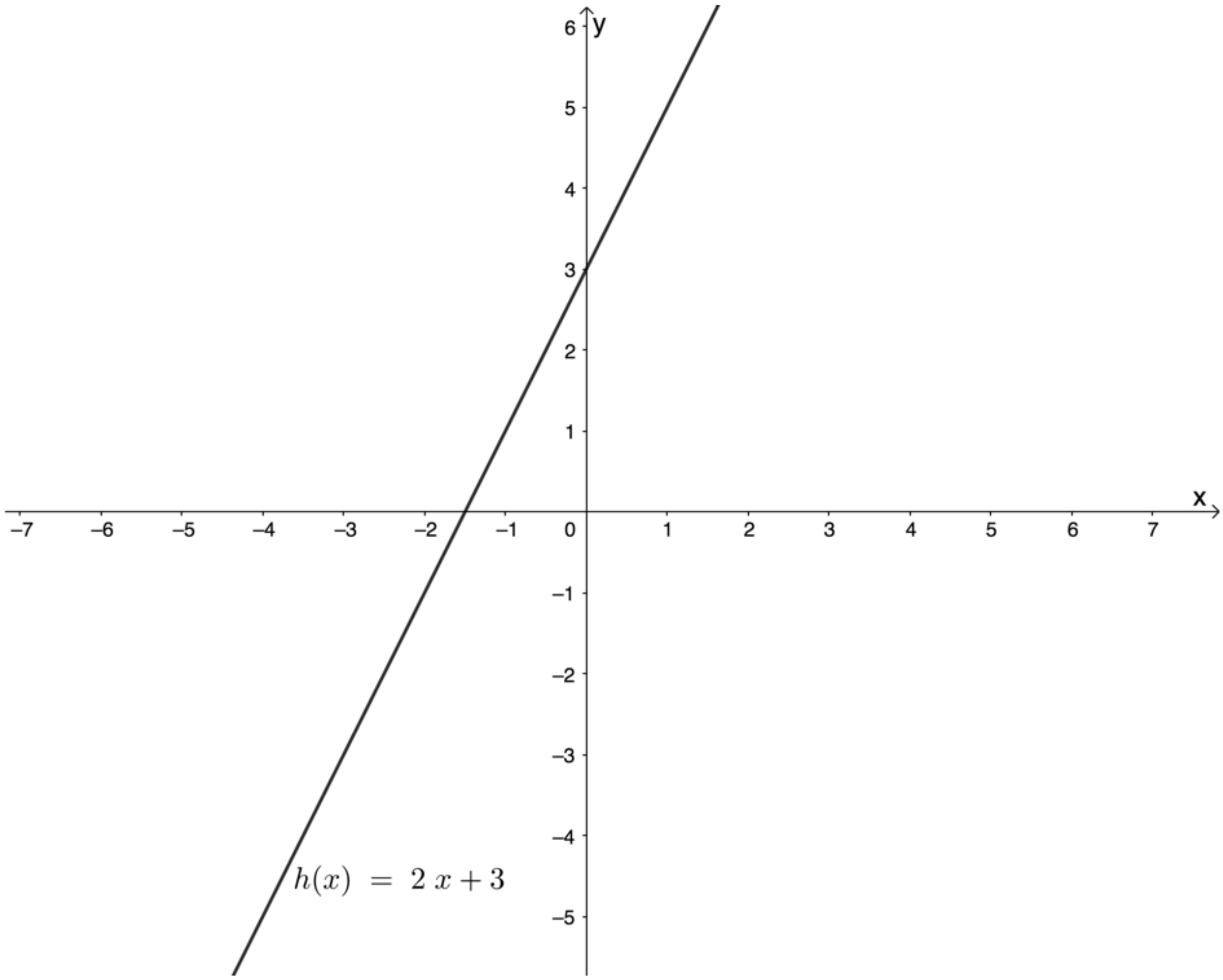
- A relation is a mathematical rule that associates elements of one set (the inputs or independent variables) with at least one element of a second set (the outputs or dependent variables).
- The set of inputs is called the domain and the set of outputs is called the range.
- A function is a special type of relation such that each input is associated with one and only one output.
- Functions can be one-to-one or many-to one.
- The inverse of a function performs the reverse operation of a function.
- If the inverse of a function is also a function, then the function is said to be invertible.
- If the inverse of a function $f(x)$ is a function it can be represented with inverse function notation $f^{-1}(x)$.

Unit 1: Assessment

Suggested time to complete: 15 minutes

Question 1 adapted from NC(V) Mathematics Level 4 Paper 1 November 2011

1. The diagram below represents the graph of $f(x) = 2x + 3$:



a. Determine f^{-1}, the inverse of f in the form $y =$.
b. Draw the graph of f^{-1}.

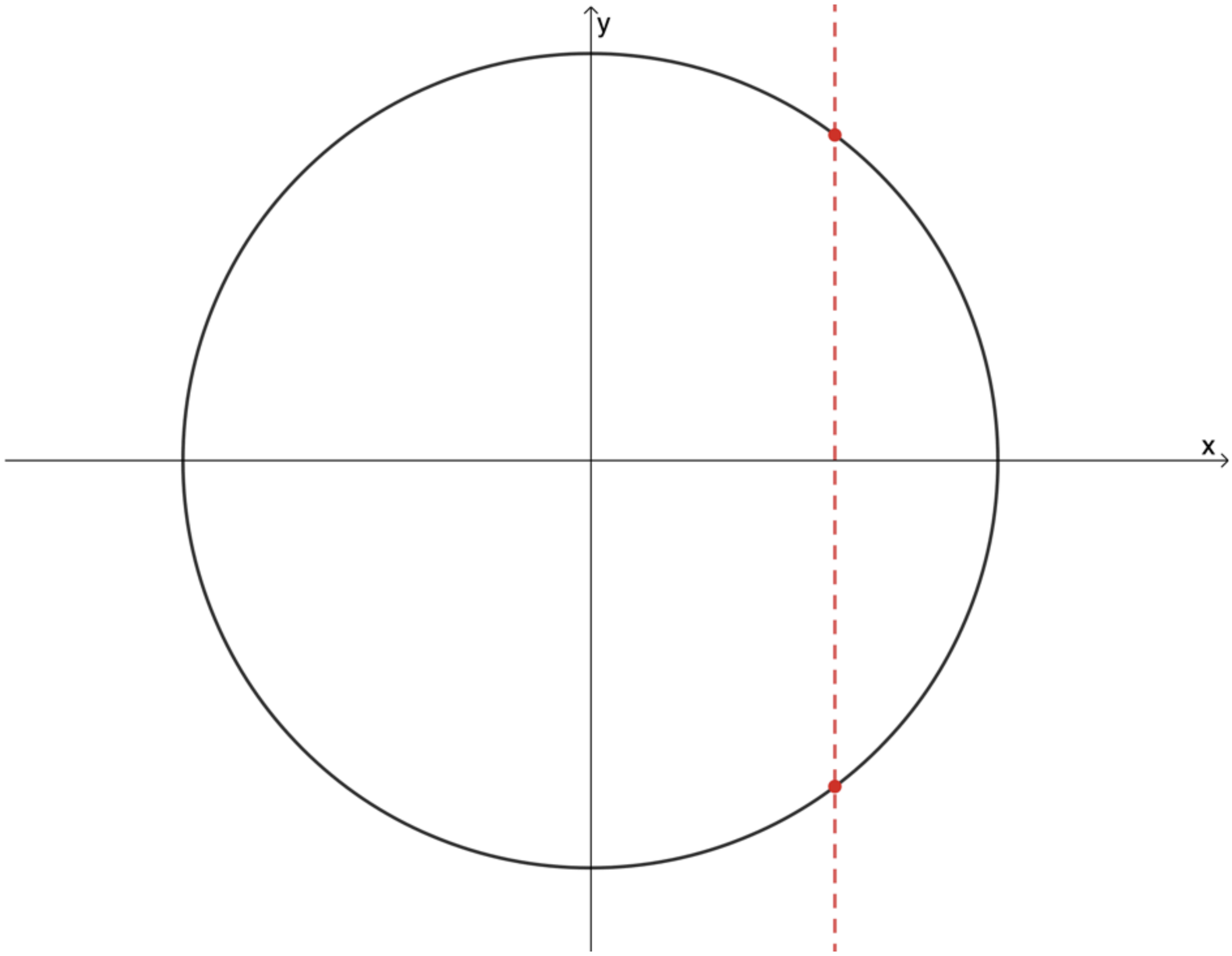
2. Given $f(x) = \frac{1}{2}x - 1$, find:
   a. $f^{-1}(x)$
   b. $[f(x)]^{-1}$
3. Given $h^{-1}(x) = \frac{1}{3}x + 2$:
   a. Determine $h(x)$.
   b. Determine the intercepts with the axes of $h(x)$ and $h^{-1}(x)$.
   c. Determine the coordinates of Q the point of intersection of $h(x)$ and $h^{-1}(x)$.
   d. State the domain and range of $h(x)$ and $h^{-1}(x)$.
   e. Sketch the graphs of $h(x)$ and $h^{-1}(x)$ on the same set of axes showing the intercepts with the axes and the point of intersection.
   f. Is $h(x)$ an increasing or decreasing function?

The full solutions are at the end of the unit.

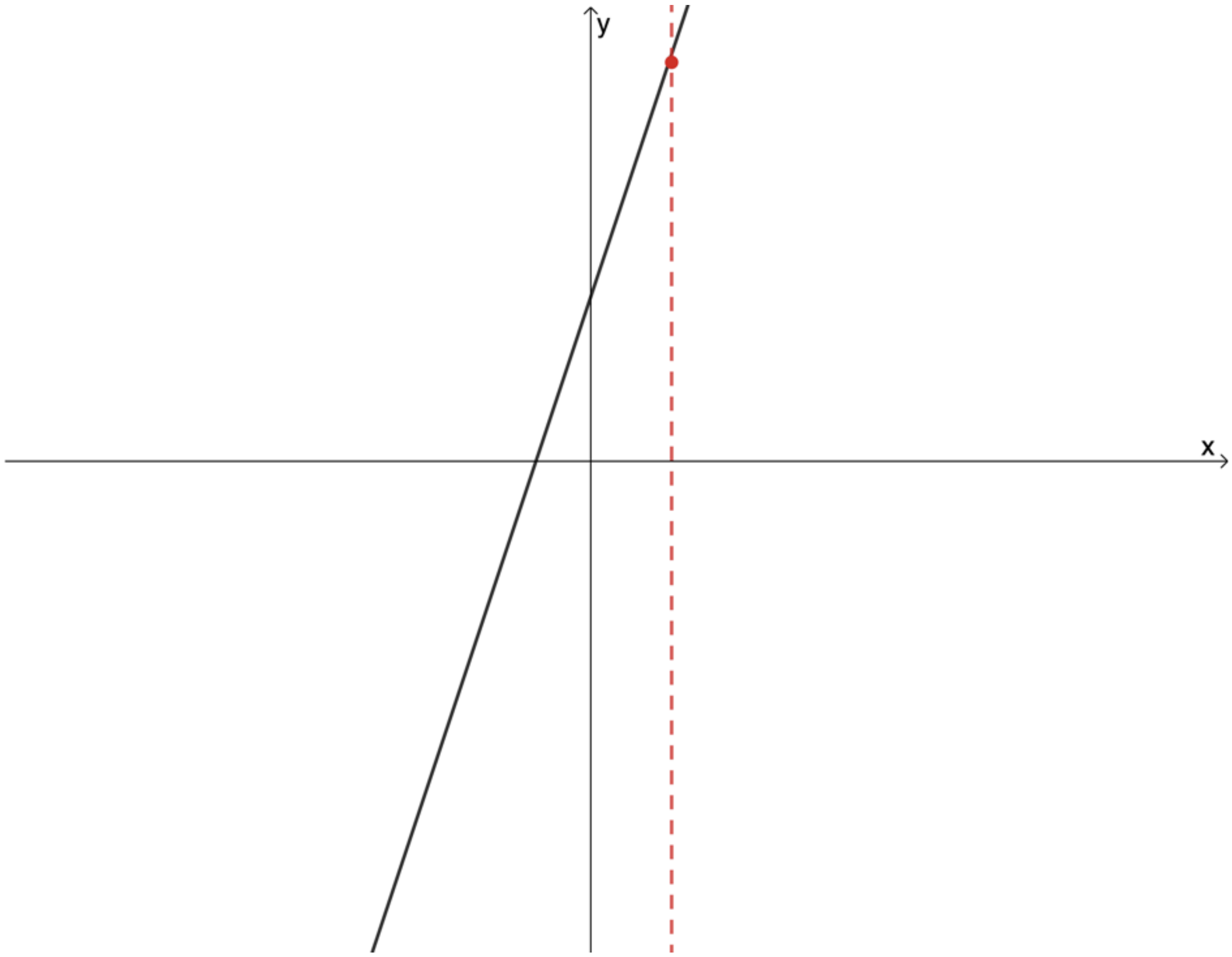
Unit 1: Solutions

Exercise 1.1

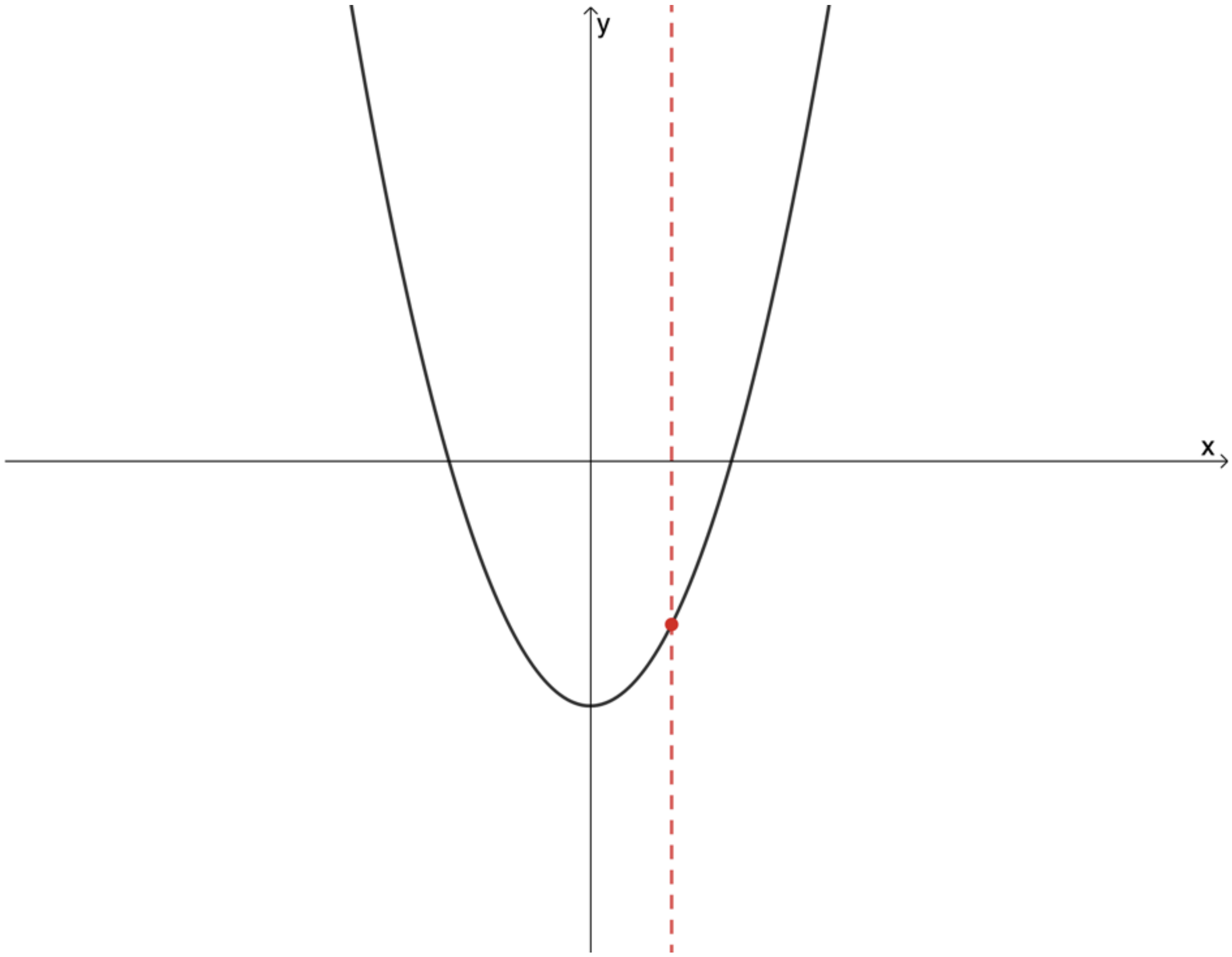
1. Not a function



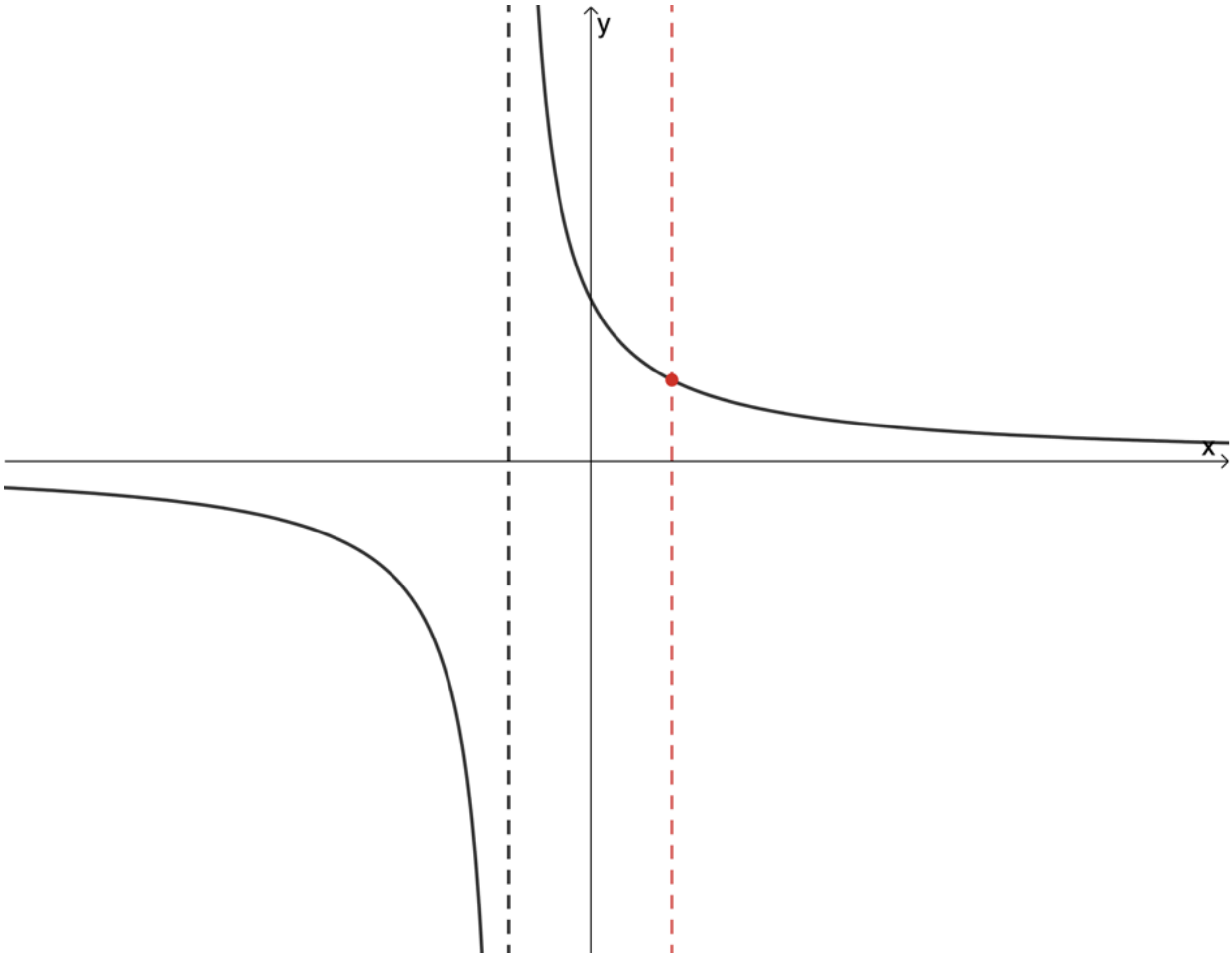
2. Function



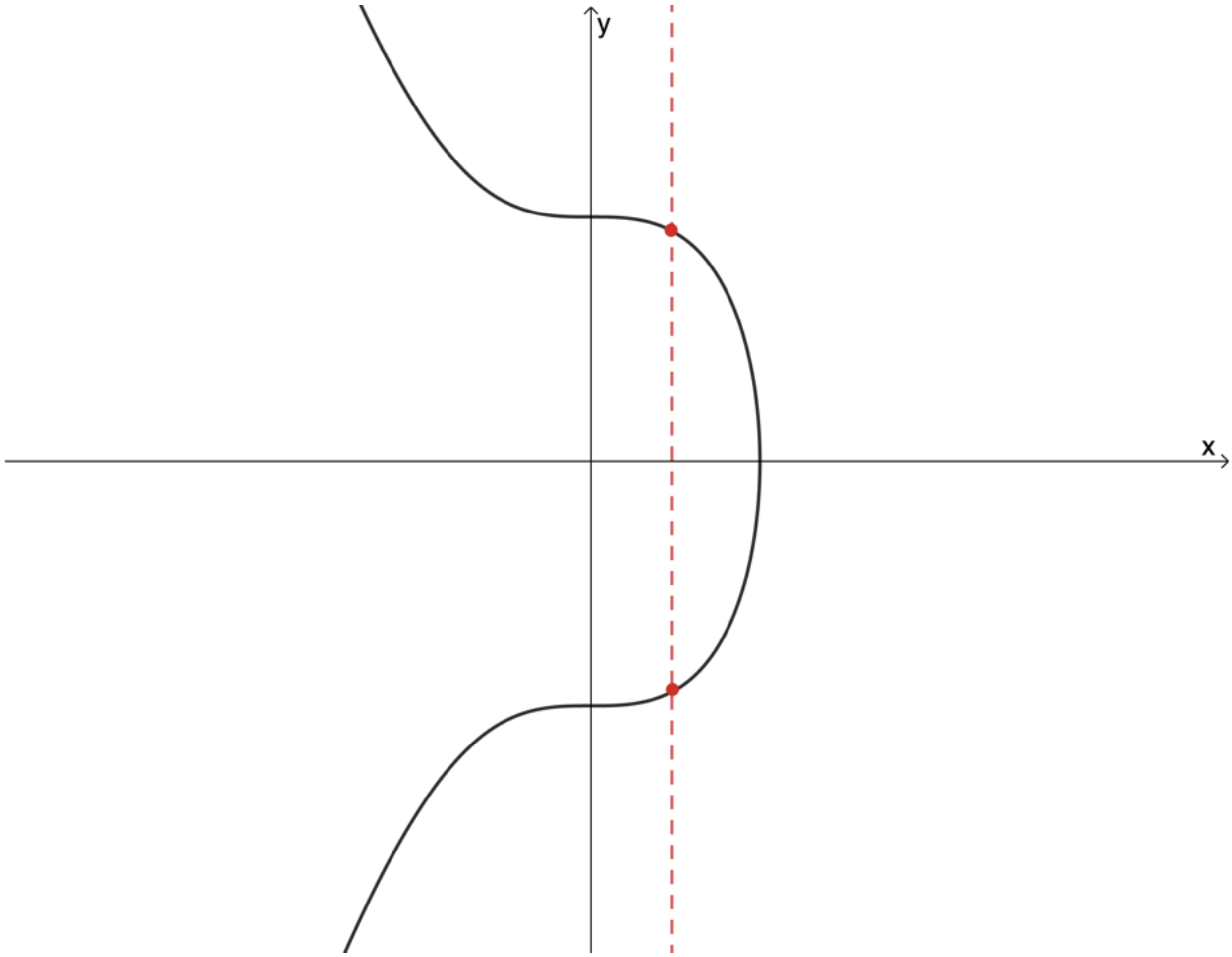
3. Function



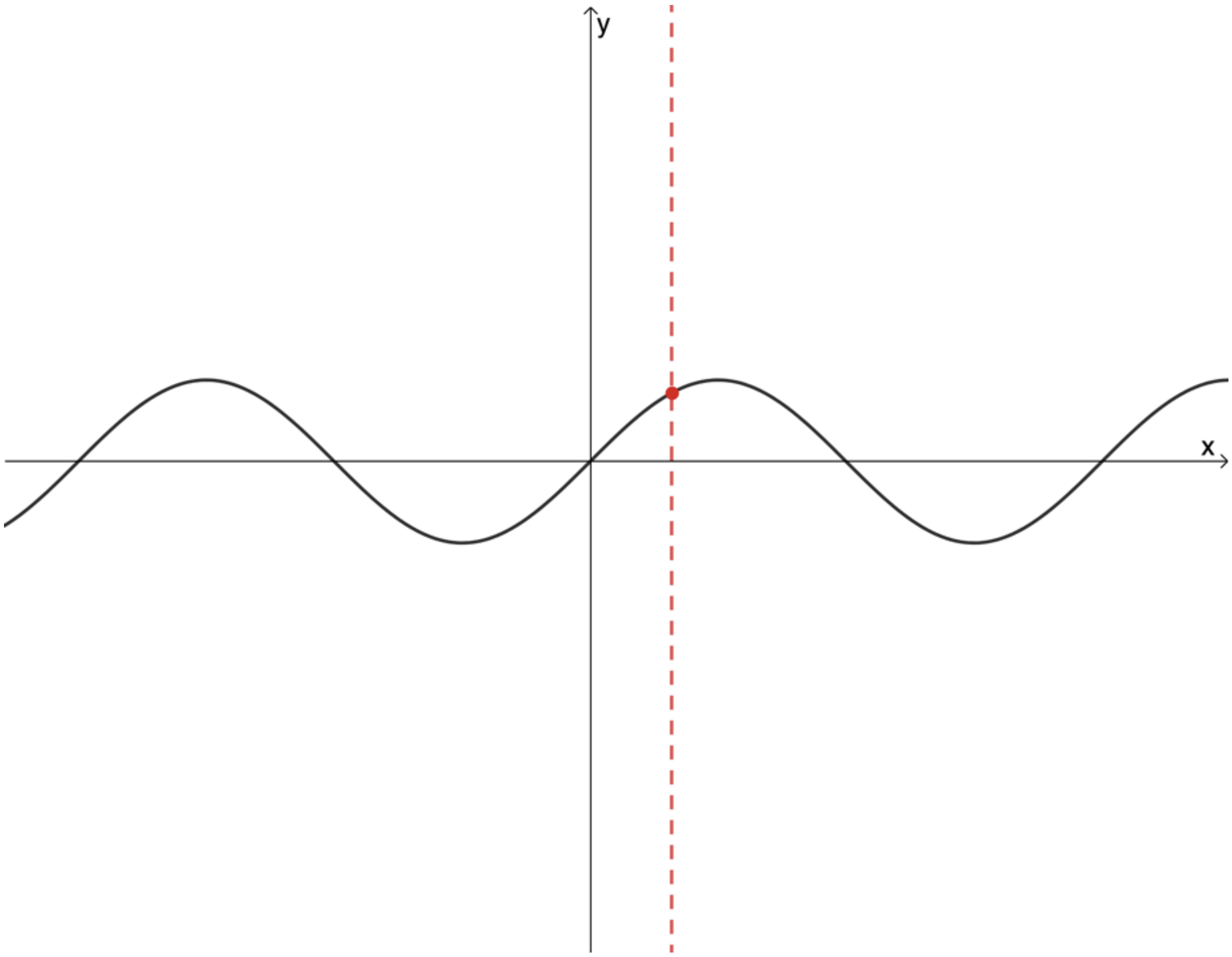
4. Function



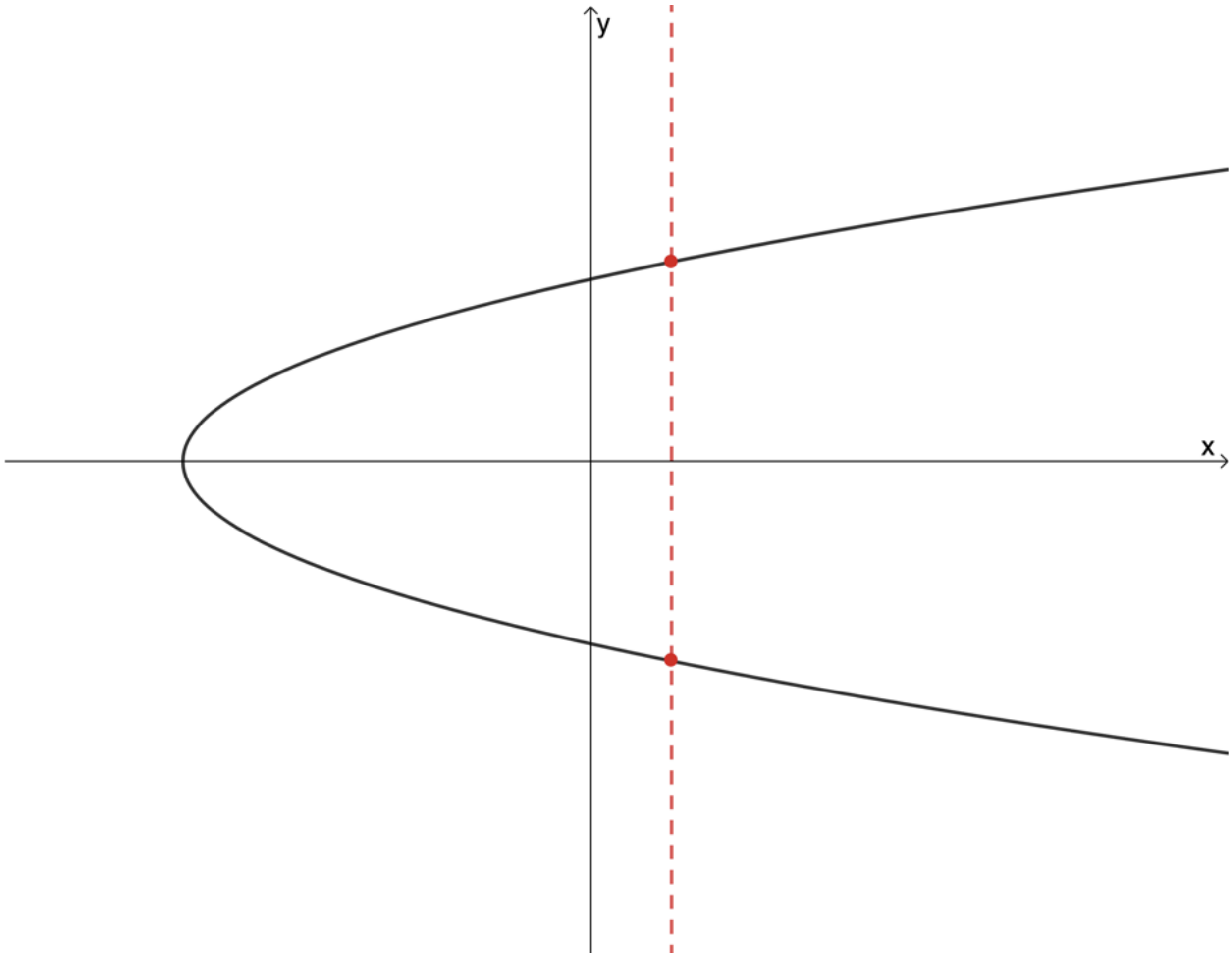
5. Not a function



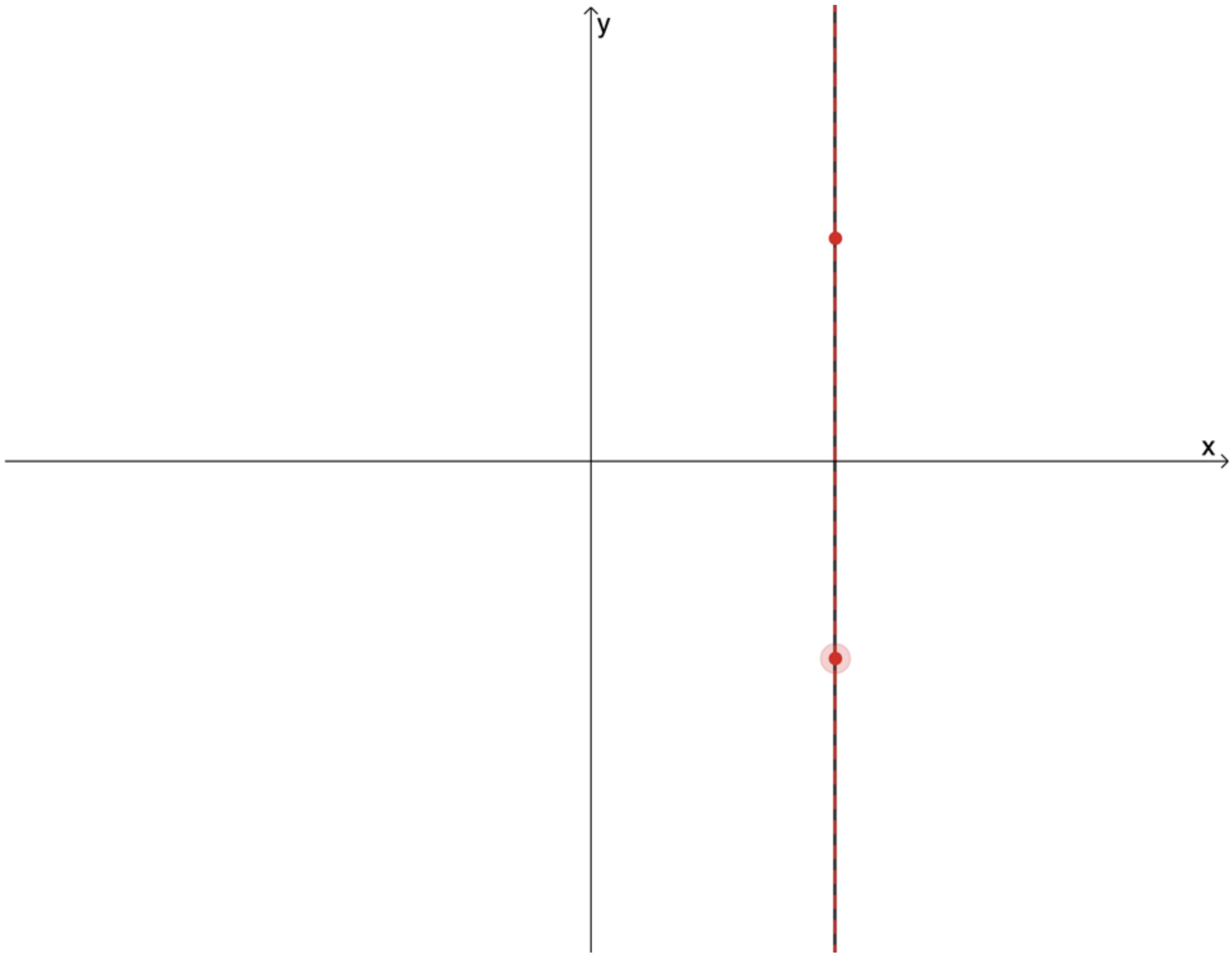
6. Function



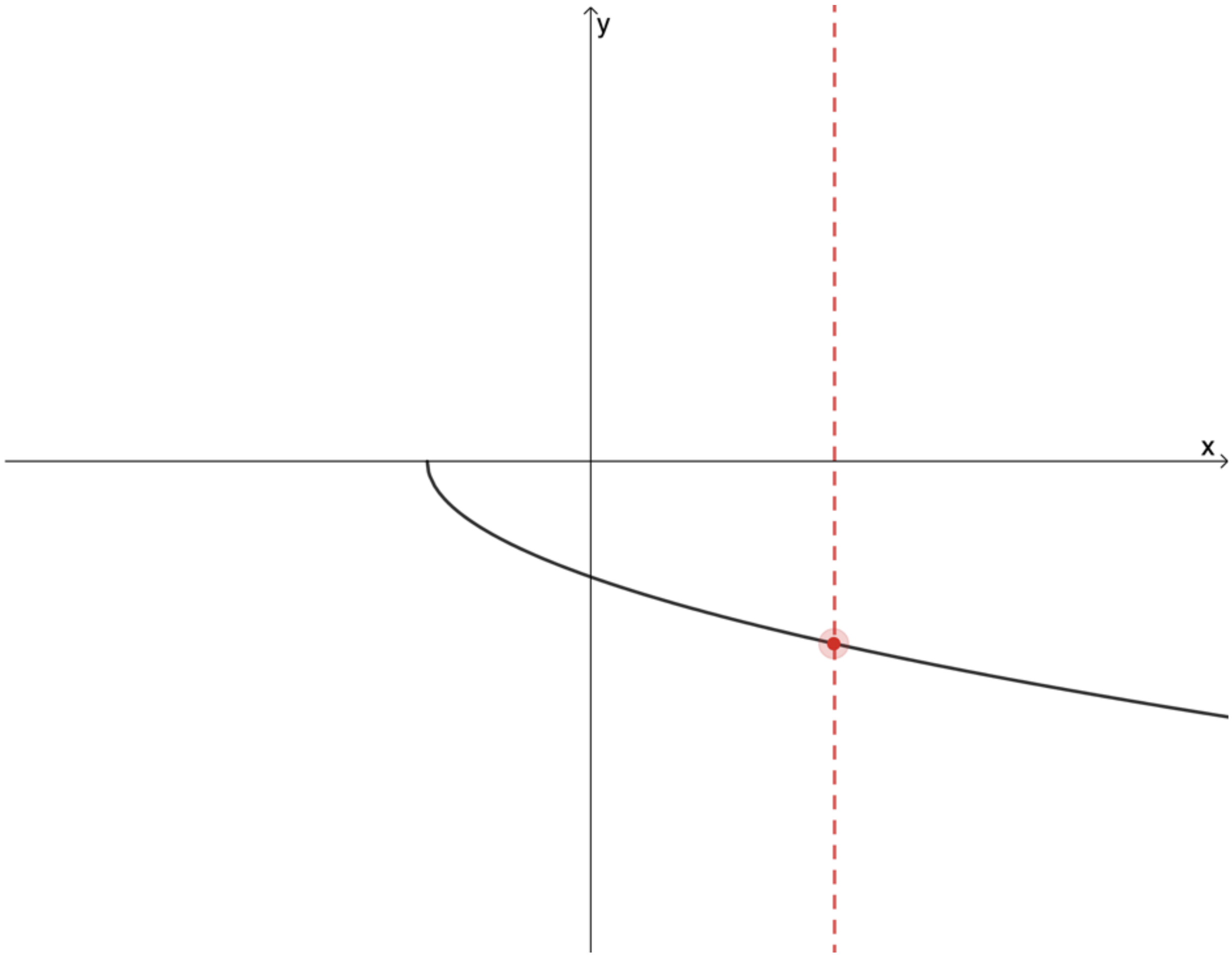
7. Not a function



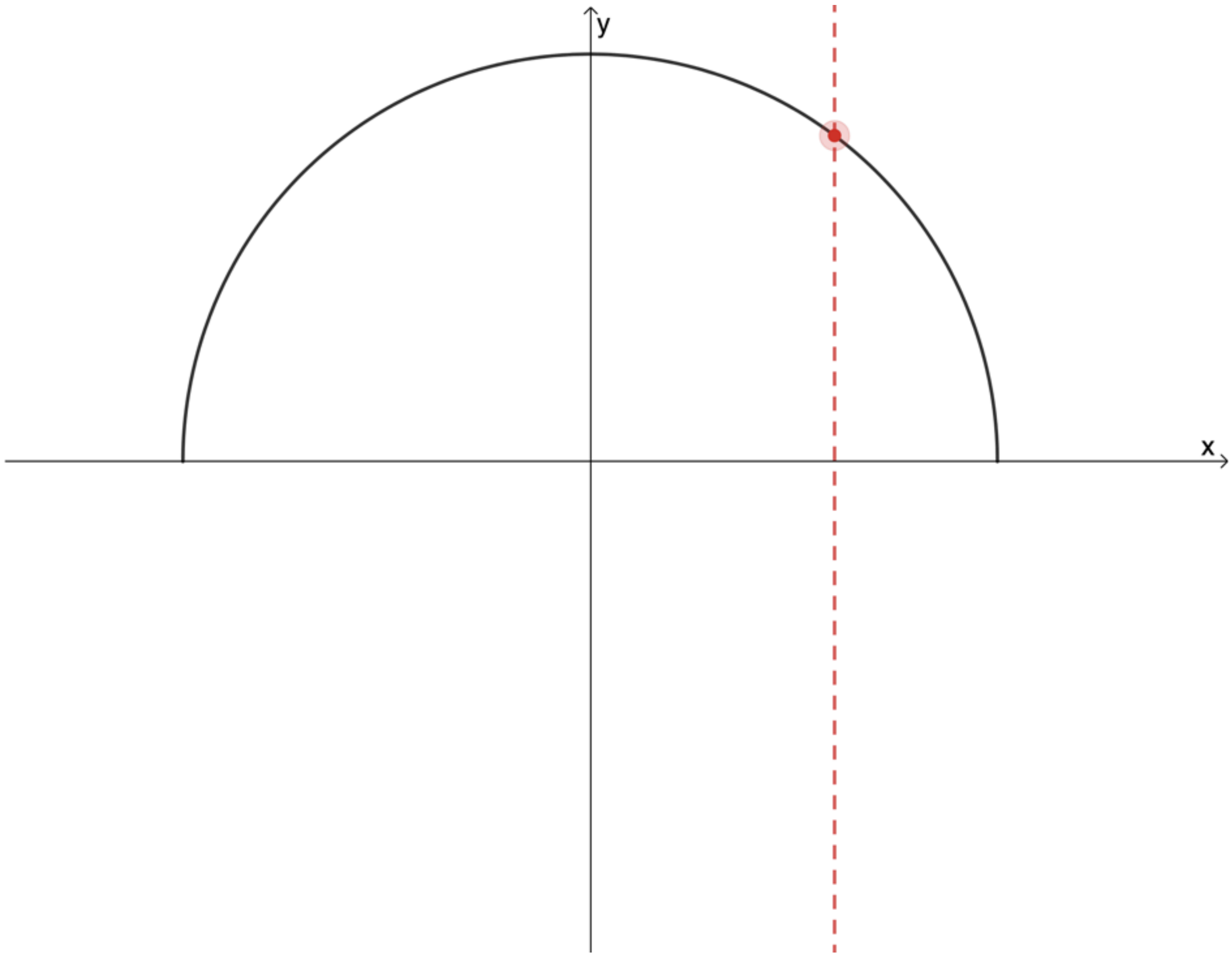
8. Not a function



9. Function



10. Function



Back to Exercise 1.1

Exercise 1.2

1. $f(x) = -2x + 3$

   a. $y = -2x + 3$
   Inverse:

   $$\begin{aligned} x &= -2y + 3 \\ \therefore 2y &= -x + 3 \\ \therefore y &= -\frac{1}{2}x + \frac{3}{2} \\ \therefore f^{-1}(x) &= -\frac{1}{2}x + \frac{3}{2} \end{aligned}$$

   b.

   $$\begin{aligned} [f(x)]^{-1} &= \frac{1}{f(x)} \\ &= \frac{1}{-2x+3} \\ &= -\frac{1}{2x-3} \end{aligned}$$

2. $y = \frac{3}{4}x + \frac{1}{2}$.

   a. The relation is a linear function.

   b.

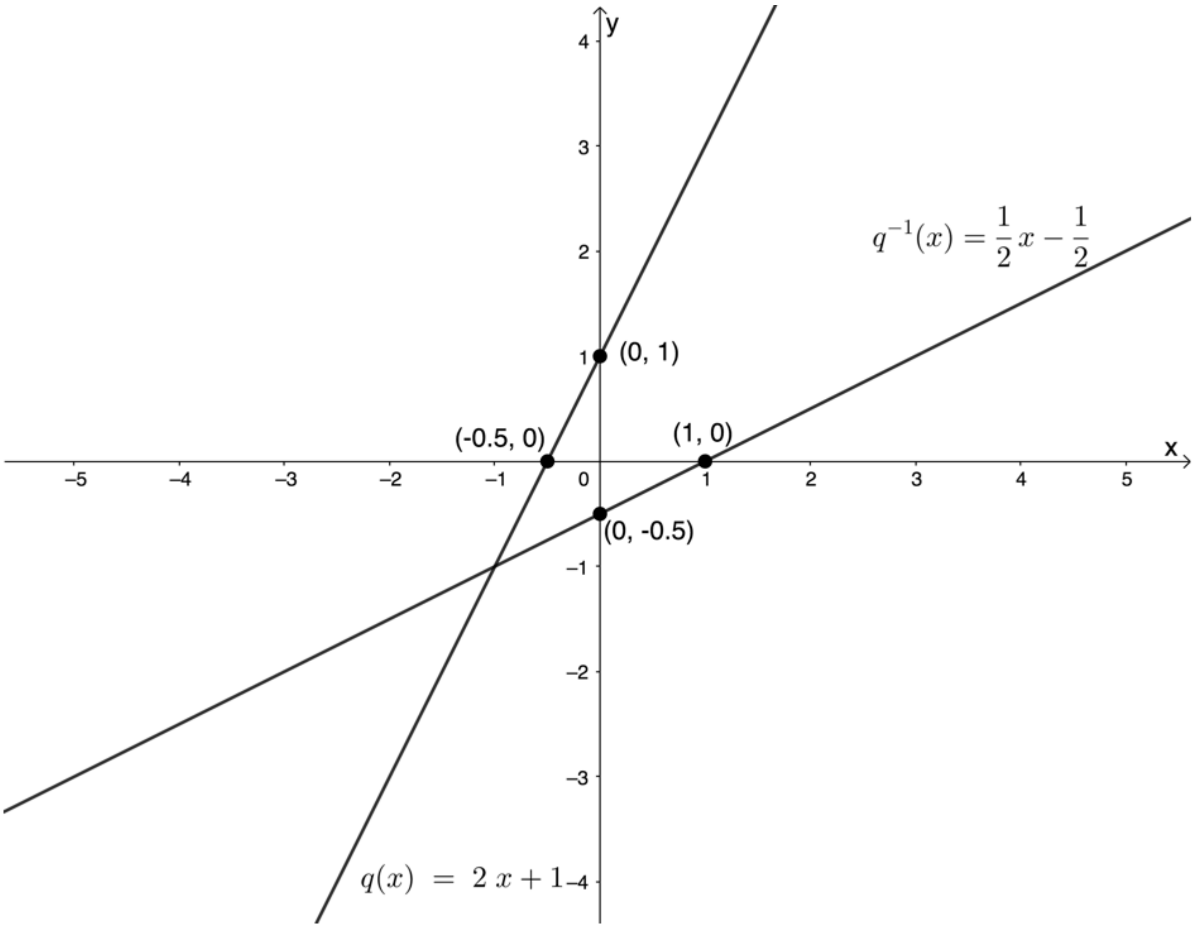
$$x = \frac{3}{4}y + \frac{1}{2}$$
$$\therefore \frac{3}{4}y = x + \frac{1}{2}$$
$$\therefore 3y = 4x - 2$$
$$\therefore y = \frac{4}{3}x - \frac{2}{3}$$

c. The inverse of the function is also a function. Therefore, the relation is an invertible function.

3. $q(x) = 2x + 1$.

a. $y = 2x + 1$
Inverse:
$$x = 2y + 1$$
$$\therefore 2y = x - 1$$
$$\therefore y = \frac{1}{2}x - \frac{1}{2}$$
$$\therefore q^{-1}(x) = \frac{1}{2}x - \frac{1}{2}$$

b.



c. $q(x)$ and $q^{-1}(x)$ are symmetrical about the line $y = x$. Therefore, T is the point $(5, 2)$.

4. $t^{-1}(x) = -3x - 3$

a. $t(x)$ is the inverse of $t^{-1}(x)$.
$y = -3x - 3$
Inverse:

$$x = -3y - 3$$
$$\therefore 3y = -x - 3$$
$$\therefore y = -\frac{1}{3}x - 1$$
$$\therefore t(x) = -\frac{1}{3}x - 1$$

b. $t(x) = -\frac{1}{3}x - 1$
x-intercept (let $t(x) = 0$):
$$-\frac{1}{3}x - 1 = 0$$
$$\therefore \frac{1}{3}x = -1$$
$$\therefore x = -3$$
x-intercept is the point $(-3, 0)$
y-intercept (let $x = 0$):
$$t(0) = -\frac{1}{3}(0) - 1$$
$$\therefore t(0) = -1$$
y-intercept is the point $(0, -1)$
$t^{-1}(x) = -3x - 3$:
$t^{-1}(x)$ is the inverse of $t(x)$. Therefore x-intercept of $t^{-1}(x)$ is $(-1, 0)$ and y-intercept of $t^{-1}(x)$ is $(0, -3)$.
Note: You could also have calculated the intercepts directly.

c.
$$t(x) = t^{-1}(x)$$
$$\therefore -\frac{1}{3}x - 1 = -3x - 3$$
$$\therefore -x - 3 = -9x - 9$$
$$\therefore 8x = -6$$
$$\therefore x = -\frac{3}{4}$$
$$t\left(-\frac{3}{4}\right) = -\frac{1}{3}\left(-\frac{3}{4}\right) - 1$$
$$= \frac{3}{12} - 1$$
$$= \frac{1}{4} - 1$$
$$= -\frac{3}{4}$$
A is the point $(-\frac{3}{4}, -\frac{3}{4})$

5. $f(x) = 3x - 2$ is an increasing function. Therefore, $f^{-1}(x)$ is an increasing function as it is symmetrical to $f(x)$ about the line $y = x$, an increasing function.

Back to Exercise 1.2

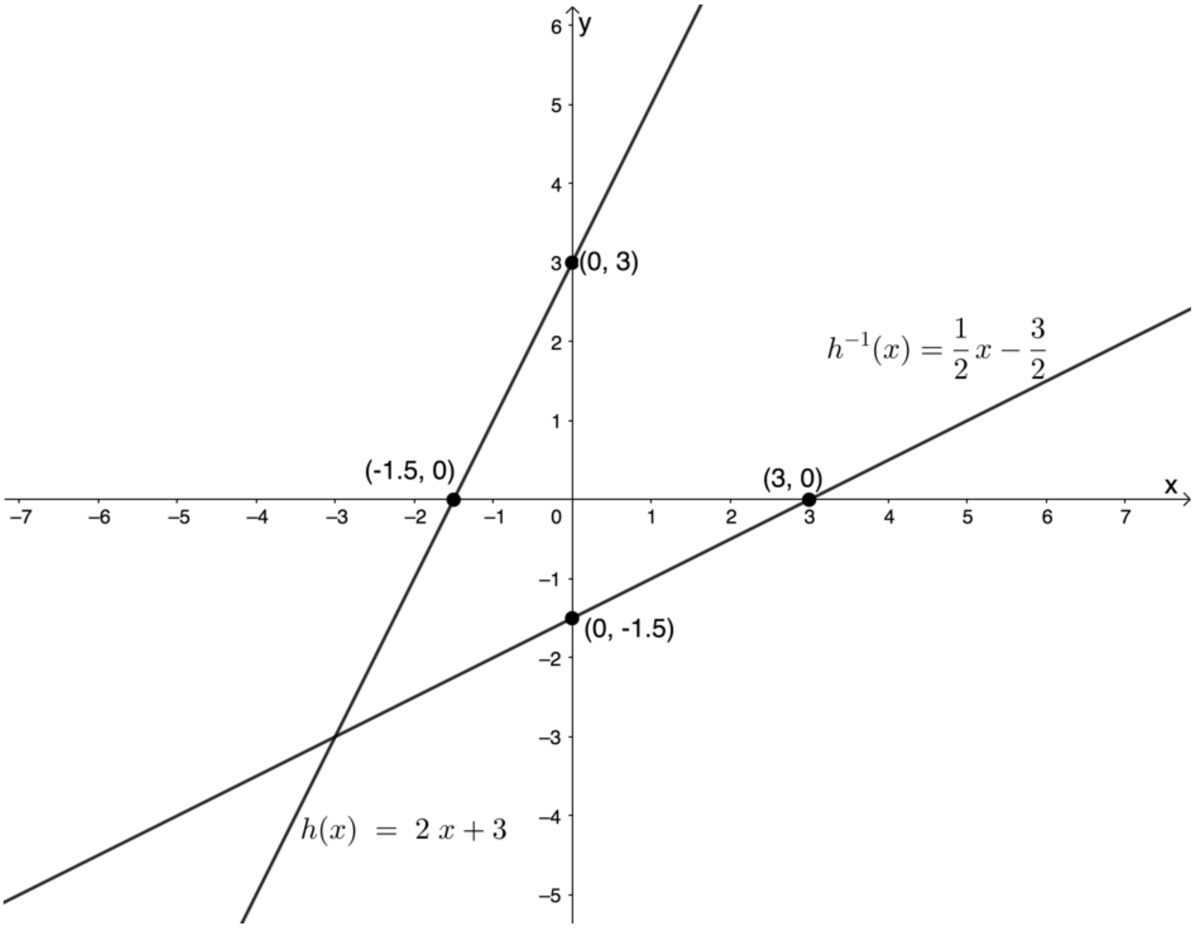
Unit 1: Assessment

1.

a. $y = 2x + 3$
Inverse:

$$x = 2y + 3$$
$$\therefore 2y = x - 3$$
$$\therefore y = \frac{1}{2}x - \frac{3}{2}$$

b.



2. $f(x) = \frac{1}{2}x - 1$

a. $y = \frac{1}{2}x - 1$

$f^{-1}(x)$:

$$x = \frac{1}{2}y - 1$$
$$\therefore \frac{1}{2}y = x + 1$$
$$\therefore y = 2x + 2$$
$$\therefore f^{-1}(x) = 2x + 2$$

b.

$$[f(x)]^{-1} = \frac{1}{f(x)}$$
$$= \frac{1}{\left(\frac{1}{2}x - 1\right)}$$
$$= \frac{1}{\left(\frac{x-2}{2}\right)}$$
$$= \frac{2}{x-2}$$

3. $h^{-1}(x) = \frac{1}{3}x + 2$.

a. $y = \frac{1}{3}x + 2$

$h(x)$:

$$x = \frac{1}{3}y + 2$$
$$\therefore \frac{1}{3}y = x - 2$$
$$\therefore y = 3x - 6$$
$$\therefore h(x) = 3x - 6$$

b. $h(x)$:

x-intercept (let $h(x) = 0$):

$3x - 6 = 0$

$\therefore 3x = 6$

$\therefore x = 2$

x-intercept is the point $(2, 0)$

y-intercept (let $x = 0$):

$h(0) = 3(0) - 6$

$\therefore h(0) = -6$

y-intercept is the point $(0, -6)$

$h^{-1}(x)$:

$h^{-1}(x)$ is the inverse of $h(x)$. Therefore, the x-intercept of $h^{-1}(x)$ is $(-6, 0)$ and y-intercept of $t^{-1}(x)$ is $(0, 2)$

c.

$$h(x) = h^{-1}(x)$$
$$\therefore 3x - 6 = \frac{1}{3}x + 2$$
$$\therefore 9x - 18 = x + 6$$
$$\therefore 8x = 24$$
$$\therefore x = 3$$
$$h(3) = 3(3) - 6$$
$$= 9 - 6$$
$$= 3$$

Q is the point $(3, 3)$.

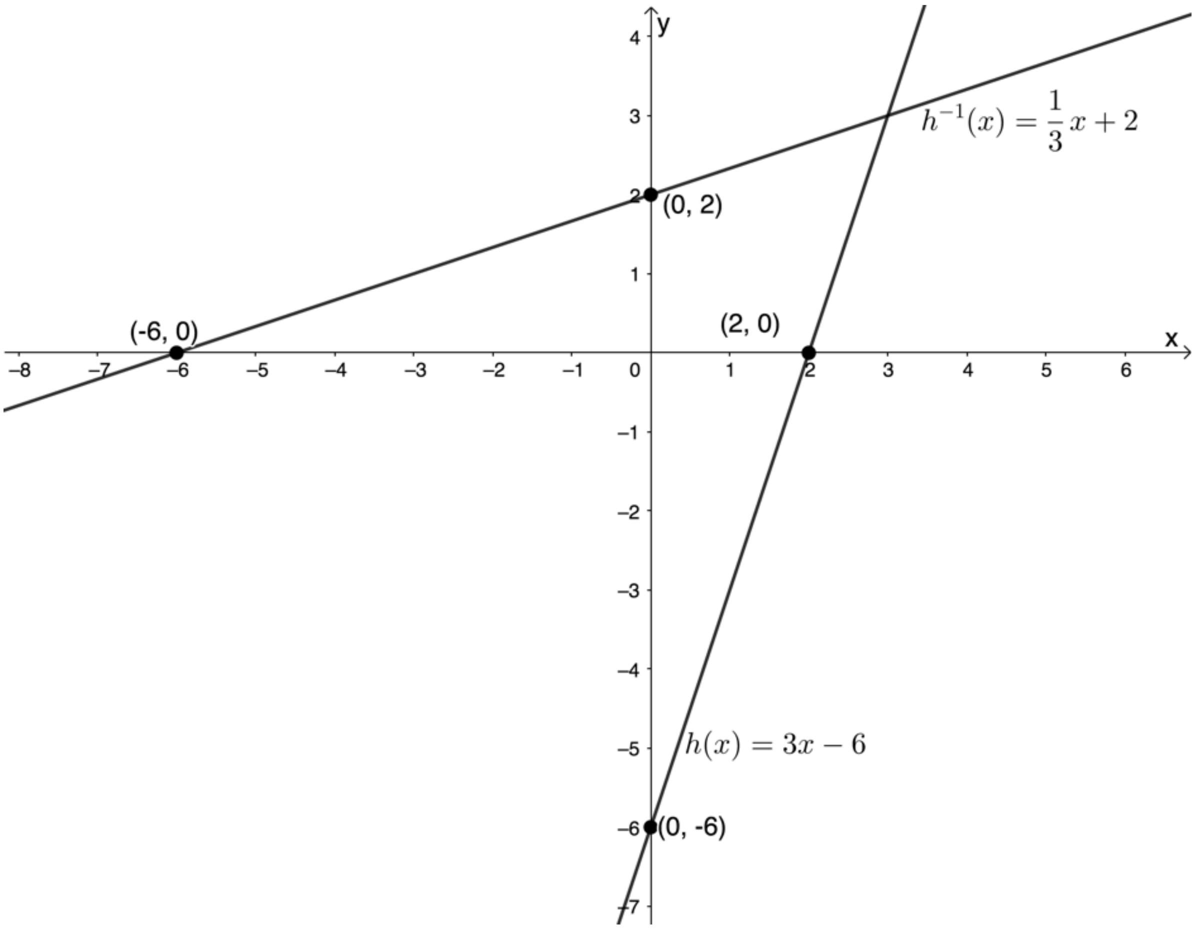
d. Domain of $h(x)$: $\{x|x \in \mathbb{R}\}$

Range of $h(x)$: $\{x|x \in \mathbb{R}\}$

Domain of $h^{-1}(x)$: $\{x|x \in \mathbb{R}\}$

Domain of $h^{-1}(x)$: $\{x|x \in \mathbb{R}\}$

e.



f. $h(x)$ is an increasing function. The function value increase as x increases.

Back to Unit 1: Assessment

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Unit 2: Determine and sketch the inverse of a quadratic function

DYLAN BUSA

Unit outcomes

By the end of this unit you will be able to:

- Find the inverse of $y = ax^2$.
- Sketch the inverse of $y = ax^2$.
- Answer questions about the domain, range, shape, continuity and other characteristics of the inverse graph.

What you should know

Before you start this unit, make sure you can:

- Determine the inverse of a function. Refer to unit 1 if you need help with this.
- Determine if a relation is a function or not using the vertical line test. Refer to unit 1 if you need help with this.
- Determine if a function is invertible or not. Refer to unit 1 if you need help with this.
- Sketch a quadratic function. Refer to level 2 subject outcome 2.1 unit 2 if you need help with this.

Introduction

We saw in the previous unit that linear functions are one-to-one functions and, therefore, are invertible – their inverses are also functions. However, is the same true for quadratic functions of the form $y = ax^2$? Are quadratic functions of the form $y = ax^2$ invertible? Is the inverse of a quadratic function of the form $y = ax^2$ also a function? If not, is there a way for us to make the inverse of a quadratic function of the form $y = ax^2$ a function?

These are the questions we will answer in this unit.

The inverse of the quadratic function

Let's explore the inverse of quadratic functions of the form $y = ax^2$.

Activity 2.1: The inverse of a quadratic function

Time required: 30 minutes

What you need:

- a blank piece of paper or graph paper
- a pen or pencil
- a ruler

What to do:

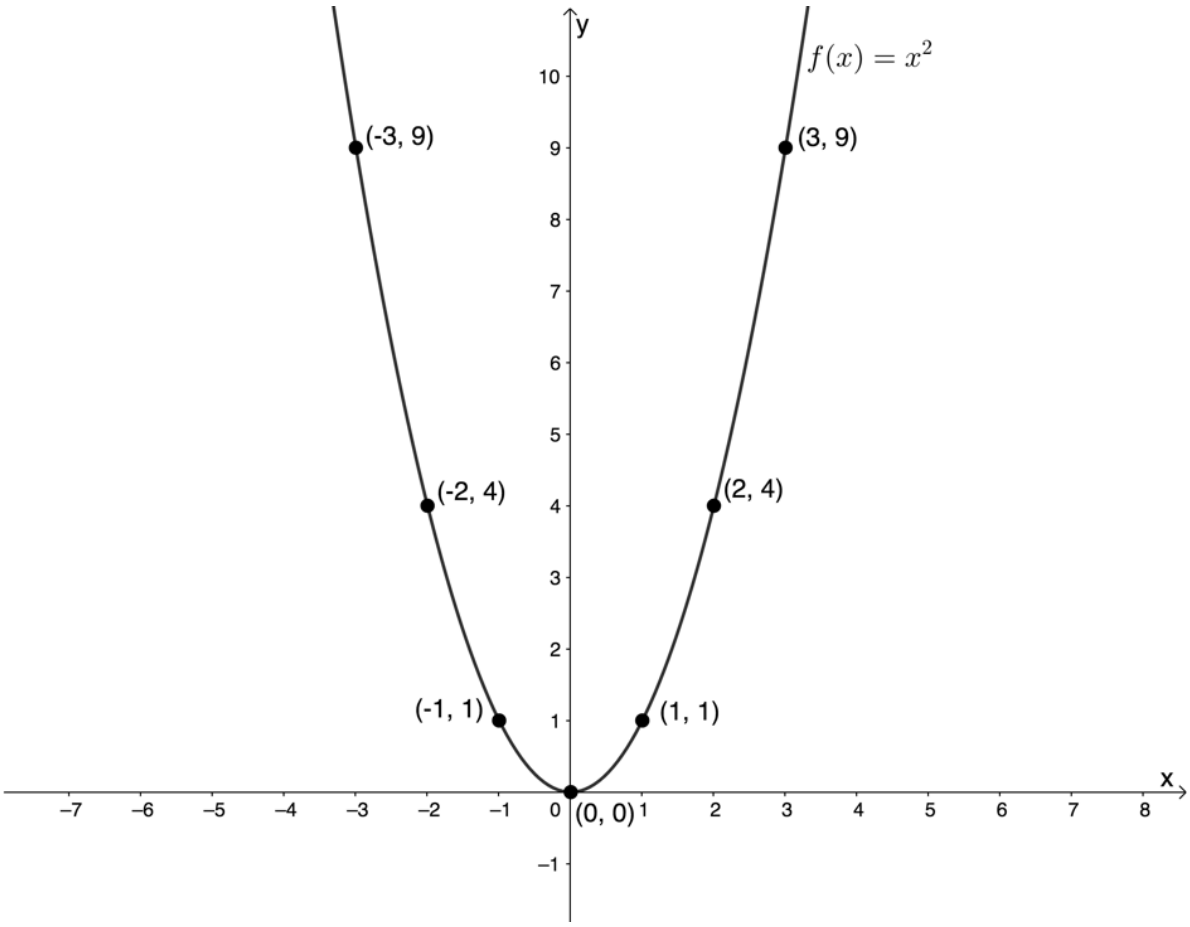
If you don't have a piece of graph paper, start by drawing a Cartesian plane on your piece of paper. Use a scale of about $1\ \text{cm}$ per unit.

1. Plot the function $f(x) = x^2$. If you need to, you can create a table of values and plot point-by-point with at least five points.
2. Now, determine the inverse of f in the form '$y =$'.
3. Plot the inverse of f on your Cartesian plane. You could do this by creating a table of values and plotting point-by-point with at least five points.
4. Is the inverse of f a function? Use the vertical line test to make sure.
5. What is the domain and range of f and of the inverse of f?
6. About what line are the graphs of f and the inverse of f symmetrical?
7. What could you do to the function $f(x) = x^2$ to make sure that its inverse was a function?

What did you find?

1. Here is a table of suitable values.

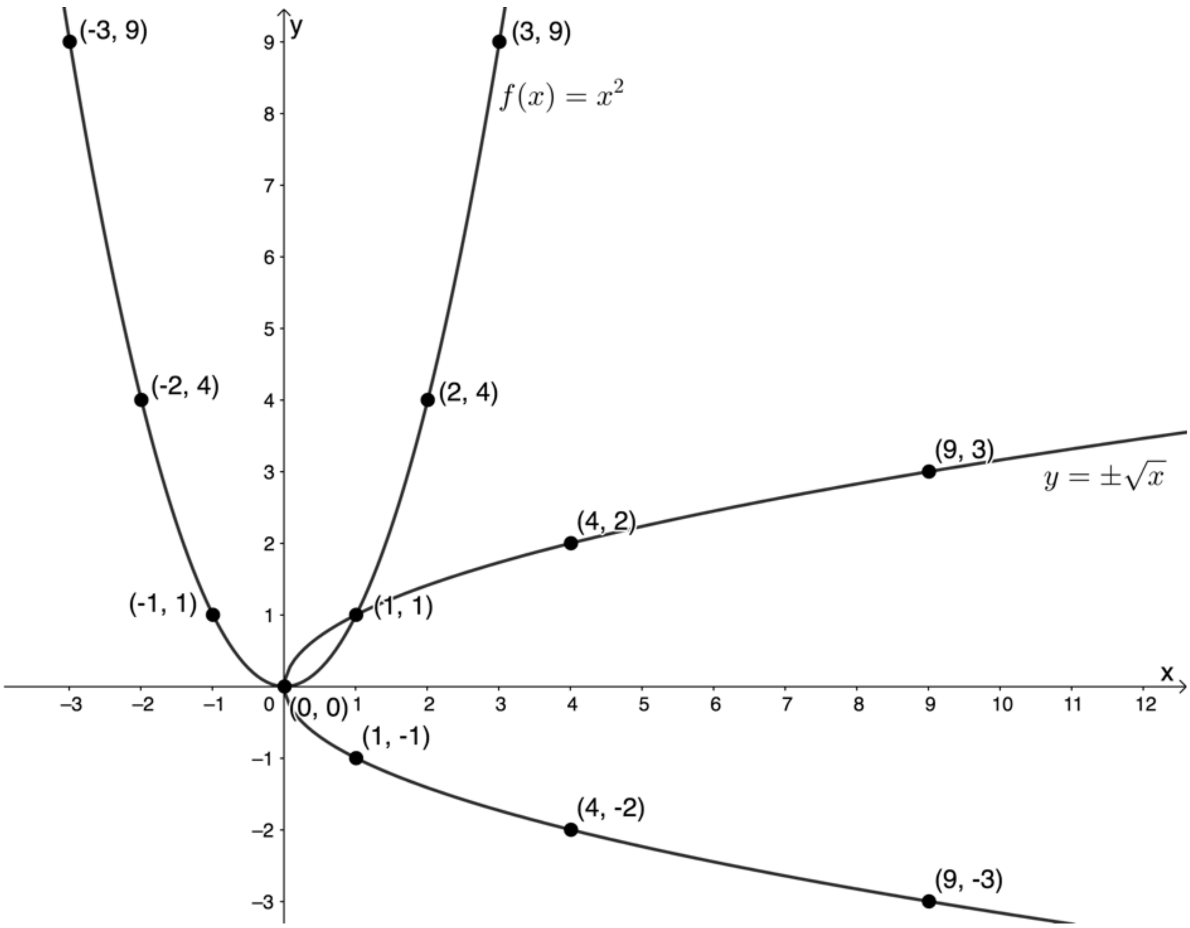
x	-3	-2	-1	0	1	2	3
$f(x) = x^2$	9	4	1	0	1	4	9



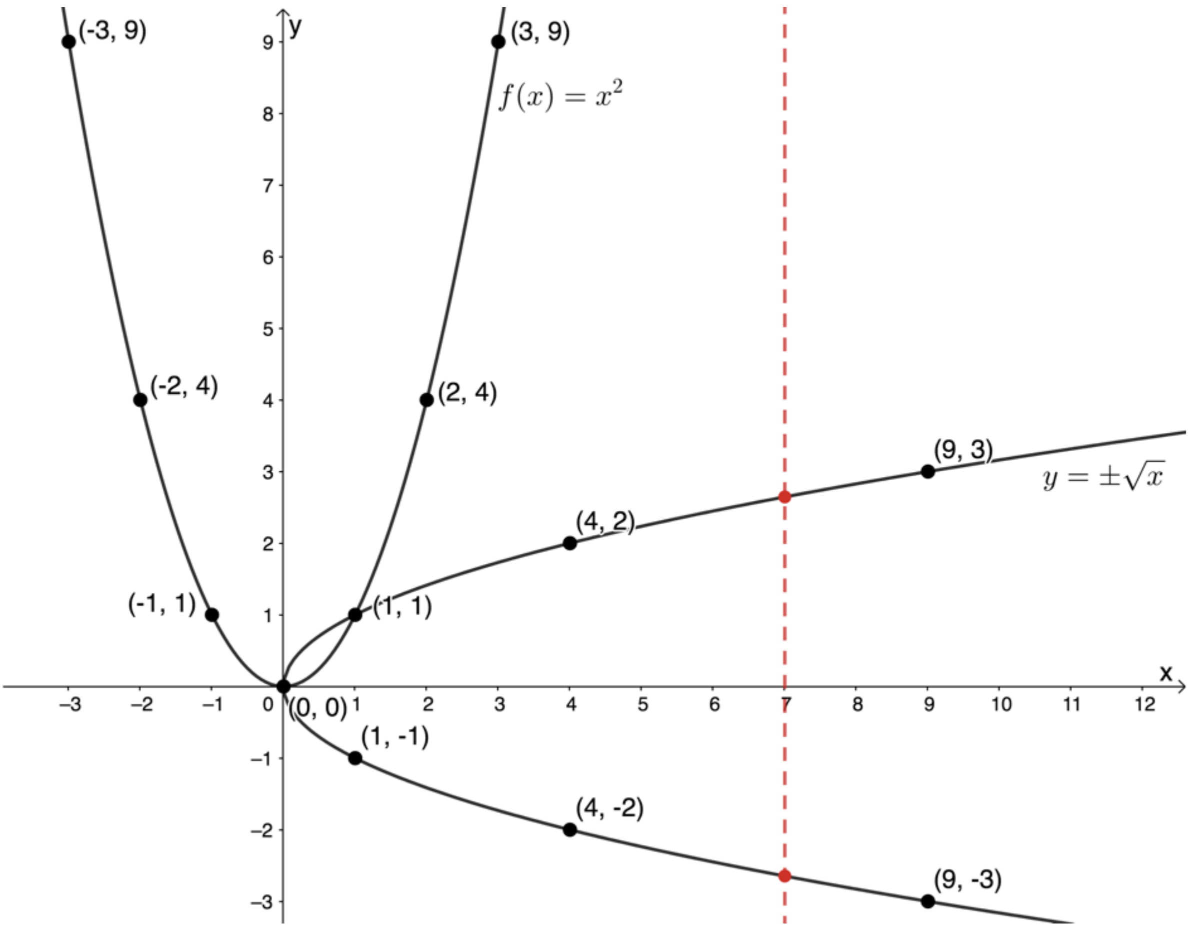
2. $y = x^2$
   Inverse:
   $x = y^2$
   $\therefore y = \pm\sqrt{x}$

3. Here is a table of suitable values.

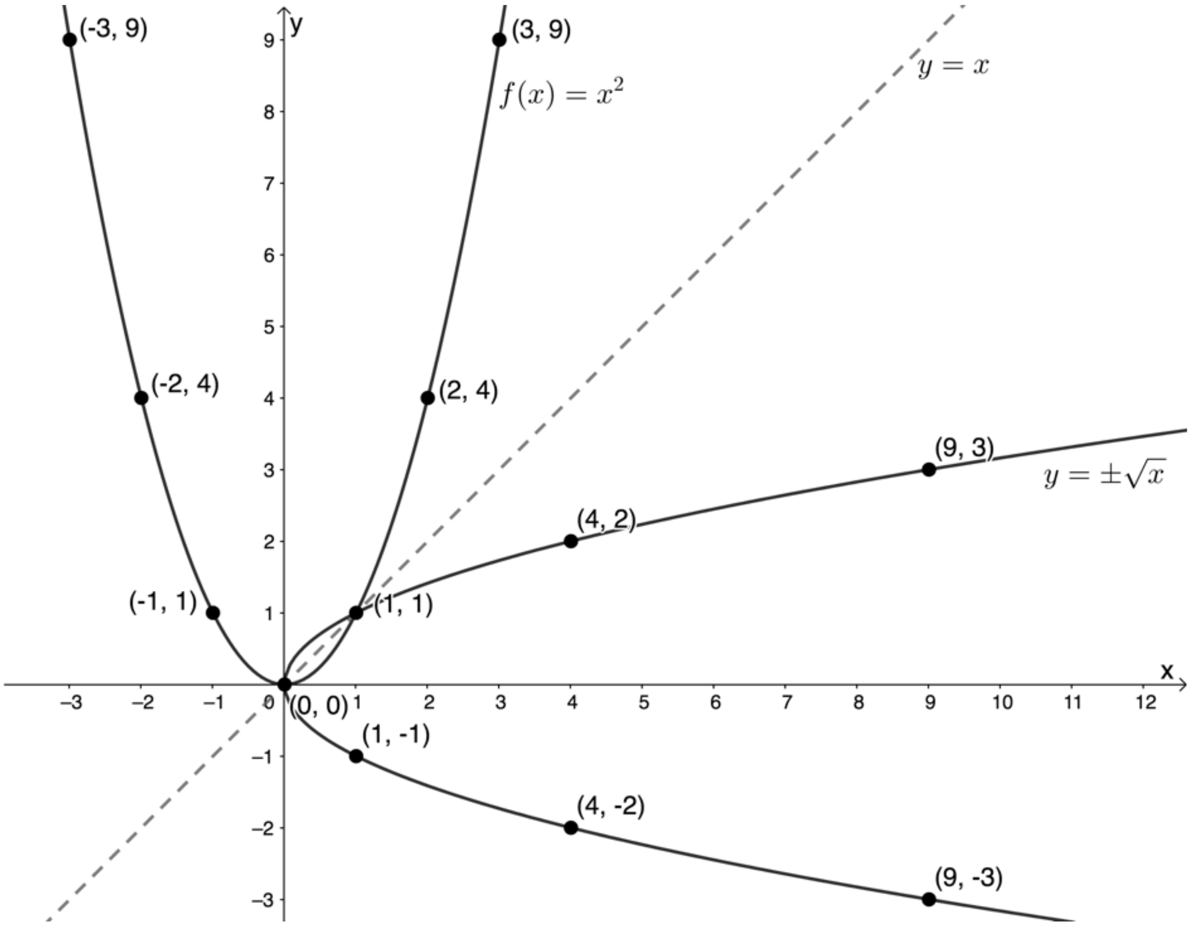
x	0	1	4	9
$y = \pm\sqrt{x}$	0	±1	±2	±3



4. The inverse of f is not a function, as demonstrated by the vertical line test.



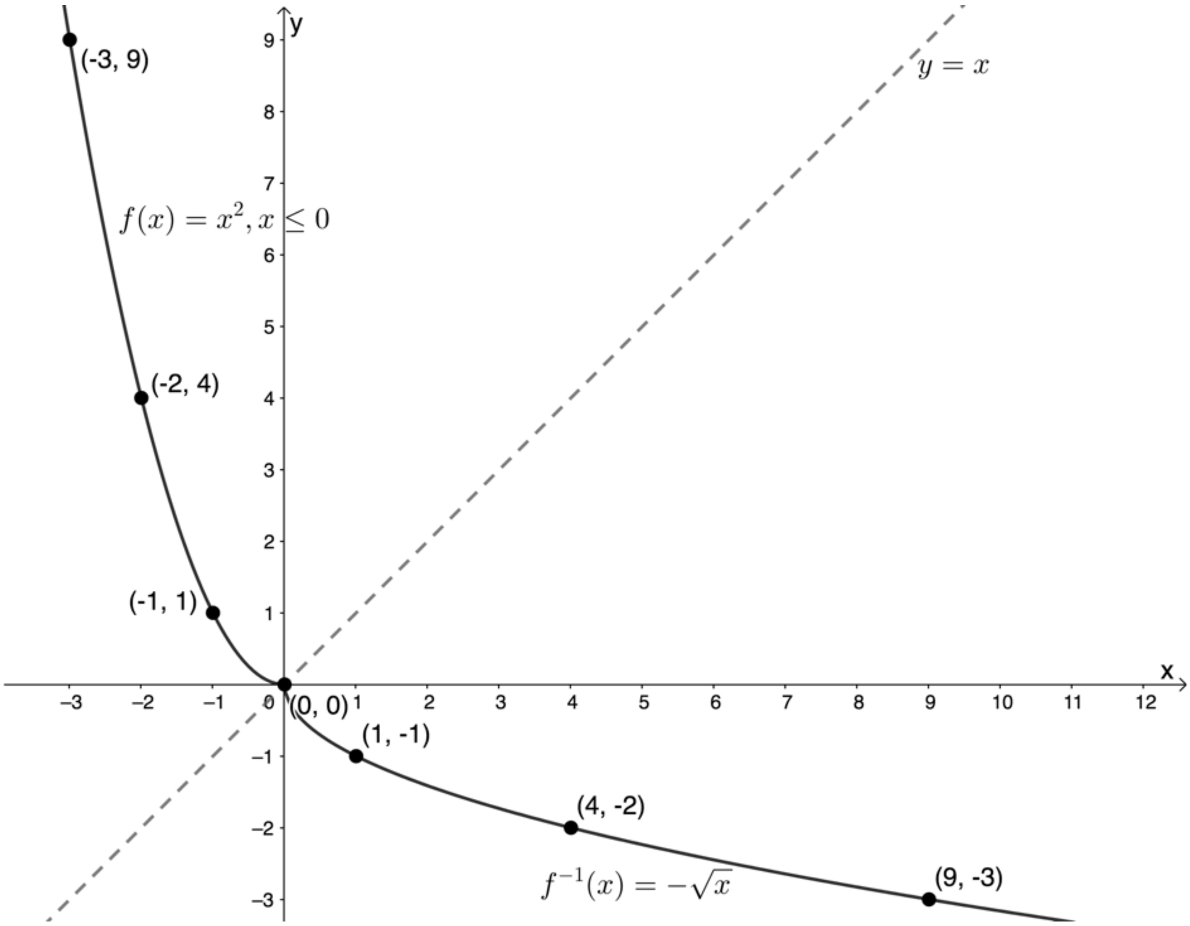
5. Domain of f: $\{x|x \in \mathbb{R}\}$
   Range of f: $\{y|y \in \mathbb{R}, y \geq 0\}$
   Domain of inverse of f: $\{x|x \in \mathbb{R}, x \geq 0\}$
   Range of inverse of f: $\{y|y \in \mathbb{R}\}$
6. The graphs of f and the inverse of f are symmetrical about the line $y = x$.



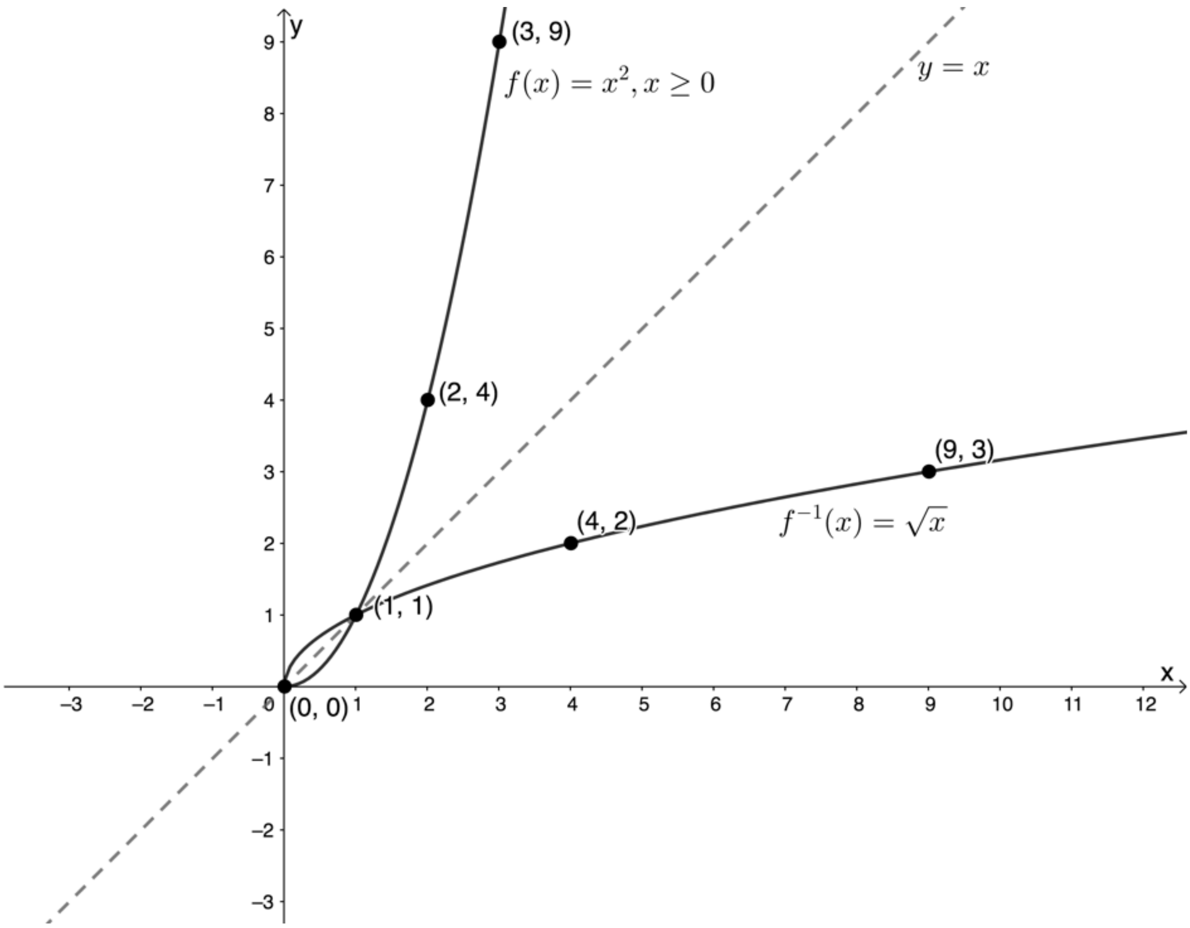
7. We can see that both the graphs of f and the inverse of f have two 'arms'. It is these two 'arms' that mean that the inverse of f is not a function. If we could remove one of the 'arms' of f, then the inverse of f would also only have one 'arm' and would be a function.

   We can remove one of the 'arms' of f by restricting the domain. We can either restrict the domain to $\{x|x \in \mathbb{R}, x \leq 0\}$ or $\{x|x \in \mathbb{R}, x \geq 0\}$.

   If we restrict the domain to $\{x|x \in \mathbb{R}, x \leq 0\}$, the graphs of f and the inverse of f are as follows. Notice that because the inverse of f is now a function we can use the $f^{-1}(x)$ notation.



If we restrict the domain to $\{x|x \in \mathbb{R}, x \geq 0\}$, the graphs of f and the inverse of f are as follows. Notice that because the inverse of f is now a function we can use the $f^{-1}(x)$ notation.



In Activity 2.1, we saw that ordinarily, the quadratic function of the form $y = ax^2$ is not invertible because its inverse is not also a function. However, if we restrict the domain to $x \leq 0$ or $x \geq 0$, we can make the inverse a function and, hence, make the original quadratic invertible. What we are really doing when we restrict the domain like this is making the function a one-to-one function.

We also saw that, like the linear function, the quadratic function and its inverse are symmetrical about the line $y = x$. Every point (x, y) on the original function, has a corresponding symmetrical point (y, x) on the inverse.

Finally, we saw that the domain of the original function becomes the range of the inverse and vice versa (see Figure 1). This remains true even when we restrict the domain of the original function. The domain of $f(x) = x^2, x \geq 0$, for example, is $\{x | x \in \mathbb{R}, x \geq 0\}$ and the range is $\{y | y \in \mathbb{R}, y \geq 0\}$. The domain and range of the inverse of $f(x) = x^2, x \geq 0$ are $\{x | x \in \mathbb{R}, x \geq 0\}$ and $\{y | y \in \mathbb{R}, y \geq 0\}$ respectively.

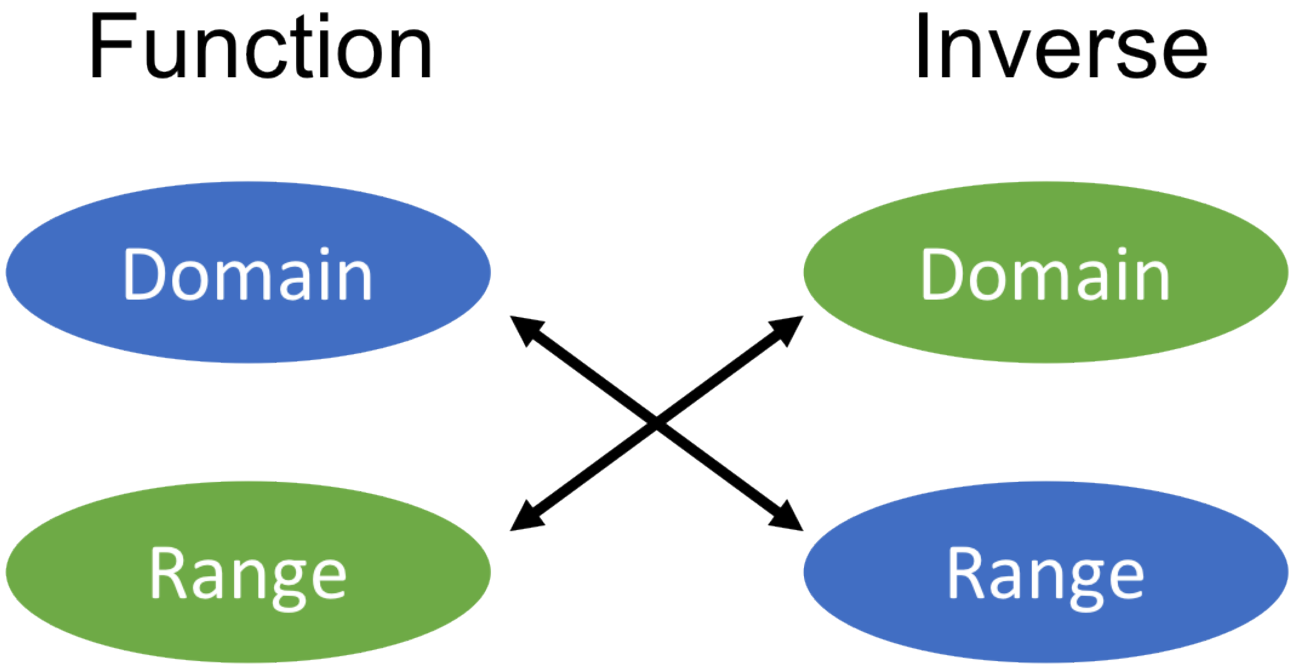


Figure 1: Relationship between the domain and range of a function and its inverse

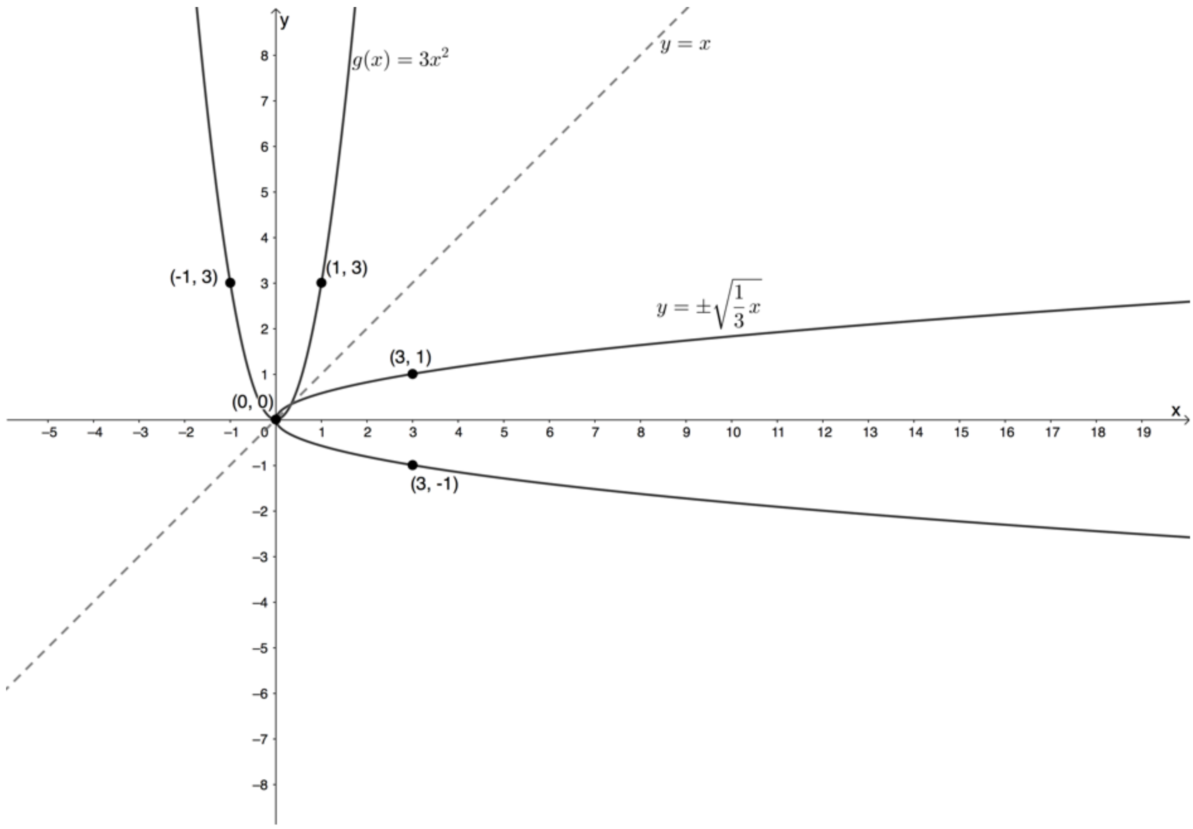
Example 2.1

Given $g(x) = 3x^2$:

1. Determine the inverse of g.
2. Sketch g and the inverse of g on the same system of axes.
3. State the domain and range of g and of the inverse of g.
4. Is the inverse of g a function?

Solutions

1. $g(x) = 3x^2$. Therefore, we can write the function equation as $y = 3x^2$. To find the inverse, we interchange x and y.
   Inverse:
   $$x = 3y^2$$
   $$\therefore y^2 = \frac{1}{3}x$$
   $$\therefore y = \pm\sqrt{\frac{1}{3}x}$$
2.



3. Domain of g: $\{x|x \in \mathbb{R}\}$
   Range of g: $\{y|y \in \mathbb{R}, y \geq 0\}$
   Domain of inverse of g: $\{x|x \in \mathbb{R}, x \geq 0\}$
   Range of inverse g: $\{y|y \in \mathbb{R}\}$
4. The inverse of g is not a function. It does not pass the vertical line test. We would need to restrict the domain of g in order to make its inverse a function.

Example 2.2

Given $r(x) = -2x^2$:

1. State the domain and range of r.
2. How must the domain be restricted so that r is invertible?
3. Find $r^{-1}(x)$.
4. State the domain and range of r^{-1}.
5. Sketch the graphs of r and r^{-1} on the same set of axes.

Solutions

1. Domain r: $\{x|x \in \mathbb{R}\}$
   Range of r: $\{y|y \in \mathbb{R}, y \leq 0\}$

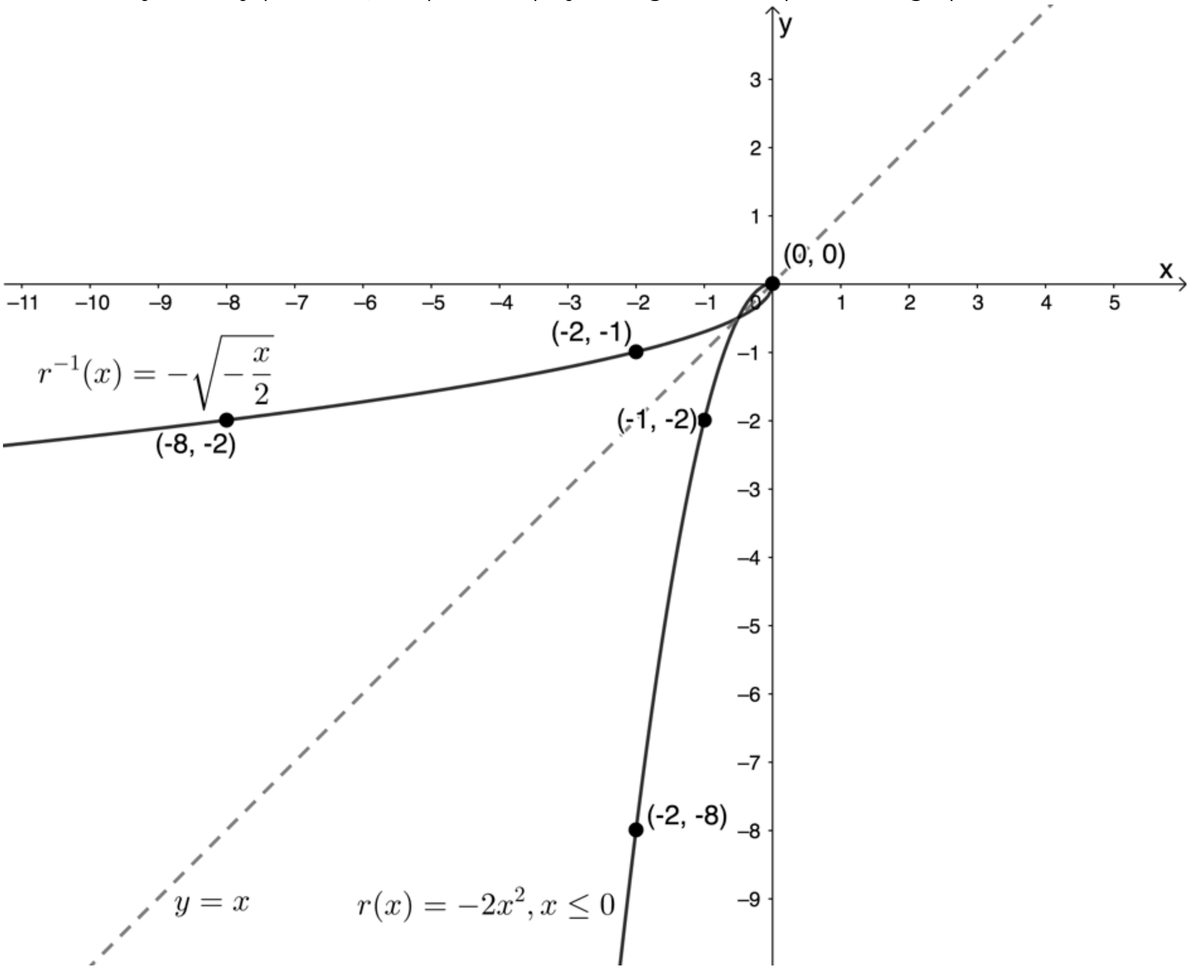
2. The domain of r must be restricted to $\{x|x \in \mathbb{R}, x \leq 0\}$ or $\{x|x \in \mathbb{R}, x \geq 0\}$ in order for the inverse to be a function and hence for r to be invertible.
3. Option 1: Restrict the domain of r to $\{x|x \in \mathbb{R}, x \leq 0\}$.
   $y = -2x^2, x \leq 0$
   Inverse:
   $$x = -2y^2$$
   $$\therefore 2y^2 = -x$$
   $$\therefore y^2 = -\frac{x}{2}$$
   $$\therefore y = -\sqrt{-\frac{x}{2}}$$
   $$\therefore r^{-1} = -\sqrt{-\frac{x}{2}}$$
   Remember that the domain of r becomes the range of r^{-1}. The domain of r was restricted to $\{x|x \in \mathbb{R}, x \leq 0\}$, therefore, the range of r^{-1} will be $\{y|y \in \mathbb{R}, y \leq 0\}$. Because the function values are restricted to $y \leq 0$, we chose the **negative** square root.

   Also note that $\sqrt{-\frac{x}{2}}$ is only defined for $x \leq 0$. However, the range of the original function r is $\{y|y \in \mathbb{R}, y \leq 0\}$. The range of r becomes the domain of r^{-1}. Hence the domain of r^{-1} is $\{x|x \in \mathbb{R}, x \leq 0\}$ and so $\sqrt{-\frac{x}{2}}$ is defined.

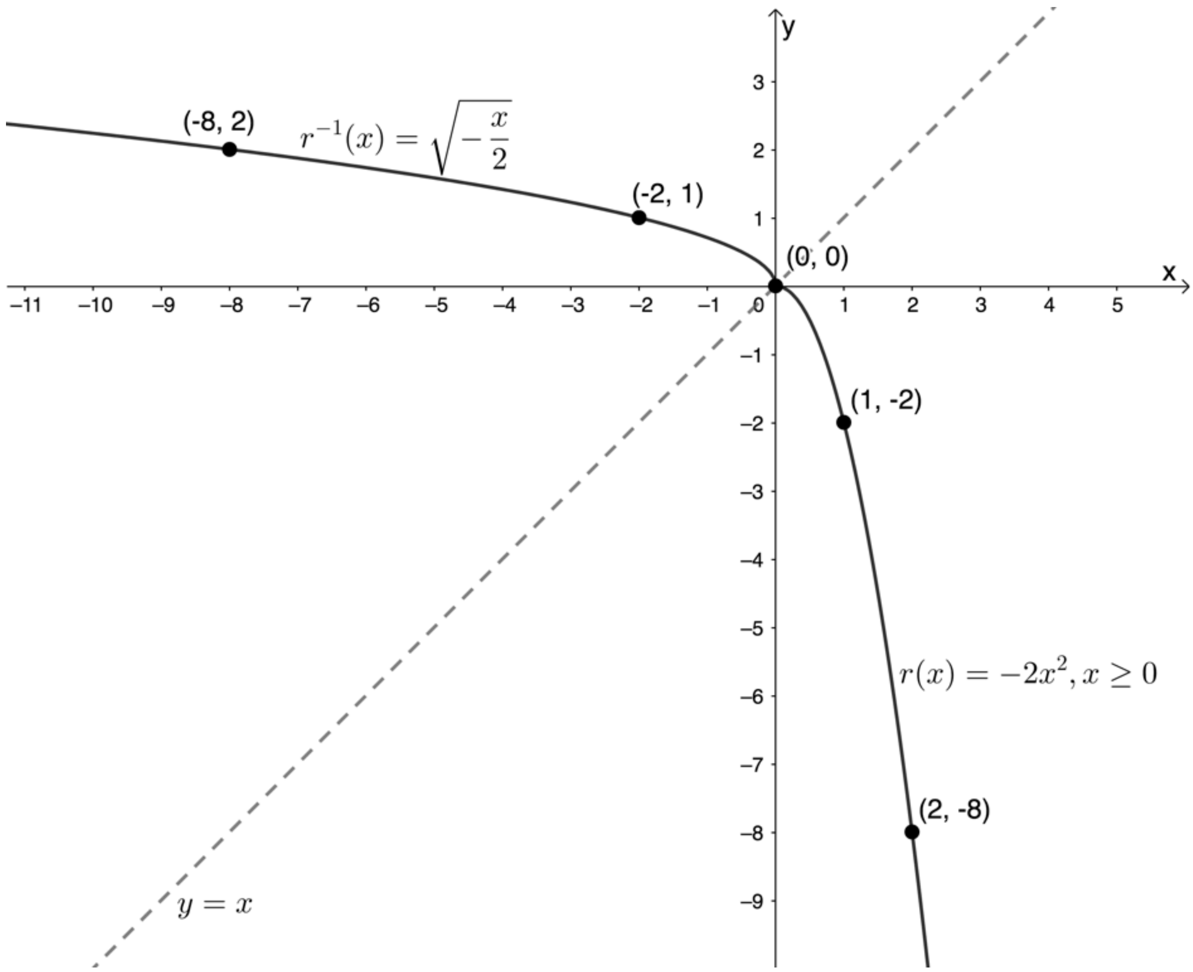
   Option 2: Restrict the domain of r to $\{x|x \in \mathbb{R}, x \geq 0\}$.
   $y = -2x^2, x \geq 0$
   Inverse:
   $$x = -2y^2$$
   $$\therefore 2y^2 = -x$$
   $$\therefore y^2 = -\frac{x}{2}$$
   $$\therefore y = +\sqrt{-\frac{x}{2}}$$
   $$\therefore r^{-1} = +\sqrt{-\frac{x}{2}}$$
   Remember that the domain of r becomes the range of r^{-1}. The domain of r was restricted to $\{x|x \in \mathbb{R}, x \geq 0\}$, therefore, the range of r^{-1} will be $\{y|y \in \mathbb{R}, y \geq 0\}$. Because the function values are restricted to $y \geq 0$, we chose the **positive** square root.

   Also note that $\sqrt{-\frac{x}{2}}$ is only defined for $x \leq 0$. However, the range of the original function r is $\{y|y \in \mathbb{R}, y \leq 0\}$. The range of r becomes the domain of r^{-1}. Hence the domain of r^{-1} is $\{x|x \in \mathbb{R}, x \leq 0\}$ and so $\sqrt{-\frac{x}{2}}$ is defined.
4. Option 1:
   Domain of r^{-1}: $\{x|x \in \mathbb{R}, x \leq 0\}$ – this was the range of r
   Range of r^{-1}: $\{y|y \in \mathbb{R}, y \leq 0\}$ – this was the restricted domain of r
   Option 2:
   Domain of r^{-1}: $\{x|x \in \mathbb{R}, x \leq 0\}$ – this was the range of r
   Range of r^{-1}: $\{y|y \in \mathbb{R}, y \geq 0\}$ – this was the restricted domain of r

5. Here is a sketch for option 1 where the domain r is restricted to $\{x | x \in \mathbb{R}, x \leq 0\}$. It is usually easiest to plot these graphs using a table of suitable values and point-by-point plotting. Sketching the axis of symmetry (the line $y = x$) also helps you to get the shapes of the graphs correct.



Here is a sketch for option 2 where the domain r is restricted to $\{x | x \in \mathbb{R}, x \geq 0\}$.

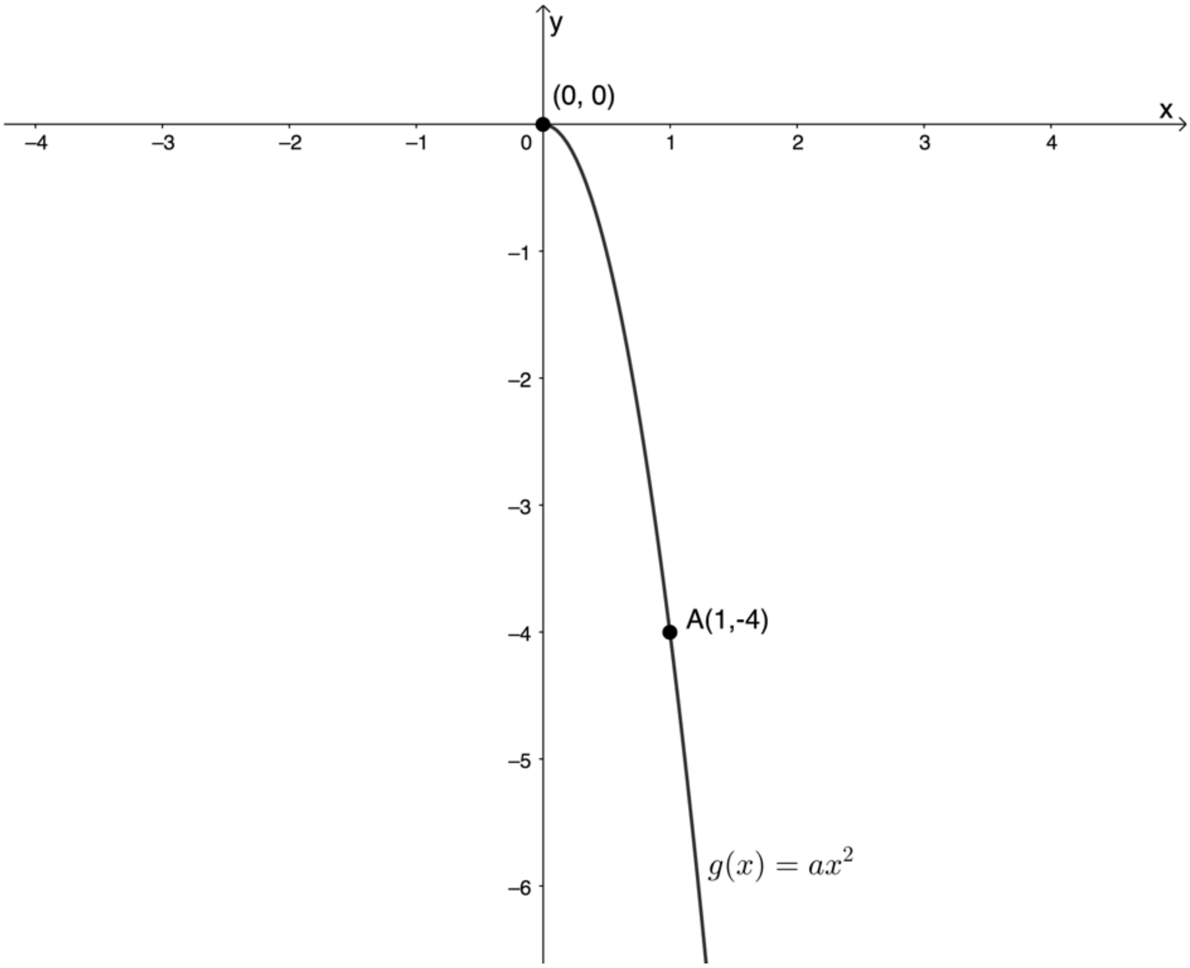


Exercise 2.1

1. Determine the inverse for each of the following, leaving your answer in '$y =$' form:
   a. $y = \frac{5}{3}x^2$
   b. $x^2 + 3y = 0$
   c. $5y + 10x^2 = 0$
   d. $2y - 25 = (x + 4)(x - 4)$
2. Given the function $f(x) = \frac{1}{3}x^2$ for $x \leq 0$:
   a. Find the inverse of f.
   b. State the domain and range of f and the inverse of f.
   c. Draw f and the inverse of f on the same set of axes, showing at least three points on each graph.
   d. Is the inverse of f a function? Explain your answer.

e. Determine the point of intersection of f and the inverse of f.

3. Given the graph of the parabola $g(x) = ax^2$ with $x \geq 0$ and passing through $A(1, -4)$:



a. Determine the equation of the parabola $g(x)$.

b. Determine the equation of the inverse of g.

c. Is the inverse of g a function?

d. Give the coordinates of the point B on the inverse of g and symmetrical to A.

e. Determine the point of intersection of g and the inverse of g.

The full solutions are at the end of the unit.

Summary

In this unit you have learnt the following:

- The inverse of the quadratic function of the form $y = ax^2$ is not a function and therefore $y = ax^2$ is not invertible.
- We can make $y = ax^2$ invertible by restricting the domain to either $\{x|x \leq 0, x \in \mathbb{R}\}$ or $\{x|x \geq 0, x \in \mathbb{R}\}$.
- The quadratic function $y = ax^2$ and its inverse are symmetrical about the line $y = x$.
- The domain of the quadratic function $y = ax^2$ becomes the range of its inverse.
- The range of the quadratic function $y = ax^2$ becomes the domain of its inverse.

Unit 2: Assessment

Suggested time to complete: 20 minutes

Question 1 adapted from NC(V) Mathematics Level 4 Paper 1 November 2015 question 4.4

1. Given $f(x) = x^2$:
   a. Draw a sketch graph of f, showing the coordinates of at least three points on the graph.
   b. Determine $f^{-1}(x)$ if the domain of f is restricted to $\{x|x \geq 0, x \in \mathbb{R}\}$.
   c. Sketch the graph of f^{-1} on the same set of axes as f.
   d. Determine algebraically the coordinates of the point(s) of intersection of f and f^{-1}.
   e. Write down the domain of f^{-1}.

Question 2 adapted from NC(V) Mathematics Level 4 Paper 1 October 2014 question 4.4

2. Given $f(x) = 2x^2$ for $x \geq 0$:
   a. Determine the equation which defines the inverse of f in the form '$y =$'.
   b. Draw sketch graphs of f and the inverse of f on the same set of axes. Clearly label the graphs and indicate at least three points on each graph.
   c. Determine algebraically the intersection of the two graphs.
   d. Is the inverse of f a function or a non-function? Give a reason for your answer.
   e. Write down the range of the inverse of f.

The full solutions are at the end of the unit.

Unit 2: Solutions

Exercise 2.1

1.
   a. $y = \frac{5}{3}x^2$

      Inverse:

      $$x = \frac{5}{3}y^2$$
      $$\therefore y^2 = \frac{3}{5}x$$
      $$\therefore y = \pm\sqrt{\frac{3}{5}x}$$

   b.
      $$x^2 + 3y = 0$$
      $$\therefore 3y = -x^2$$
      $$\therefore y = -\frac{1}{3}x^2$$

      Inverse:

$$x = -\frac{1}{3}y^2$$
$$\therefore y^2 = -3x$$
$$\therefore y = \pm\sqrt{-3x}$$

c.

$$5y + 10x^2 = 0$$
$$\therefore 5y = -10x^2$$
$$\therefore y = -2x^2$$

Inverse:

$$x = -2y^2$$
$$\therefore y^2 = -\frac{1}{2}x$$
$$\therefore y = \pm\sqrt{-\frac{1}{2}x}$$

d.

$$2y - 25 = (x+4)(x-4))$$
$$\therefore 2y - 25 = x^2 - 16$$
$$\therefore 2y = x^2 + 9$$
$$\therefore y = \frac{1}{2}x^2 + \frac{9}{2}$$

Inverse:

$$x = \frac{1}{2}y^2 + \frac{9}{2}$$
$$\therefore y^2 = 2x - 9$$
$$\therefore y = \pm\sqrt{2x-9}$$

2. $f(x) = \frac{1}{3}x^2$ for $x \leq 0$.

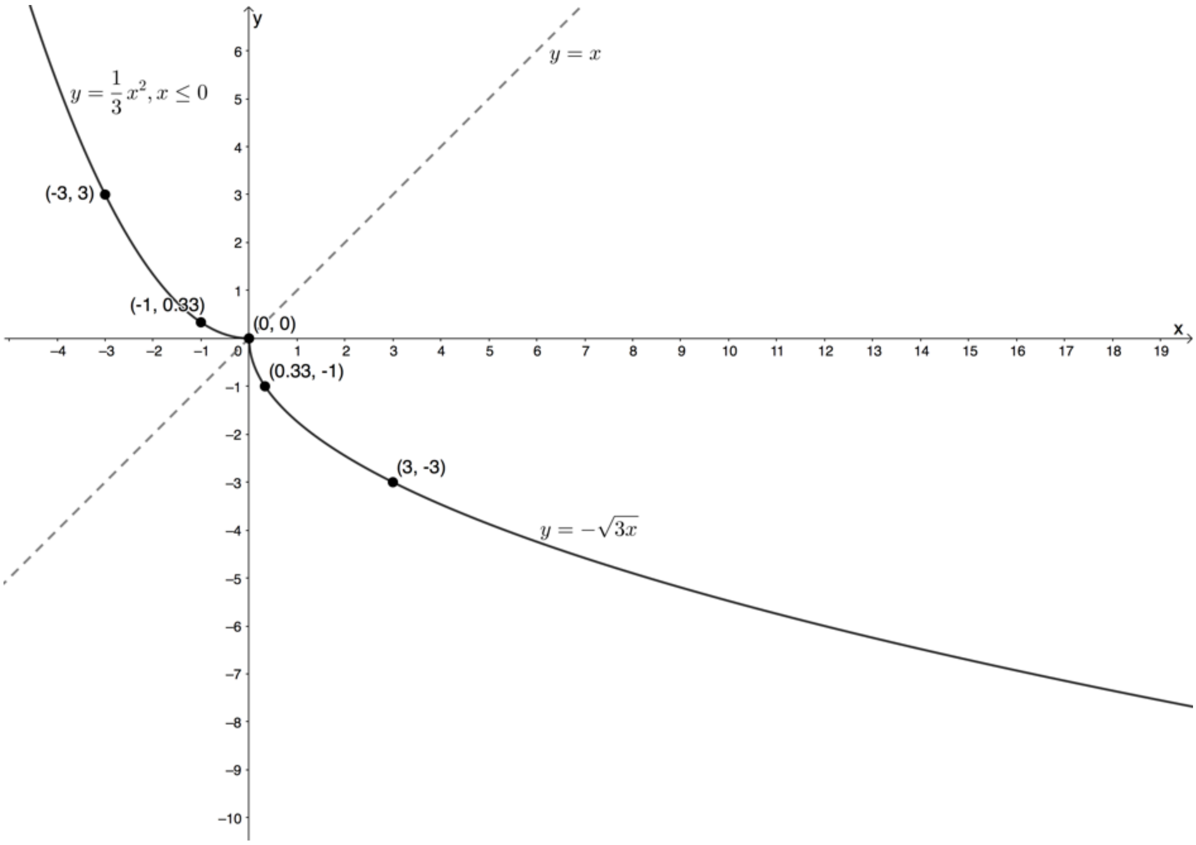
   a. $y = \frac{1}{3}x^2$

   Inverse:

$$x = \frac{1}{3}y^2$$
$$\therefore y^2 = 3x$$
$$\therefore y = -\sqrt{3x} \quad \text{Original domain was } x \leq 0 \text{ so inverse range will be } y \leq 0$$

   b. Domain of f: $\{x|x \leq 0, x \in \mathbb{R}\}$
   Range of f: $\{y|y \geq 0, y \in \mathbb{R}\}$
   Domain of inverse of f: $\{x|x \geq 0, x \in \mathbb{R}\}$
   Range of inverse of f: $\{y|y \leq 0, y \in \mathbb{R}\}$

   c.



d. The inverse of f is a function as there is only ever one output value for each input value. Any vertical line will intercept the graph only once.

e.

$$f(x) = f^{-1}(x)$$
$$\therefore \frac{1}{3}x^2 = -\sqrt{3x}$$
$$\therefore \frac{1}{9}x^4 = 3x$$
$$\therefore x^4 - 27x = 0$$
$$\therefore x(x^3 - 27) = 0$$
$$\therefore x = 0 \text{ or } x^3 = 27$$
$$\therefore x = 0 \text{ or } x = 3$$

But the domain of $f(x)$ is restricted to $x \leq 0$. Therefore, $x = 0$ is the only solution.

$f(0) = \frac{1}{3}(0)^2 = 0$. Therefore, the point of intersection is $(0, 0)$.

3.

a. $g(x) = ax^2$ and passes through $(1, -4)$. Therefore:

$$g(1) = -4$$
$$\therefore a(1)^2 = -4$$
$$\therefore a = -4$$
$$g(x) = -4x^2$$

b. $y = -4x^2$

Inverse:

$$x = -4y^2$$
$$\therefore y^2 = -\frac{1}{4}x$$
$$\therefore y = +\sqrt{-\frac{1}{4}x} \quad \text{Original domain was } x \geq 0 \text{ so inverse range will be } y \geq 0$$

c. The inverse of g is a function.

d. B symmetrical to A about $y = x$. Therefore, $B(-4, 1)$.

e.

$$\begin{aligned} g(x) &= g^{-1}(x) \\ \therefore -4x^2 &= \sqrt{-\frac{1}{4}x} \\ \therefore 16x^4 &= -\frac{1}{4}x \\ \therefore 64x^4 + x &= 0 \\ \therefore x(64x^3 + 1) &= 0 \\ \therefore x = 0 \text{ or } x^3 &= -\frac{1}{64} \\ \therefore x = 0 \text{ or } x &= -\frac{1}{4} \end{aligned}$$

But the domain of $g(x)$ is restricted to $x \geq 0$. Therefore, $x = 0$ is the only solution.

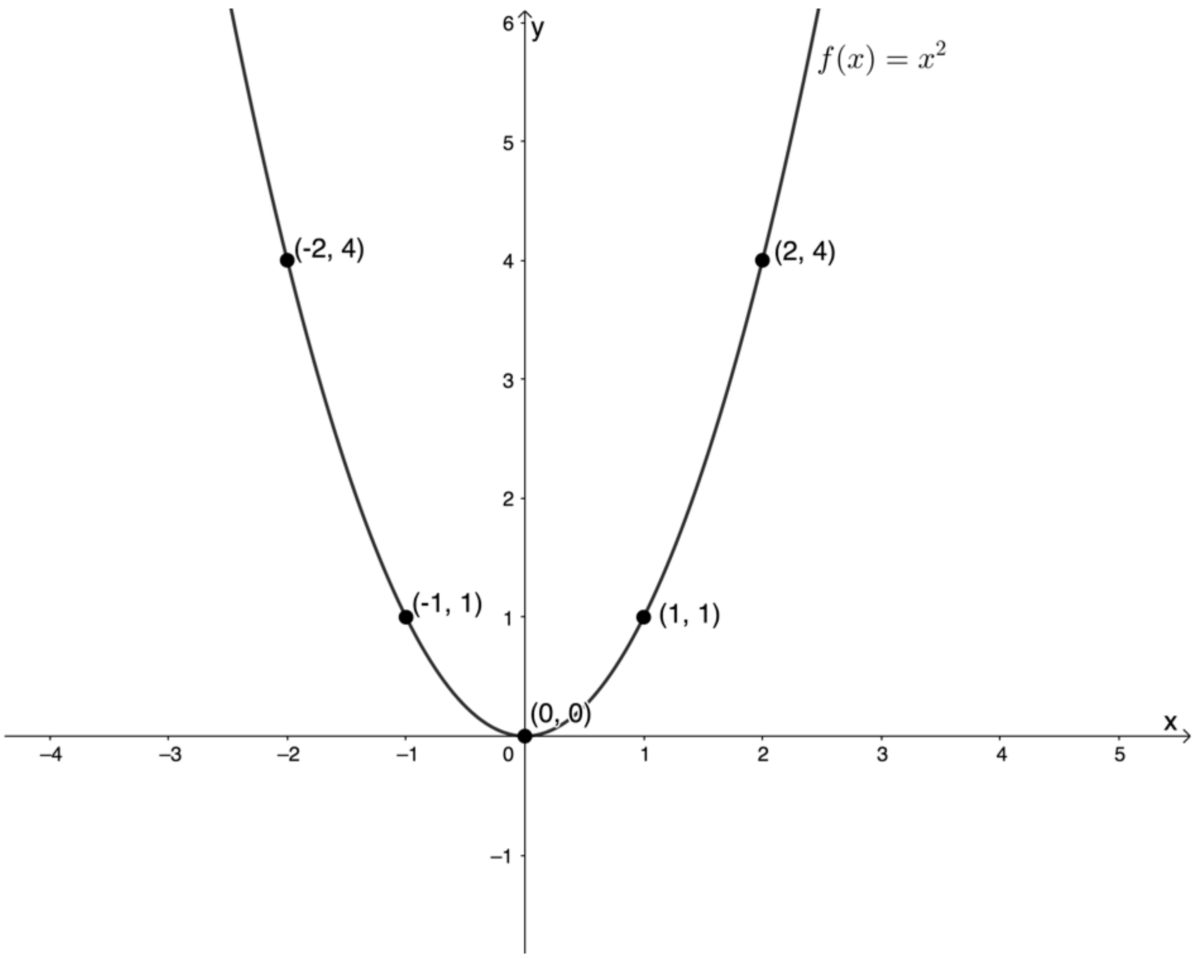
$g(0) = -4(0)^2 = 0$. Therefore, the point of intersection is $(0, 0)$.

Back to Exercise 2.1

Unit 2: Assessment

1. $f(x) = x^2$

a.

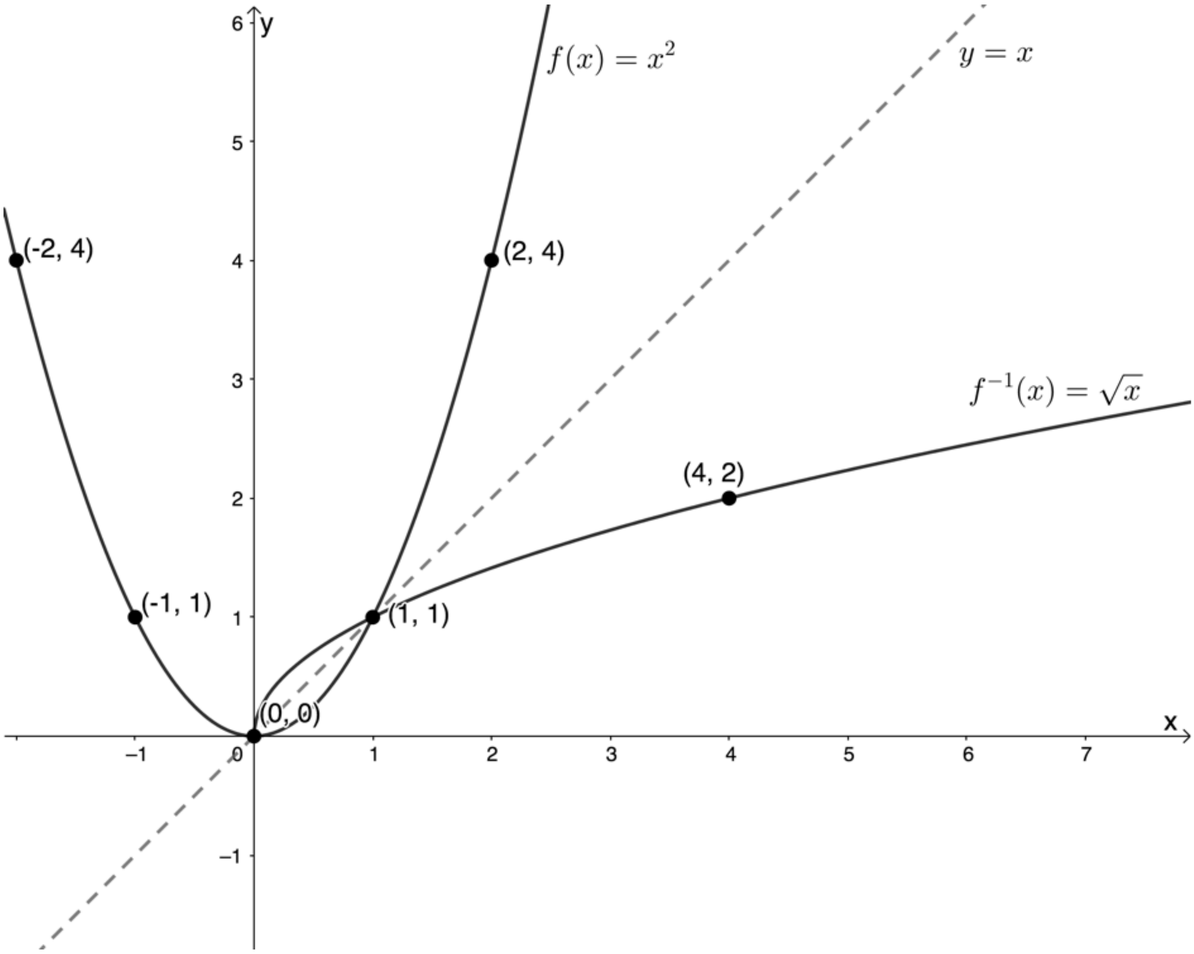


Note that the domain was not restricted, therefore, the full parabola must be sketched.

b. Domain of f is restricted to $\{x | x \geq 0, x \in \mathbb{R}\}$

$y = x^2$
Inverse:
$x = y^2$
$\therefore y = +\sqrt{x}$ Original domain was $x \geq 0$ so inverse range will be $y \geq 0$

c.



d.

$$f(x) = f^{-1}(x)$$
$$\therefore x^2 = \sqrt{x}$$
$$\therefore x^4 = x$$
$$\therefore x^4 - x = 0$$
$$\therefore x(x^3 - 1) = 0$$
$$\therefore x = 0 \text{ or } x^3 = 1$$
$$\therefore x = 0 \text{ or } x = 1$$

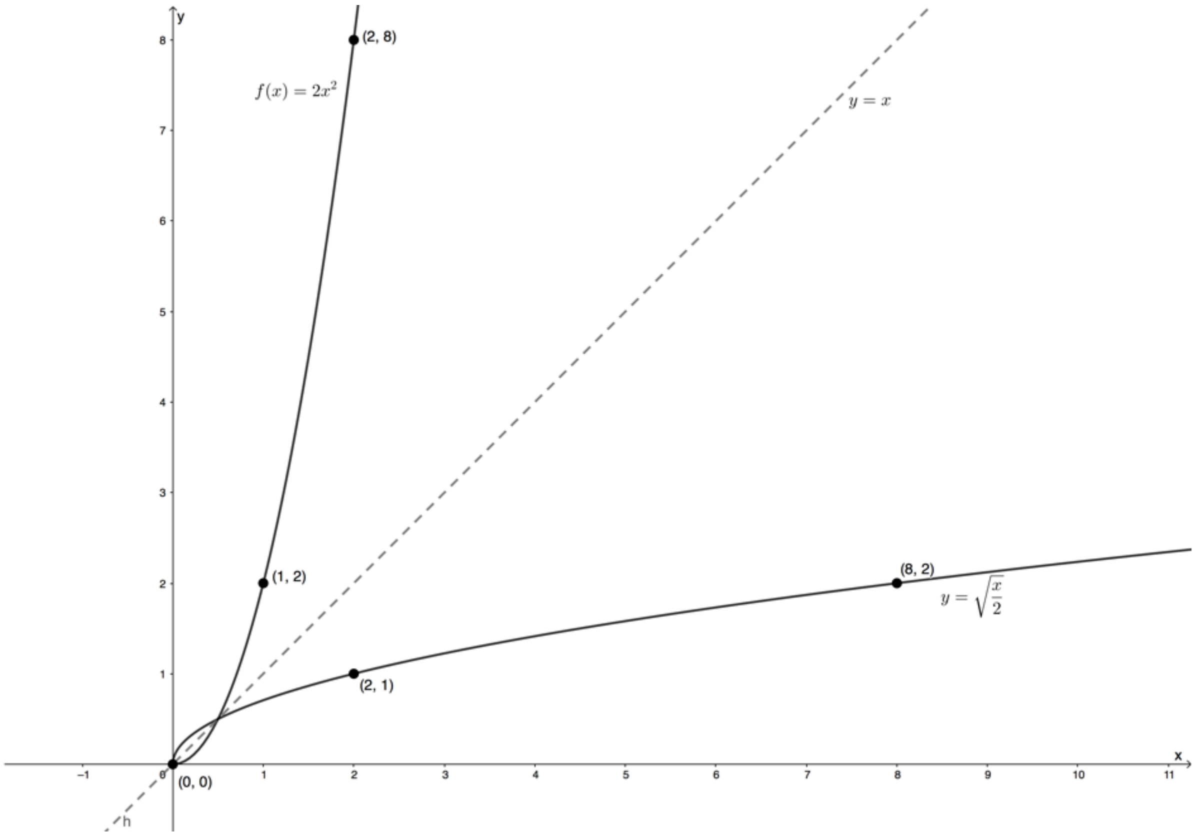
The restricted domain of f allows both solutions.
Therefore, the points of intersection are $(0, 0)$ and $(1, 1)$.

e. Domain of f^{-1}: $\{x | x \geq 0, x \in \mathbb{R}\}$

2. Given $f(x) = 2x^2$ for $x \geq 0$

a. $f(x) = 2x^2$ or $y = 2x^2$
Inverse:
$$x = 2y^2$$
$$\therefore y^2 = \frac{x}{2}$$
$$\therefore y = +\sqrt{\frac{x}{2}} \quad \text{Domain of } f(x) \text{ is } x \geq 0, \text{ therefore, range is } y \geq 0$$

b.



c.

$$\begin{aligned}
f(x) &= f^{-1}(x) \\
\therefore 2x^2 &= \sqrt{\frac{x}{2}} \\
\therefore 4x^4 &= \frac{x}{2} \\
\therefore 8x^4 &= x \\
\therefore 8x^4 - x &= 0 \\
\therefore x(8x^3 - 1) &= 0 \\
\therefore x = 0 \text{ or } x^3 &= \frac{1}{8} \\
\therefore x = 0 \text{ or } x &= \frac{1}{2}
\end{aligned}$$

The restricted domain of f allows both solutions.

Therefore, the points of intersection are $(0, 0)$ and $(1, \frac{1}{2})$.

d. The inverse of f is a function. Each input value is associated with one and only one output value.

e. Range of f^{-1}: $\{y|y \geq 0, y \in \mathbb{R}\}$

Back to Unit 2: Assessment

Media Attributions

Unit 3: Determine and sketch the inverse of the exponential function

DYLAN BUSA

Unit outcomes

By the end of this unit you will be able to:

- Find the inverse of $y = a^x$ where $a > 0$.
- Sketch the inverse of $y = a^x$ where $a > 0$.
- Answer questions about the domain, range, shape, continuity and other characteristics of the inverse graph.

What you should know

Before you start this unit, make sure you can:

- Determine the inverse of a function. Refer to unit 1 if you need help with this.
- Determine if a relation is a function or not using the vertical line test. Refer to unit 1 if you need help with this.
- Determine if a function is invertible or not. Refer to unit 1 if you need help with this.
- Sketch an exponential function. Refer to level 2 Subject outcome 2.1 unit 4 if you need help with this.

Introduction

We saw in unit 1 that linear functions are one-to-one functions and, therefore, are invertible – their inverses are also functions. We saw in unit 2 that if we restrict the domain of quadratic functions of the form $y = ax^2$ to be either $x \leq 0$ or $x \geq 0$, we can make the quadratic function one-to-one and hence make the inverse a function as well.

However, is the same true for exponential functions of the form $y = a^x, a > 0$? Are these exponential functions invertible? If not, is there a way for us to make the inverse of an exponential function of the form $y = a^x, a > 0$ a function?

These are the questions we will answer in this unit.

The inverse of the exponential function

Let's explore the inverse of exponential functions of the form $y = a^x, a > 0$.

Activity 3.1: The inverse of an exponential function

Time required: 30 minutes

What you need:

- a blank piece of paper or graph paper
- a pen or pencil
- a ruler

What to do:

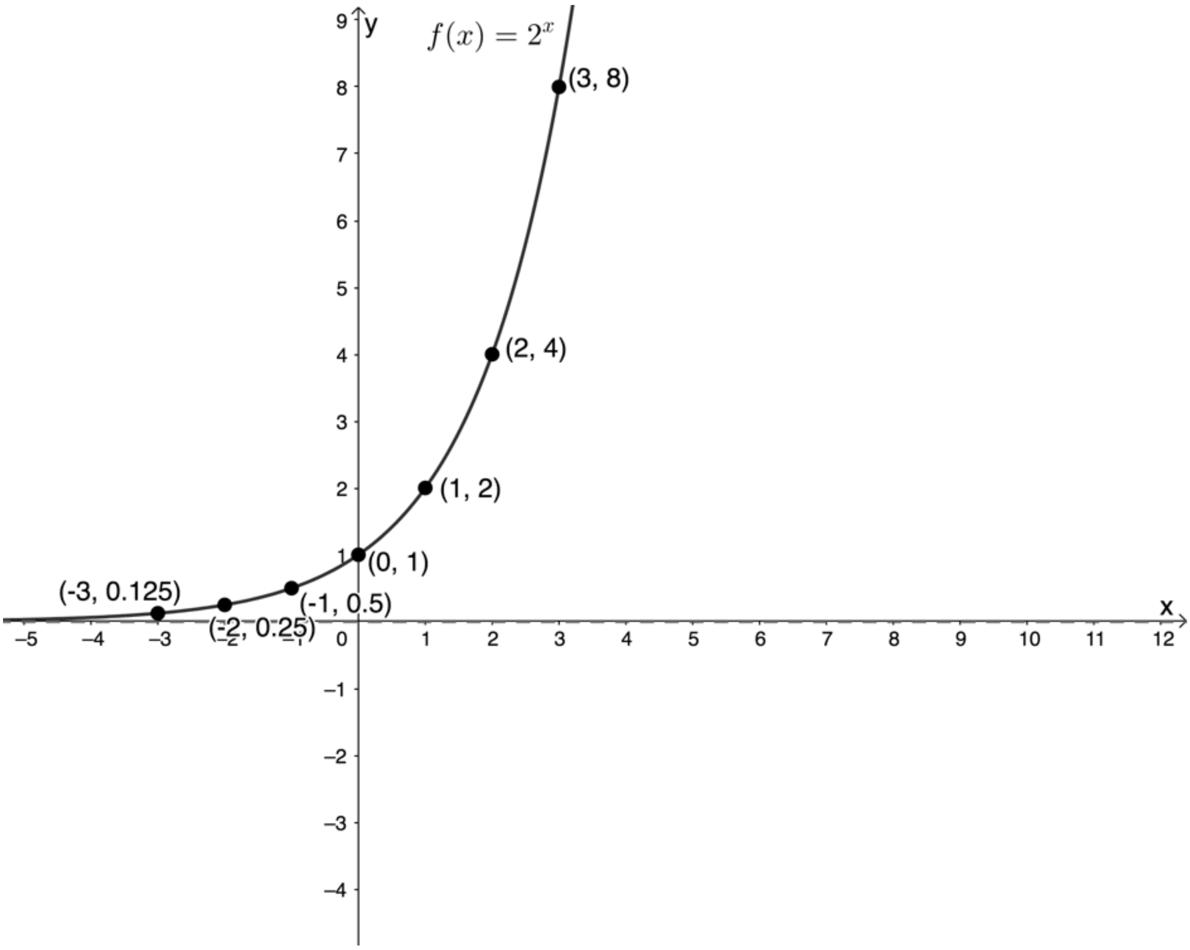
Start by drawing a Cartesian plane on your piece of paper. Use a scale of about 1 cm per unit.

1. Plot the function $f(x) = 2^x$. If you need to, you can create a table of values and use a point-by-point plot with at least five points.
2. Now, determine the inverse of f in the form '$x =$'.
3. Plot the inverse of f on your Cartesian plane. It is suggested you create a table of values and plot point-by-point with at least five points.
4. Does the inverse of f have an asymptote? If so, what is it and how does it relate to the asymptote of f?
5. Is the inverse of f a function? Use the vertical line test to make sure.
6. What are the domain and range of f and of the inverse of f?
7. About what line are the graphs of f and the inverse of f symmetrical?
8. Is the function f increasing or decreasing?
9. Is the inverse of f increasing or decreasing?

What did you find?

1. Here is a table of suitable values.

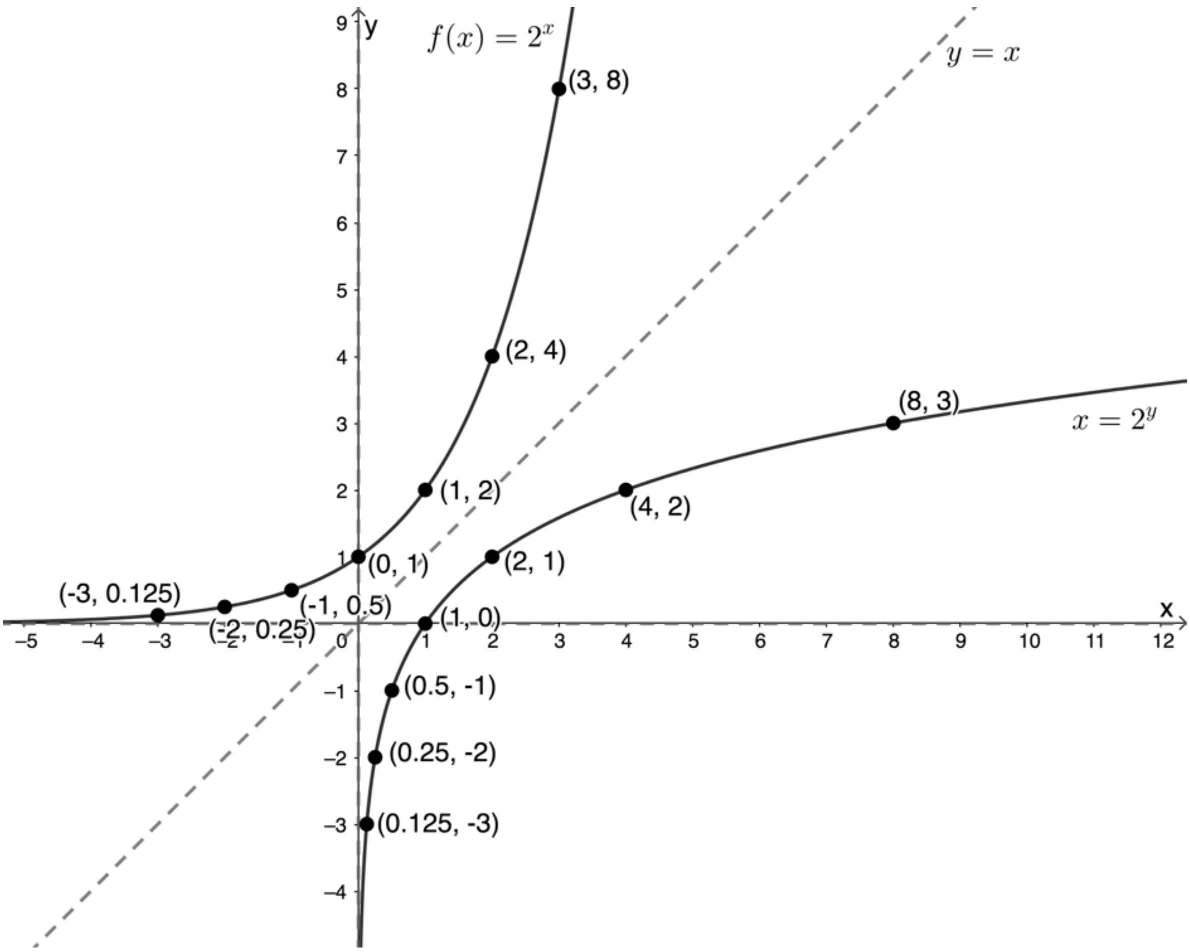
x	-3	-2	-1	0	1	2	3
$f(x) = 2^x$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8



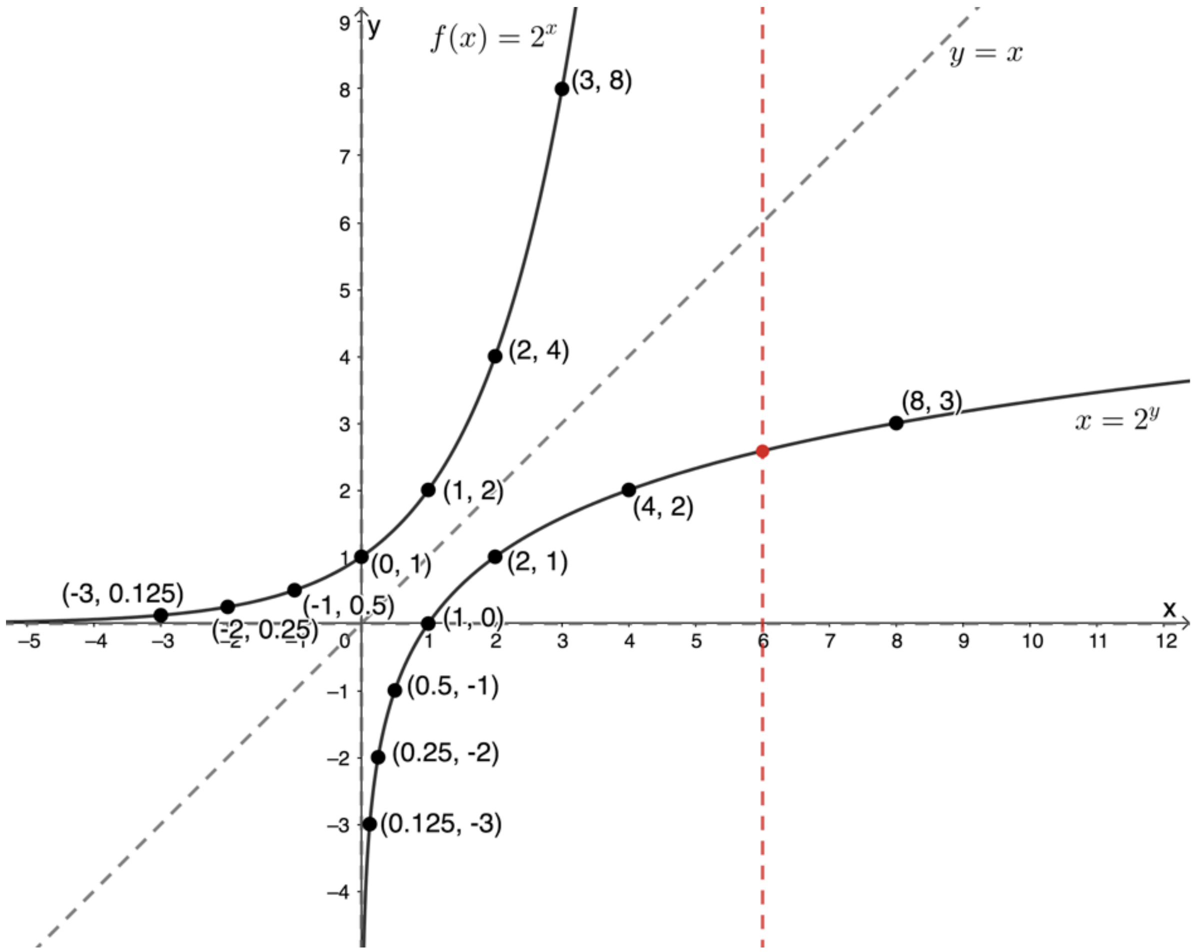
Remember that the exponential function has a horizontal asymptote, in this case the line $y = 0$.

2. $y = 2^x$
   Inverse:
   $x = 2^y$. We can leave the inverse with x as the subject of the formula. We have no way of rearranging this equation to make y the subject of the equation.
3. Here is a table of suitable values. In this case, because x is still the subject of the equation, it is easier to choose and substitute in different values for y.

x	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
y if $x = 2^y$	-3	-2	-1	0	1	2	3



4. The inverse of f does have an asymptote of $x = 0$. The asymptote of the original function f is $y = 0$. Therefore, the asymptotes are symmetrical about the line $y = x$.
5. The inverse of f is a function. It cuts the vertical line only once.



6. Domain of f: $\{x|x \in \mathbb{R}\}$
   Range of f: $\{y|y \in \mathbb{R}, y > 0\}$ – remember that the line $y = 0$ is a horizontal asymptote. The function approaches this value but never reaches it. Therefore, the range cannot include zero.
   Domain of inverse of f: $\{x|x \in \mathbb{R}, x > 0\}$ – the range of the function is the domain of the inverse.
   Range of inverse of f: $\{y|y \in \mathbb{R}\}$ – the domain of the function is the range of the inverse.
7. The graphs of f and the inverse of f are symmetrical about the line $y = x$. This is the same line of symmetry for all inverses we have studied.
8. The function f is increasing. This means that as x increases the function increases.
9. The inverse of f is also increasing. Once again, as x increases, the function value increases.

In Activity 3.1 we saw that the inverse of the exponential function $y = a^x$ is also a function. Therefore, the exponential function is invertible.

Like the inverses of the linear and quadratic functions, the inverse of the exponential function is symmetrical to the original function about the line $y = x$. This symmetry extends to the intercepts with the axes and the asymptotes.

The y-intercept of the function becomes the x-intercept of the inverse. Where the exponential function of the form $y = a^x, a > 0$ has a horizontal asymptote of $y = 0$, its inverse has a vertical asymptote of $x = 0$.

Also, the domain of the original function becomes the range of the inverse function and the range of the original function becomes the domain of the inverse function. This is also the same as the inverses for the linear and quadratic functions.

However, we were not able to write the equation for the inverse of the exponential function in the normal '$y =$' form. We had to leave it with x as the subject.

Did you know?

There is a way to rewrite the inverse of an exponential function, $x = a^y$ as '$y =$'. To do so, mathematicians defined the **logarithm** (or log). It is defined as follows:
If $x = a^y$, then $y = \log_a x$.

This means that $x = a^y$ and $y = \log_a x$ are both expressions for the inverse of the function $y = a^x$. Logarithms are not part of the NC(V) curriculum.

In Activity 3.1, we dealt with $f(x) = 2^x$ and we saw that the function and its inverse were increasing. In other words, as x **increases**, the function value **increases**. This is always the case for $a > 1$ (see Figure 1).

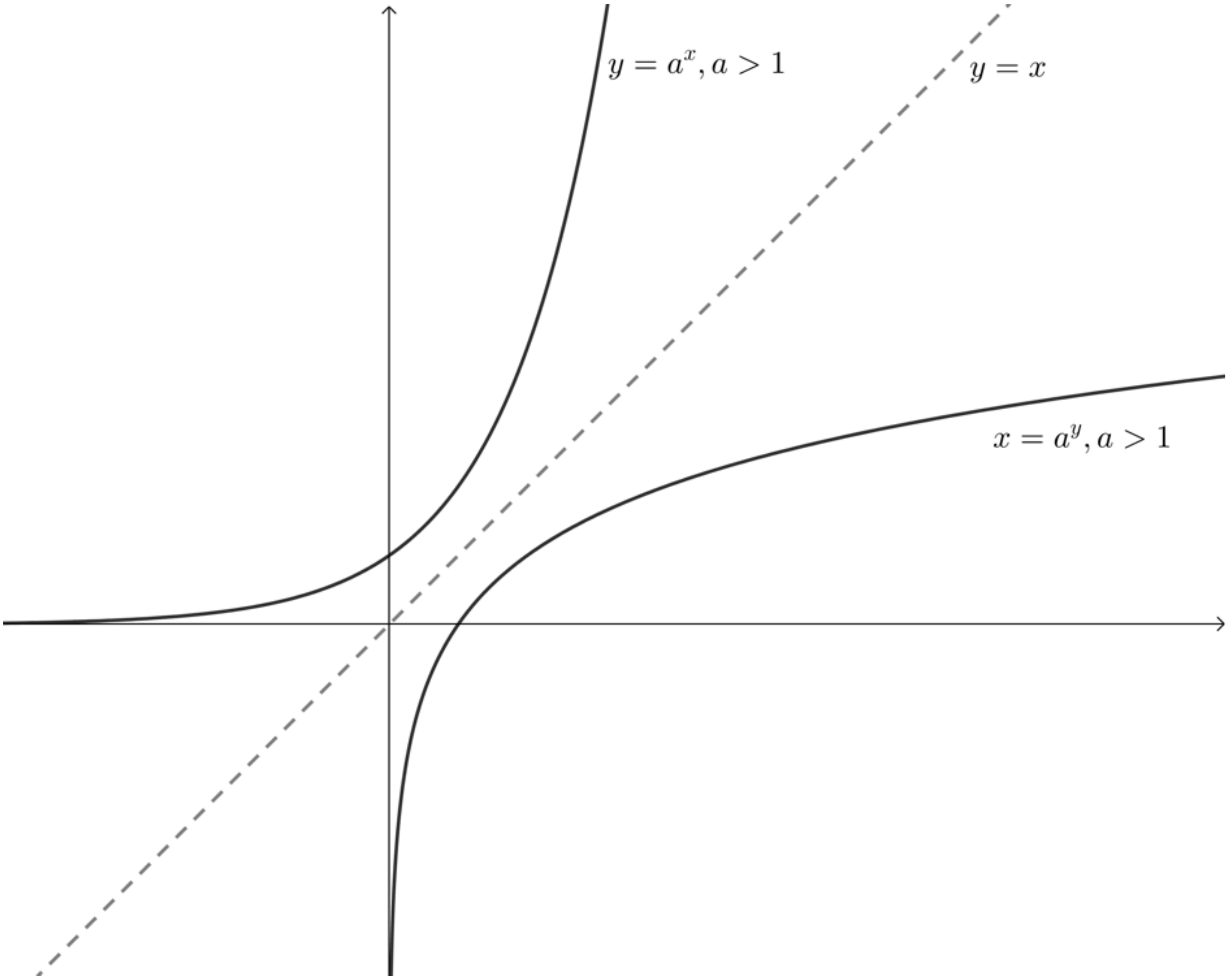


Figure 1: The increasing exponential function and its inverse when $a > 1$

However, we know that if $0 < a < 1$ then the exponential function is decreasing. As x increases, the function value decreases. The inverse of the exponential function where $0 < a < 1$ is also decreasing (see Figure 2).

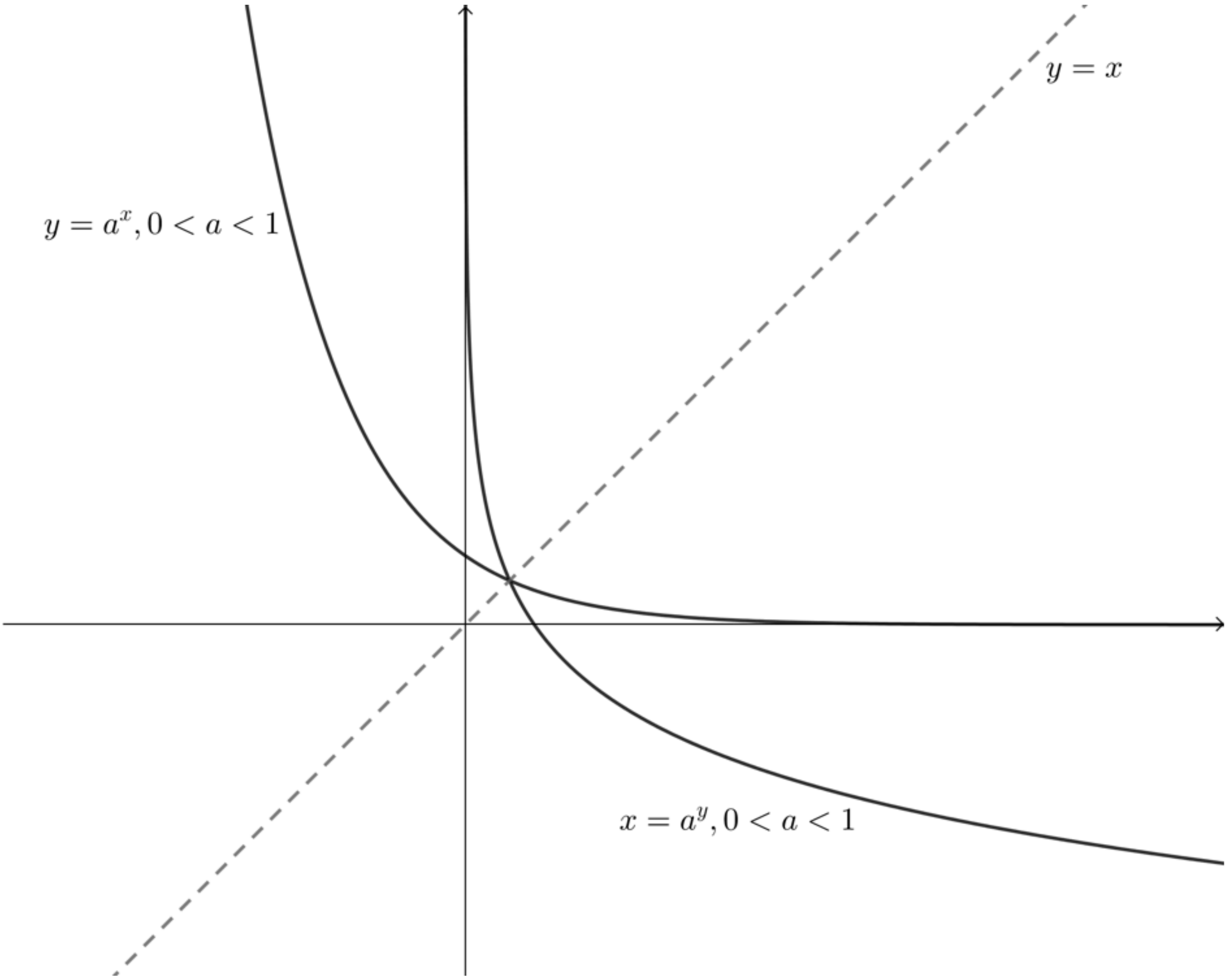


Figure 2: The decreasing exponential function and its inverse when $0 < a < 1$

Example 3.1

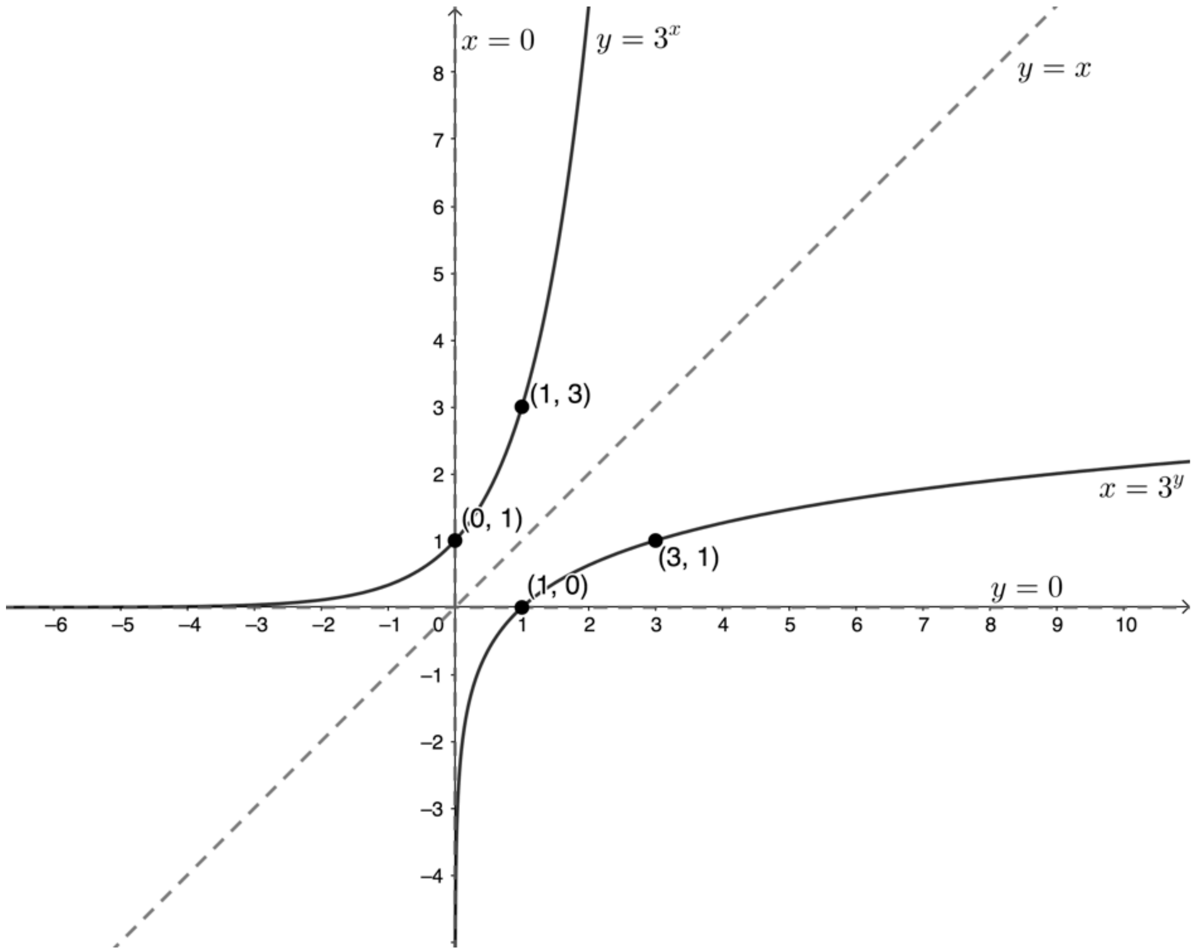
Given $f(x) = 3^x$:

1. Is f an increasing or decreasing function?
2. State the domain and range of f.
3. Determine f^{-1}, leaving the equation as '$x =$'.
4. Is f^{-1} an increasing or decreasing function?
5. State the domain and range of f^{-1}.
6. Draw sketches of f and f^{-1} on the same system of axes and mark the intercepts with the axes, at least one other point on the graph, and the asymptotes.

Solutions

1. $a > 1$. Therefore, f is an increasing function.

2. Domain of f: $\{x|x \in \mathbb{R}\}$
   Range of f: $\{y|y \in \mathbb{R}, y > 0\}$
3. $y = 3^x$
   Inverse:
   $x = 3^y$
4. $a > 1$. Therefore, $f^{-1}$is an increasing function.
5. Domain of f^{-1}: $\{x|x \in \mathbb{R}, x > 0\}$
   Range of f^{-1}: $\{y|y \in \mathbb{R}\}$
6.



Exercise 3.1

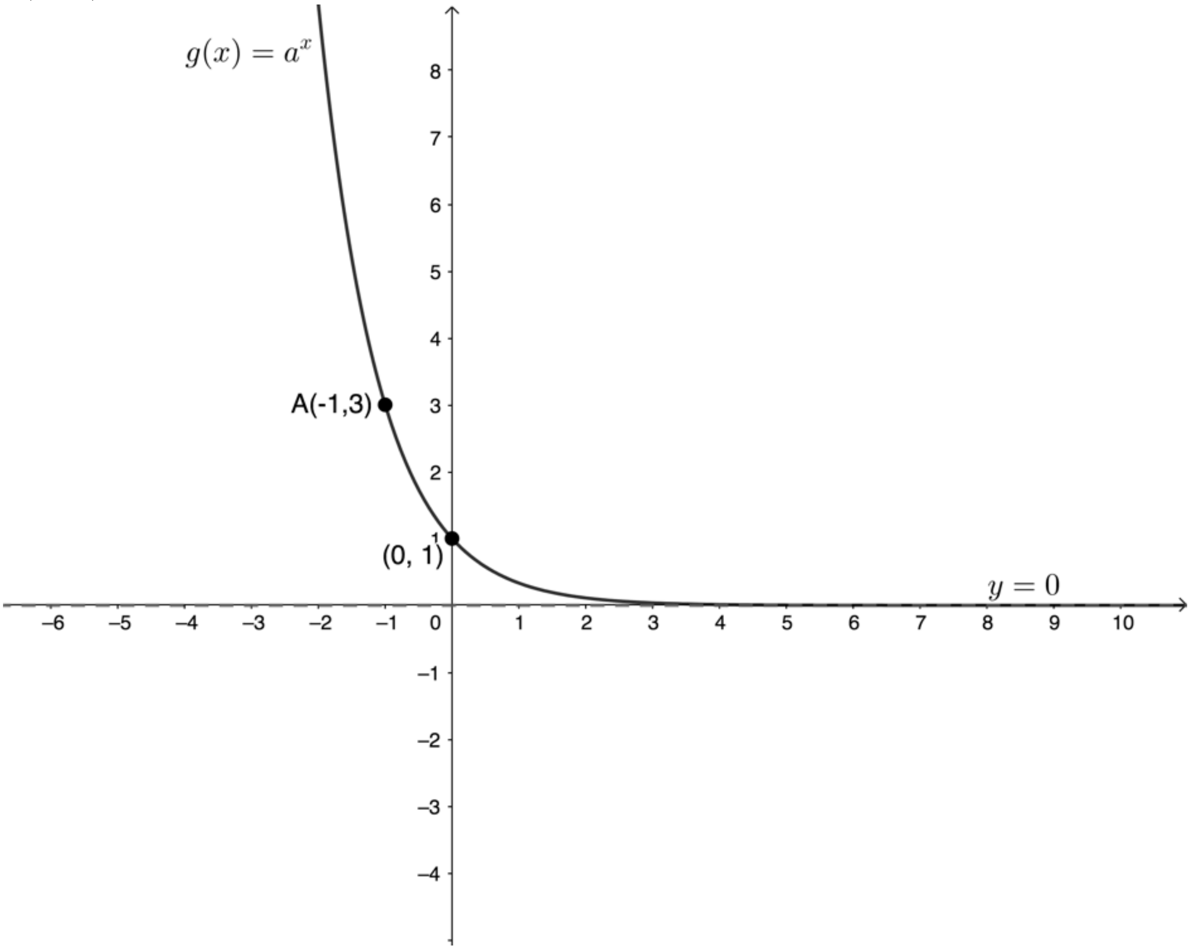
1. Given $f(x) = \left(\frac{1}{2}\right)^x$:
   a. Is f an increasing or decreasing function?
   b. State the domain and range of f.
   c. Determine f^{-1}, leaving the equation as '$x =$'.

d. Is f^{-1} an increasing or decreasing function?

e. State the domain and range of f^{-1}.

f. Draw sketches of f and f^{-1} on the same system of axes and mark the intercepts with the axes, at least one other point on the graph, and the asymptotes.

2. The graph below shows an exponential function of the form $g(x) = a^x, a > 0$. It passes through $A(-1, 3)$.



a. Determine the equation of the function $g(x)$.

b. Determine the equation of the inverse of g.

c. Is the inverse of g a function?

d. Give the coordinates of the point B on the inverse of g and symmetrical to A.

e. Is the inverse of g increasing or decreasing?

The full solutions are at the end of the unit.

Summary

In this unit you have learnt the following:

- The exponential function of the form $y = a^x, a < 0$ is invertible – its inverse is also a function.
- When $a > 1$, the exponential function and its inverse are increasing – the function value increases as x increases.

- When $0 < a < 1$, the exponential function and its inverse are decreasing – the function value decreases as x increases.
- The function $y = a^x, a < 0$ and its inverse are symmetrical about the line $y = x$.
- The y-intercept of the function $y = a^x, a < 0$ becomes the x-intercept of the inverse.
- The function $y = a^x, a < 0$ has a horizontal asymptote of $y = 0$ and its inverse has a vertical asymptote of $x = 0$.

Unit 3: Assessment

Suggested time to complete: 15 minutes

Question adapted from NC(V) Mathematics Level 4 Paper 1 November 2016 question 2.3

Given: $f(x) = 4^x$

1. $P(a, 4)$ is a point on f. Determine the value of a.
2. Make a neat sketch graph of f, showing clearly the point P and other key features.
3. Write down the range of the function f.
4. Write down the equation(s) of any asymptote(s) of f.
5. Determine f^{-1} and write your answer in the form '$x =$'.
6. Using your graph in 2, write down the coordinates of two points that lie on f^{-1}.
7. On the same set of axes as the graph of f, sketch the graph of f^{-1}.
8. Is f^{-1} a function or non-function? Give a reason for your answer.
9. The graphs of f and $f^{-1}$are symmetrical about a line. Write down the equation of this line.

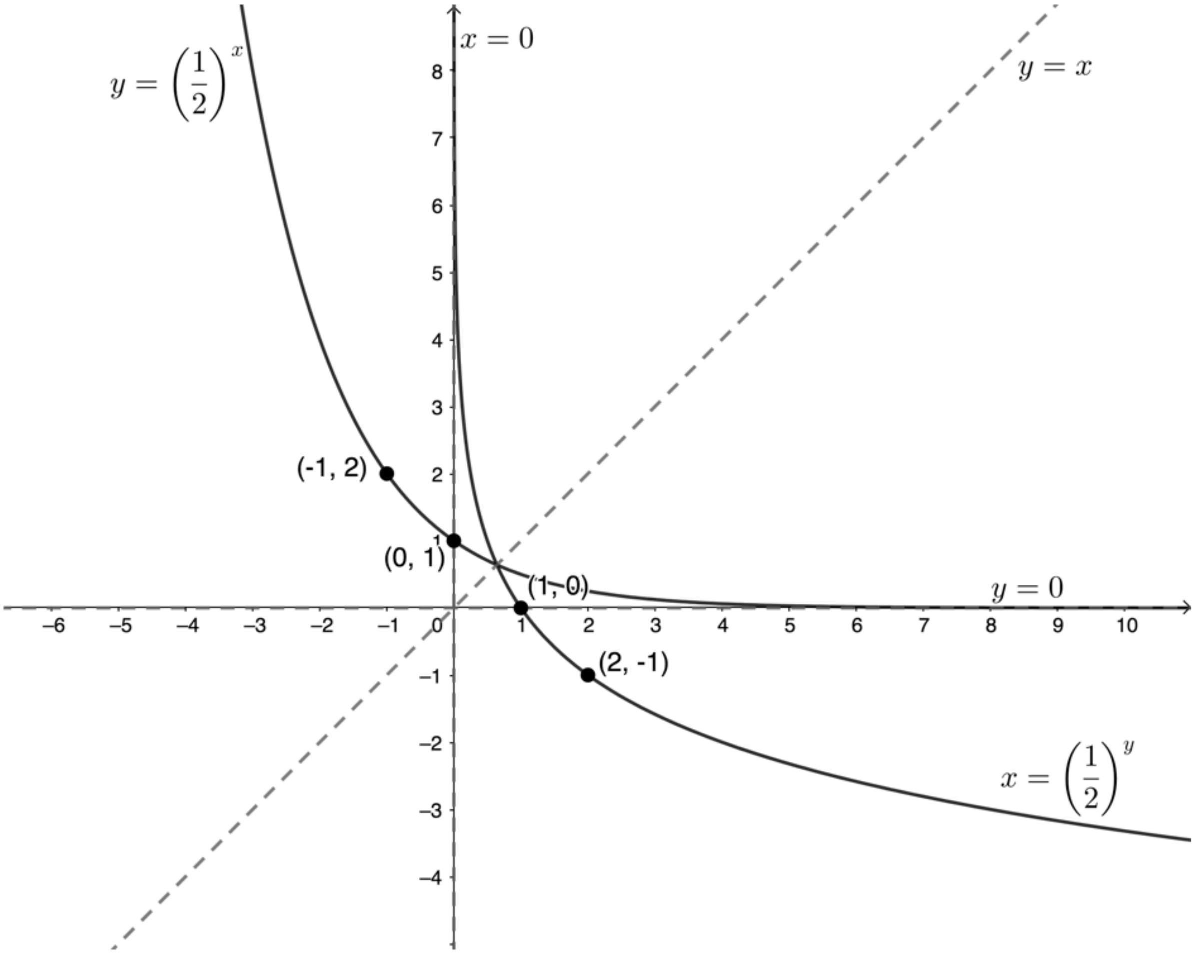
The full solutions are at the end of the unit.

Unit 3: Solutions

Exercise 3.1

1.
   a. $0 < a < 1$. Therefore, f is a decreasing function.
   b. Domain of f: $\{x|x \in \mathbb{R}\}$
      Range of f: $\{y|y \in \mathbb{R}, y > 0\}$
   c. $y = \left(\frac{1}{2}\right)^x$
      Inverse:
      $x = \left(\frac{1}{2}\right)^y$
   d. $0 < a < 1$. Therefore, f^{-1} is a decreasing function.
   e. Domain of f^{-1}: $\{x|x \in \mathbb{R}, x > 0\}$
      Range of f^{-1}: $\{y|y \in \mathbb{R}\}$

f.



2.

a. $g(x) = a^x$ and passes through $A(-1, 3)$. Therefore:
$g(-1) = 3$
$\therefore a^{-1} = 3$
$\therefore \frac{1}{a} = 3$
$\therefore a = \frac{1}{3}$
$g(x) = \left(\frac{1}{3}\right)^x$

b. $y = \left(\frac{1}{3}\right)^x$
Inverse:
$x = \left(\frac{1}{3}\right)^y$

c. The inverse of g is a function.

d. B is symmetrical to A about $y = x$. Therefore $B(3, -1)$.

e. The inverse of g is a decreasing function since $0 < a < 1$.

Back to Exercise 3.1

Unit 3: Assessment

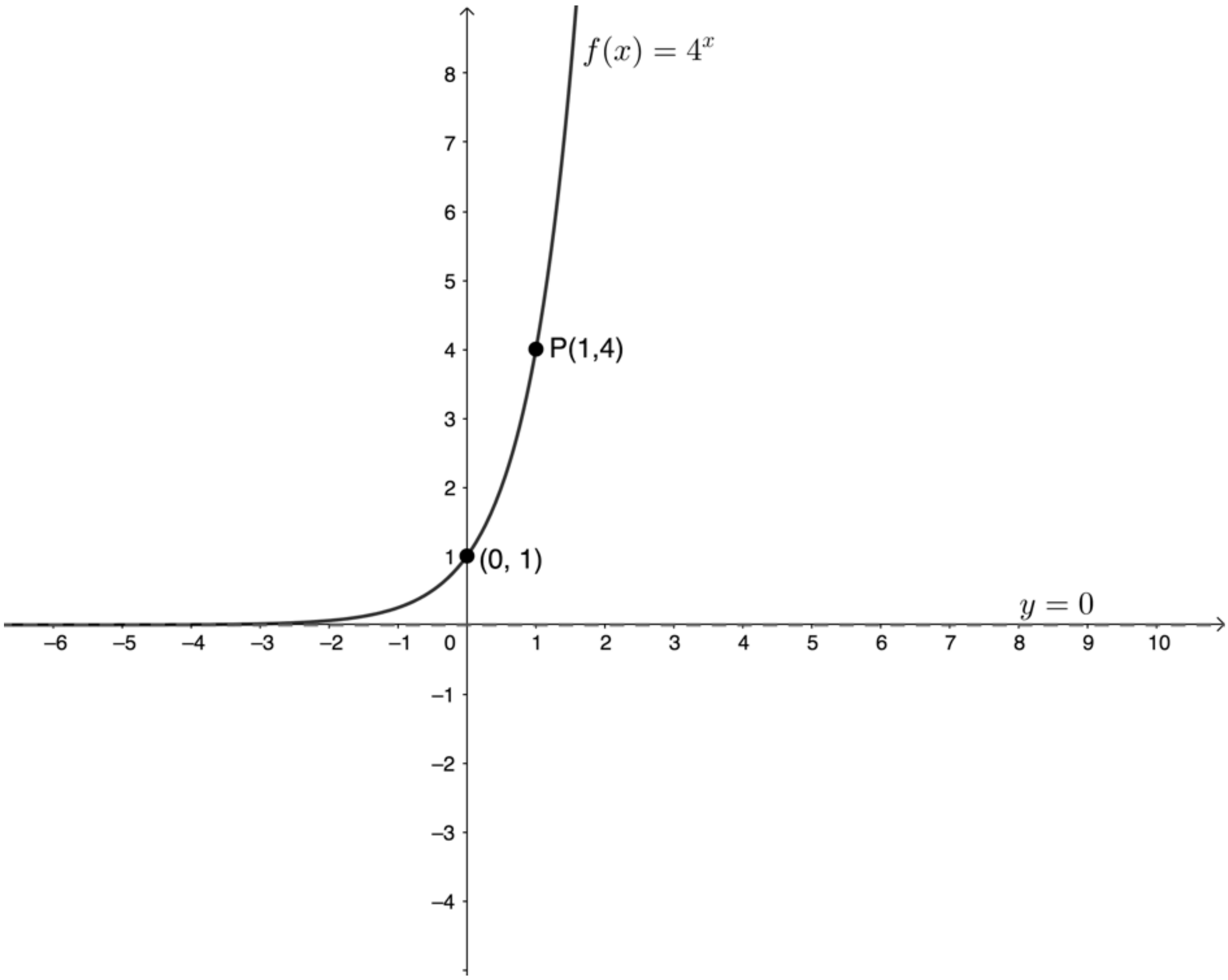
1.

$f(x) = 4^x = 4$

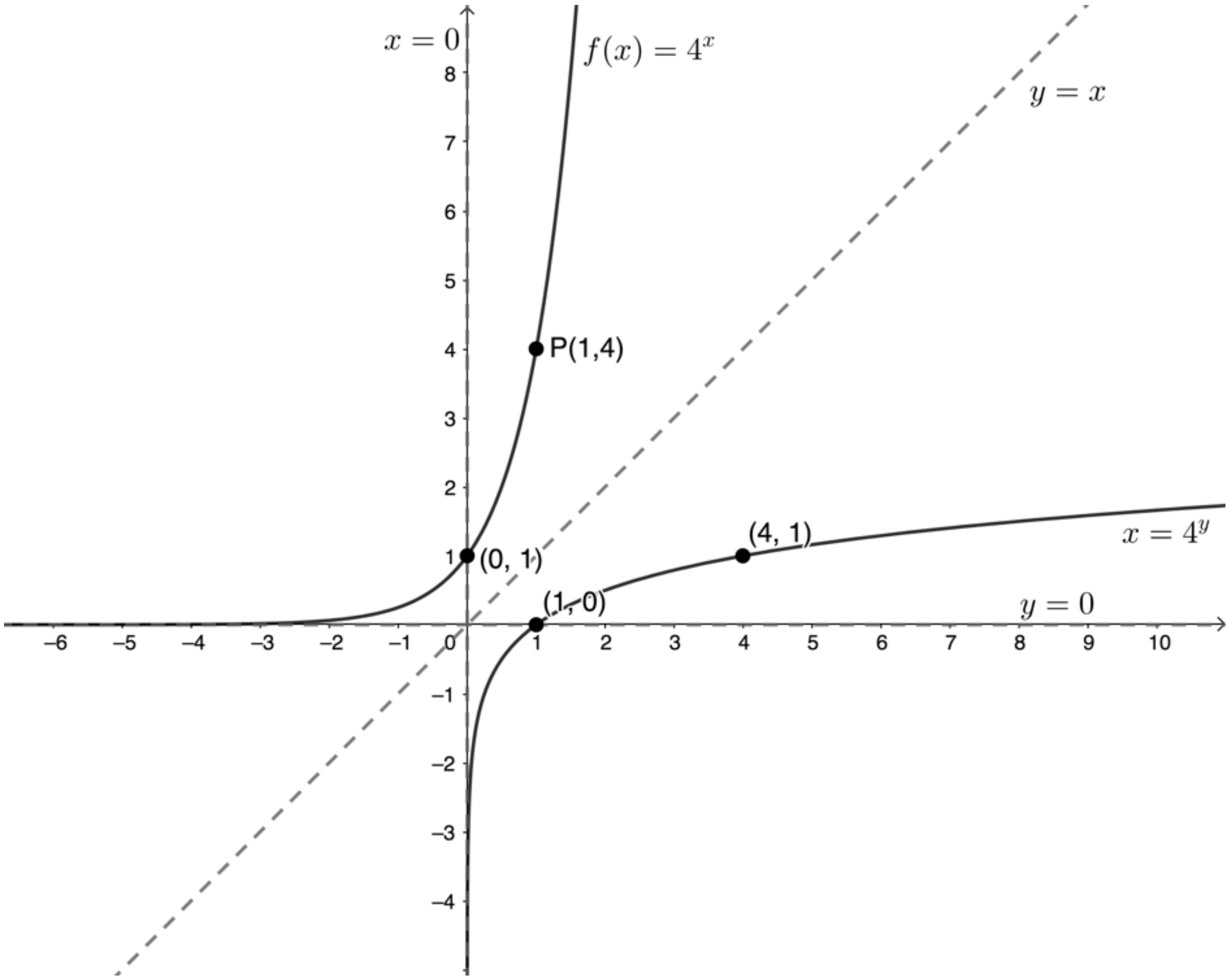
$\therefore x = 1$

$a = 1$

2.



3. Range of f: $\{y|y > 0\}$
4. Horizontal asymptote of f is $y = 0$.
5. $y = 4^x$
   Inverse:
   $x = 4^y$
6. $(1, 0)$ and $(4, 1)$
7.



8. f^{-1} is a function. Each input produces one and only one output.
9. $y=x$

Back to Unit 3: Assessment

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SUBJECT OUTCOME V

FUNCTIONS AND ALGEBRA: USE MATHEMATICAL MODELS TO INVESTIGATE LINEAR PROGRAMMING PROBLEMS

Subject outcome

Subject outcome 2.3: Use mathematical models to investigate linear programming problems

Learning outcomes

- Find and formulate the linear constraints from a given problem.
- Solve linear programming problems by optimising a function in two variables, subject to one or more linear constraints, using the search line method.

Unit 1 outcomes

By the end of this unit you will be able to:

- Determine the linear constraints for a given problem.
- Sketch the linear constraints.
- Find the feasible region.
- Use a search line to optimise the constraints.

Unit 1: Solve linear programming problems by optimising a function

DYLAN BUSA

Unit outcomes

By the end of this unit you will be able to:

- Determine the linear constraints for a given problem.
- Sketch the linear constraints.
- Find the feasible region.
- Use a search line to optimise the constraints.

What you should know

Before you start this unit, make sure you can:

- Simplify algebraic expressions including those with fractions. Refer to level 2 subject outcome 2.2 unit 1 and 3 if you need help with this.
- Solve linear equations. Refer to level 2 subject outcomes 2.3 unit 1 if you need help with this.
- Solve linear inequalities. Refer to level 3 subject outcome 2.3 unit 3 if you need help with this.
- Sketch graphs of linear functions. Refer to level 2 subject outcome 2.1 unit 1 if you need help with this.
- Sketch constraints given as linear inequalities. Refer to level 3 subject outcome 2.4 unit 1 if you need help with this.
- Find the feasible region defined by the given constraints. Refer to level 3 subject outcome 2.4 unit 1 if you need help with this.
- Define an objective function and optimise this objective function. Refer to level 3 subject outcome 2.4 unit 1 if you need help with this.

Introduction

In level 3 subject outcome 2.4 we were introduced to Qhubeka, the bicycle company that makes and donates bicycles to children in rural areas to make things such as getting to and from school easier and quicker.

We saw how, given a set of constraints, we could optimise how many of each of two types of bicycles the company should make in order to achieve the lowest costs. We called these kinds of problems **optimisation problems** – considering several given constraints that all need to be taken into account to find an optimal solution.

We also saw that an excellent way to solve optimisation problems was using a graphical technique called **linear programming**. Linear programming involves:

1. assigning variables to the unknowns which we would like to optimise (in the Qhubeka example this was the number of each type of bicycle that should be produced to minimise costs)

2. determining and expressing the various constraints in terms of these variables (usually as inequalities)
3. sketching these constraints on a graph
4. determining the feasible region (the set of all the variables that obey or fulfil all the constraints at the same time)
5. determining the objective function or search line to optimise the constraints,
6. finding the optimal solution.

In level 3, the expressions for the constraints were always given to you. In this unit, you will need to interpret the given information to determine and express these constraints for yourself. Other than that, everything is the same as you learnt in level 3.

Did you know?

Linear programming is just one of many mathematical modelling techniques. A mathematical model is simply a description of a real-world system or problem using mathematical concepts and language that helps to solve problems, make decisions or predict what might happen in the future.

Watch this informative video that describes mathematical modelling in a little more detail, "What is Math Modeling?".

What is Math Modeling? (Duration: 3.12)

Solve optimisation problems graphically using linear programming

The best way to learn how to solve optimisation problems using linear programming is to work through a few examples before trying some questions on your own.

Example 1.1

A group of learners plans to sell hamburgers and chicken burgers at a rugby match. Their butcher has a total of 300 hamburger patties and 400 chicken burger patties. Each burger needs to be sold in a packet but there are only 500 packets available. Based on past events, the group predicts that demand is likely to be such that the number of chicken burgers sold will be at least half the number of hamburgers sold.

1. Write the constraint inequalities and draw a graph of the feasible region.
2. Looking at the cost of ingredients, the group plans to make a profit of $\text{R}3.00$ on each hamburger and $\text{R}2.00$ on each chicken burger sold. Write an equation which represents the total profit P.

3. How many of each type of burger should the group plan to sell if they want to maximise their profits?

Solutions

1. We need to write down mathematical expressions of the constraints and draw the feasible region.

   Step 1: Assign variables
   In this problem, there are two types of burgers to be sold and we want to find out how many of each kind should be made and sold. We need to assign a variable to each type of burger. You can assign any variables you like but, because we will be sketching linear functions, it is often simplest to use x and y.

   Let the number of hamburgers be x.
   Let the number of chicken burgers be y.

   In both cases, the values of x and y are limited to natural numbers. We cannot sell a fraction of a burger. Therefore $\{x|x \in \mathbb{N}\}$ and $\{y|y \in \mathbb{N}\}$.

   Step 2: Express the constraints
   In level 4, you need to interpret the information given to you to express your own mathematical statements of the constraints.

   We know that the butcher only has 300 hamburger patties and 400 chicken burger patties. Therefore, our first two constraints are $x \leq 300$ and $y \leq 400$.

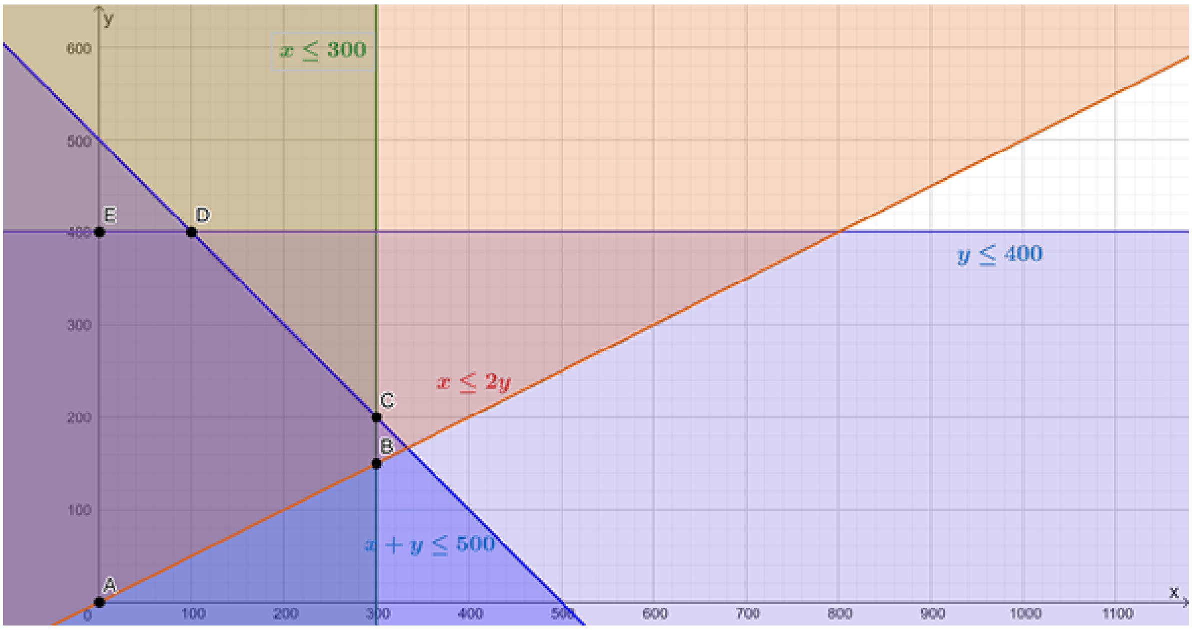
   We also know that the total number of burgers sold is limited by the number of packets available. Therefore, our next constraint is $x + y \leq 500$.

   Finally, the group makes a prediction that at least half as many chicken burgers as hamburgers will be sold. Therefore, our final constraint is $x \leq 2y$ (or $\frac{1}{2}x \leq y$).

   Step 3: Graph the constraints
   It is time to start graphing our constraints so that we can optimise the objective function. Here they are again.

   - $x \leq 300$
   - $y \leq 400$
   - $x + y \leq 500$
   - $x \leq 2y$

   Here are these constraints graphed. The feasible region is that contained by points A, B, C, D and E



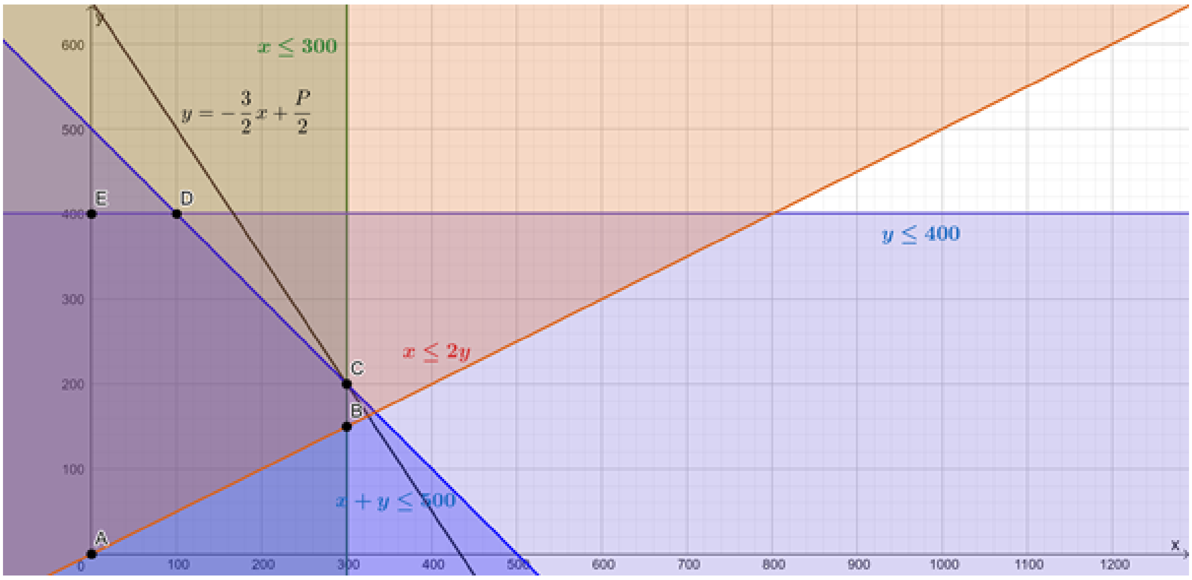
2. Now, if a profit of $\text{R}3.00$ is made on each hamburger sold and a profit of $\text{R}2.00$ is made on each chicken burger sold, the profit equation will be $P = 3x + 2y$
3. To find the maximum profit (the optimum solution), we need to graph our profit function (also called the objective function or the search line) to find what the maximum values for x and y are such that the objective function is still within the feasible region.

   To sketch the objective function/search line, it is normally best to get the function into standard form.

   $$P = 3x + 2y$$
   $$\therefore 2y = -3x + P$$
   $$\therefore y = -\frac{3}{2}x + \frac{P}{2}$$

   In this form, we can see that to maximise the value of P we need to maximise the value of the y-intercept of this straight line. We do not yet know what the value of P is but we do know that the gradient of the graph is $m = -\frac{3}{2}$. Therefore, we need to sketch any straight line with $m = -\frac{3}{2}$ and then move this line up or down until we get the greatest y-intercept while still having the line pass through at least one point in the feasible region.

   Here is the objective function/search line graphed such that the profit is maximised. In order to maximise P, while still remaining in the feasible region, the objective function must pass through the point $C(300, 200)$.



We can either substitute the values for x and y from C into the objective function or we can read off the y-intercept of the function and hence determine the value of P.

Substituting $C(300, 200)$:

$$\begin{aligned} P &= 3(300) + 2(200) \\ &= 900 + 400 \\ &= 1\ 300 \end{aligned}$$

Reading off the y-intercept:
y-intercept is 650

$$\therefore \frac{P}{2} = 650$$
$$\therefore P = 1\ 300$$

This means that to make a maximum profit, the group must sell 300 hamburgers and 200 chicken burgers. This will generate a maximum profit of $\text{R}1\ 300.00$.

Note

If you would like to play with an interactive version of the solution to Example 1.1, please visit this link.

Example 1.2

Question adapted from NC(V) Mathematics Level 4 Paper 1 October 2013 question 4.5

A patient is required to take at least 18 grams of protein, 6 milligrams of vitamin C, and 5 milligrams of iron per meal which consists of two types of food, A and B. Type A contains 9 grams of protein, 6 milligrams of vitamin C and no iron per mass unit. Type B contains 3 grams of protein, 2 milligrams of vitamin C and 5 milligrams of iron per mass unit. The energy value of A is 800 kilojoules and that of B is 400 kilojoules per mass unit. A patient is not allowed to have more than 4 mass units of A and 5 mass units of B. There are x mass units of A and y mass units of B on the patient's plate. The patient cannot eat a fraction of a mass unit of either type of food.

1. Write down all the constraints with respect to the above information in terms of x and y.
2. What is the kilojoule intake per meal?
3. Represent the constraints graphically.
4. Deduce from the graphs the values of x and y that give the **minimum** kilojoule intake per meal for the patient.

Solutions

1. We are told that the mass units of food type A are x and the mass units of food type B are y. We are also told that the patient can have at most 4 mass units of A and 5 mass units of B. Therefore, $x \leq 4$ and $y \leq 5$. Remember, however, that we cannot have fractions of mass units. Therefore, $x \in \mathbb{N}$ and $y \in \mathbb{N}$.

   We are also told about the nutrient content of each food and how much of each nutrient the patient needs. The patient needs at least 18 grams of protein per meal. Food type A gives 9 grams and food type B gives 3 grams. Therefore, $9x + 3y \geq 18$.

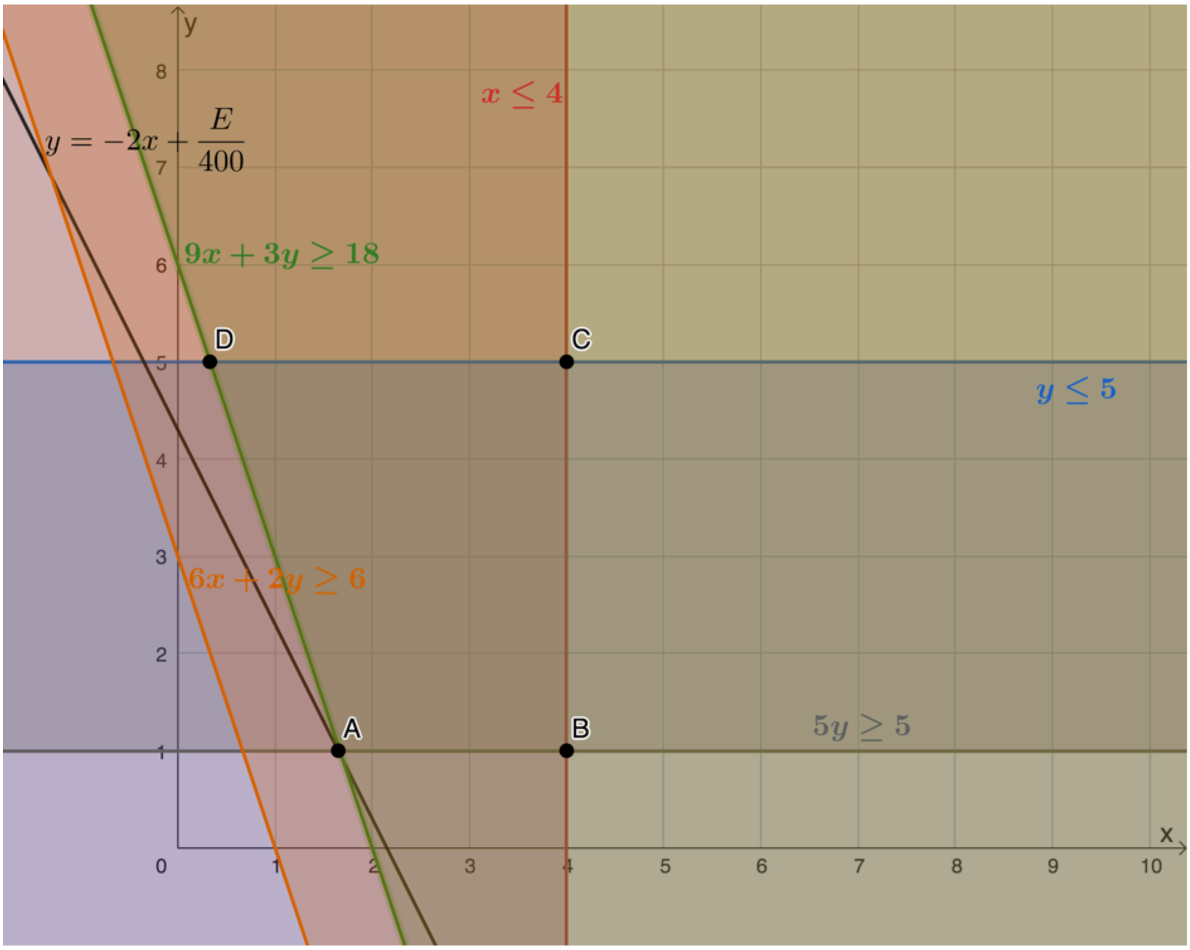
   The patient needs at least 6 milligrams of vitamin C per meal. Food type A gives 6 milligrams and food type B gives 2 milligrams. Therefore, $6x + 2y \geq 6$.

   Finally, the patient needs at least 5 milligrams of iron per meal. Food type A gives 0 milligrams and food type B gives 5 milligrams. Therefore, $5y \geq 5$
2. Food type A has 800 kilojoules and food type B has 400 kilojoules. Therefore, the total energy (E) intake is $E = 800x + 400y$.
3. It makes it easier to sketch the constraints if they are all in standard form.
   $9x + 3y \geq 18$
   $\therefore 3y \geq -9x + 18$
   $\therefore y \geq -3x + 6$
   $6x + 2y \geq 6$
   $\therefore 2y \geq -6x + 6$
   $\therefore y \geq -3x + 3$
   $5y \geq 5$
   $\therefore y \geq 1$
   Here are these constraints graphed. The feasible region is that contained by points A, B, C and D.

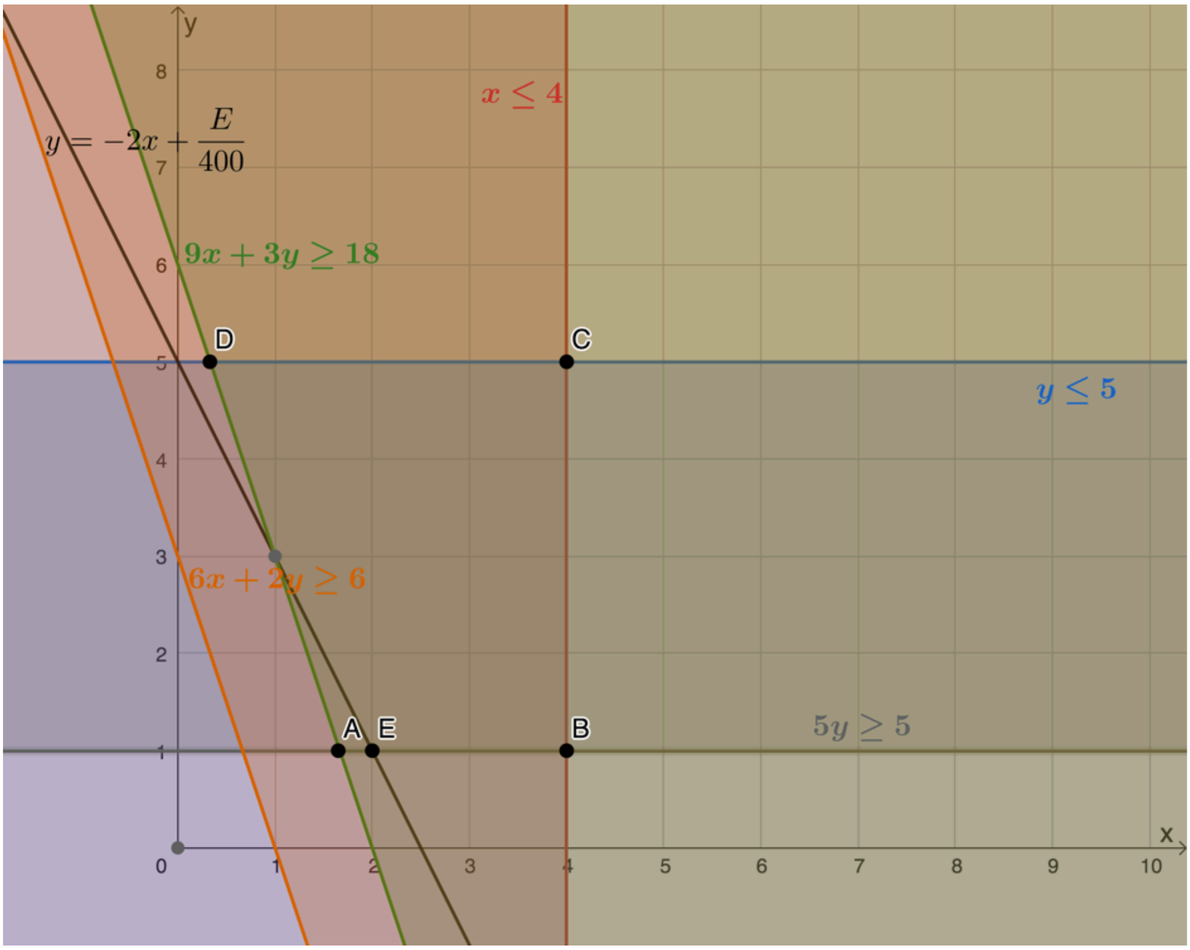
4. We have an equation for the kilojoule intake (the objective function) and we are asked to find the minimum intake. In other words, we need to minimise the function. If we write the objective function in standard form, we will see that the minimum value for E corresponds to the minimum y-intercept that the function or search line can achieve while still passing through the feasible region. Here is the objective function in standard form.

$$E = 800x + 400y$$
$$\therefore 400y = -800x + E$$
$$\therefore y = -2x + \frac{E}{400}$$

Here is a plot of the objective function. We can see that the minimum value is achieved when the function passes through A.



However, we need to remember that $x \in \mathbb{N}$ and $y \in \mathbb{N}$. Point A represents a value of x that is not an integer, therefore, we need to move the objective function so that it passes through the next available point in the feasible region where both coordinates are integers. This is represented by point $E(2, 1)$.



Once again, we can determine what the minimum value of E is by substituting the coordinates of point E into the objective function or by determining the y-intercept and then using the fact that the y-intercept is equal to $\frac{E}{400}$.

Substituting $E(2, 1)$:

$$\begin{aligned} E &= 800(2) + 400(1) \\ &= 1\ 600 + 400 \\ &= 2\ 000 \end{aligned}$$

Reading off the y-intercept:

y-intercept is 5

$\therefore \frac{E}{400} = 5$

$\therefore E = 2\ 000$

The minimum kilojoule intake the patient can have is $2\ 000$ kilojoules and this corresponds to eating 2 mass units of food type A and 1 mass unit of food type B.

Note

If you would like to play with an interactive version of the solution to example 1.2, please visit this link.

Exercise 1.1

Question 1 adapted from NC(V) Mathematics Level 4 Paper 1 November 2016 question 3

1. A small sugar company produces brown sugar and white sugar. In order for the company to be economically viable, it must produce at least 200 boxes of sugar daily. The company can produce a maximum of 500 boxes of sugar daily. At least 100 boxes of brown sugar are to be produced per day. At least 50 boxes and at most 300 boxes of white sugar must be produced daily.

   Let x be the number of boxes of brown sugar and y be the number of boxes of white sugar that are produced each day.

   a. Write down all the algebraic inequalities (in terms of x and y) which describe the constraints related to the production of sugar.
   b. Represent the inequalities in a. graphically and clearly indicate (shade and label) the feasible region.
   c. A profit of $\text{R}20.00$ is made on a box of brown sugar and profit of $\text{R}30.00$ on a box of white sugar. Use this information to draw, on the graph for b., a profit search-line.
   d. Use the profit search-line to calculate the minimum and maximum profit the company can make each day.

Question 2 adapted from NC(V) Mathematics Level 4 Paper 1 November 2015 question 5

2. A transport company uses buses and minibuses to transport a minimum of 400 passengers and a maximum of 600 passengers each day. At least 200 passengers per day must be transported by bus. The number of passengers travelling by bus cannot be more than three times the number of passengers travelling by minibus. Let x represent the number of passengers transported by bus and y be the number of passengers transported by minibus.

   a. Determine the constraints, in terms of x and y, under which the transport company operates.
   b. Represent the constraints graphically and clearly indicate the feasible region.
   c. The profit per day per passenger travelling by bus is $\text{R}2.00$ and the profit per day per passenger travelling by minibus is $\text{R}1.60$. Determine from your graph by means of a search line, the values of x and y that will yield a maximum profit.
   d. Calculate the maximum possible profit per day.

The full solutions are at the end of the unit.

Summary

In this unit you have learnt the following:

- How to determine and express the constraints for a given problem as linear inequalities.
- How to sketch the constraints on a Cartesian plane.
- How to find and mark the feasible region.
- How to define the objective function (or search line) from given information
- How to use this objective function (or search line) to determine the optimal solution.

Unit 1: Assessment

Suggested time to complete: 15 minutes

Question 1 adapted from NC(V) Mathematics Level 4 Paper 1 November 2012 question 1.3

1. A carpenter makes two types of tables, A and B.

    - He has $400\ \text{m}^2$ of floor space available.
    - Each table A requires $30\ \text{m}^2$ of floor space and each table B requires $40\ \text{m}^2$ of floor space.
    - He does not have enough skilled labourers to make more than 8 tables of type A and 6 of type B.
    - According to public demand he has to make at most two type A tables for each type B table.

      Let x be the number of tables of type A and y the number of tables of type B.

    b. Write the constraints with respect to the above information in terms of x and y.
    c. Represent graphically ALL the constraint inequalities and clearly indicate the feasible region.
    d. If the profit made on each table is the same, determine by means of a search line, how many of each type of table should be made for maximum profit.

Question 2 adapted from NC(V) Mathematics Level 4 Paper 1 November 2011 question 1.3

2. Jackson's clothing factory manufactures jackets and jerseys. A maximum of 60 jackets and 40 jerseys can be manufactured daily while in total not more than 70 pieces of clothing can be manufactured per day. It takes 5 hours to manufacture a jacket and 10 hours to manufacture a jersey, while there are at most 500 working hours available per day.

    Let x represent the number of jackets and y represents the number of jerseys manufactured per day.

    a. Write down the constraints with respect to the above information in terms of x and y.
    b. Represent all the constraint inequalities on the grid and clearly indicate the feasible region.
    c. If the profit is $\text{R}15$ on a jacket and $\text{R}25$ on a jersey, give the equation that indicates the profit.
    d. Using the search line method, determine how many jackets and how many jerseys should be manufactured in order to obtain the maximum profit?
    e. Determine the maximum profit.

Question 3 adapted from NC(V) Mathematics Level 4 Paper 1 October 2014 question 5

3. A trucking company wants to purchase a maximum of 15 new trucks that will provide at least 36 tons of additional shipping capacity. A model A truck holds 2 tons and costs $\text{R}150\ 000$. A model B truck holds 3 tons and costs $\text{R}240\ 000$.

    Let x be the number of model A trucks and let y be the number of model B trucks.

a. Write down the constraints that represent the above information.
b. Represent the system of constraints on the grid indicating the feasible region by means of shading.
c. How many trucks of each model should the company purchase in order to provide the additional shipping capacity at minimum cost?
d. Calculate the minimum cost.

The full solutions are at the end of the unit.

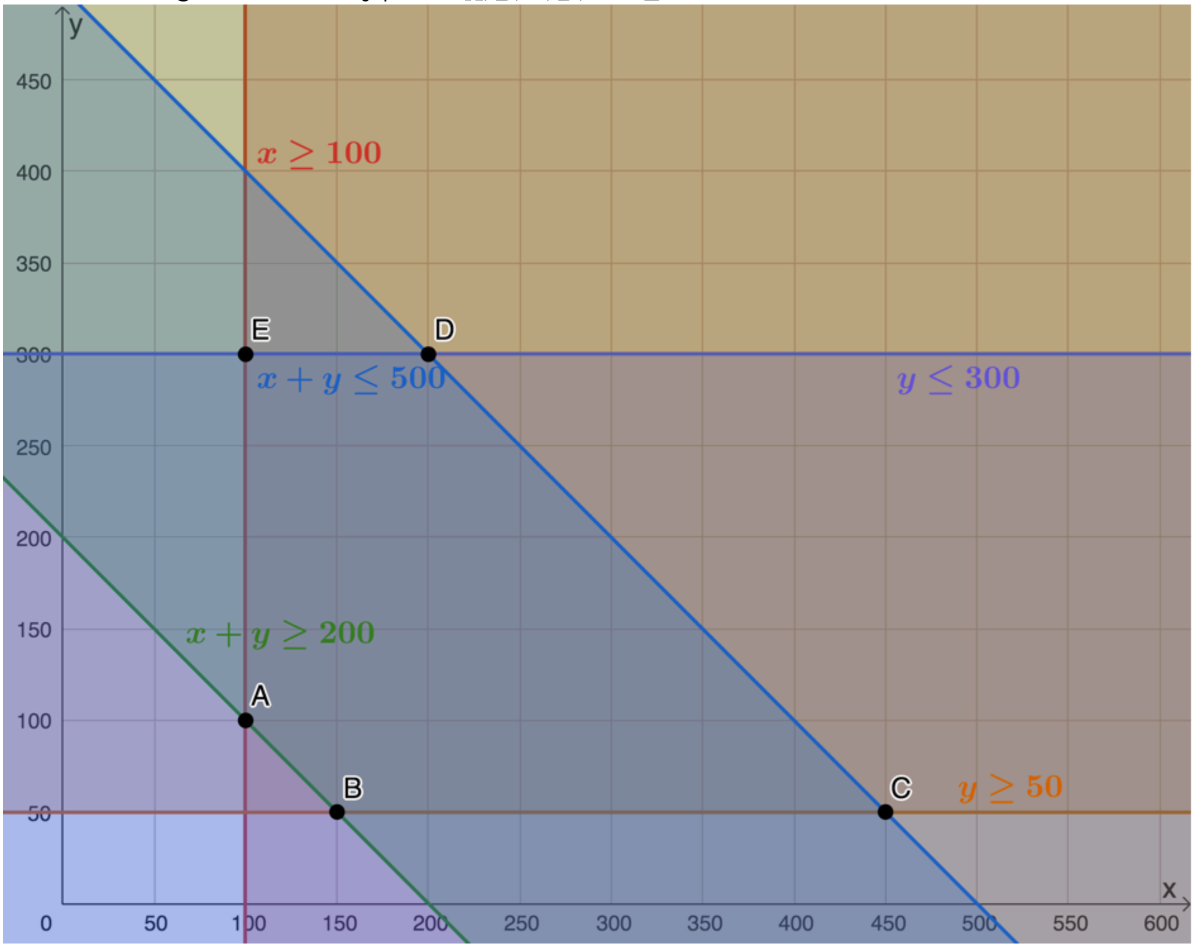
Unit 1: Solutions

Exercise 1.1

1.

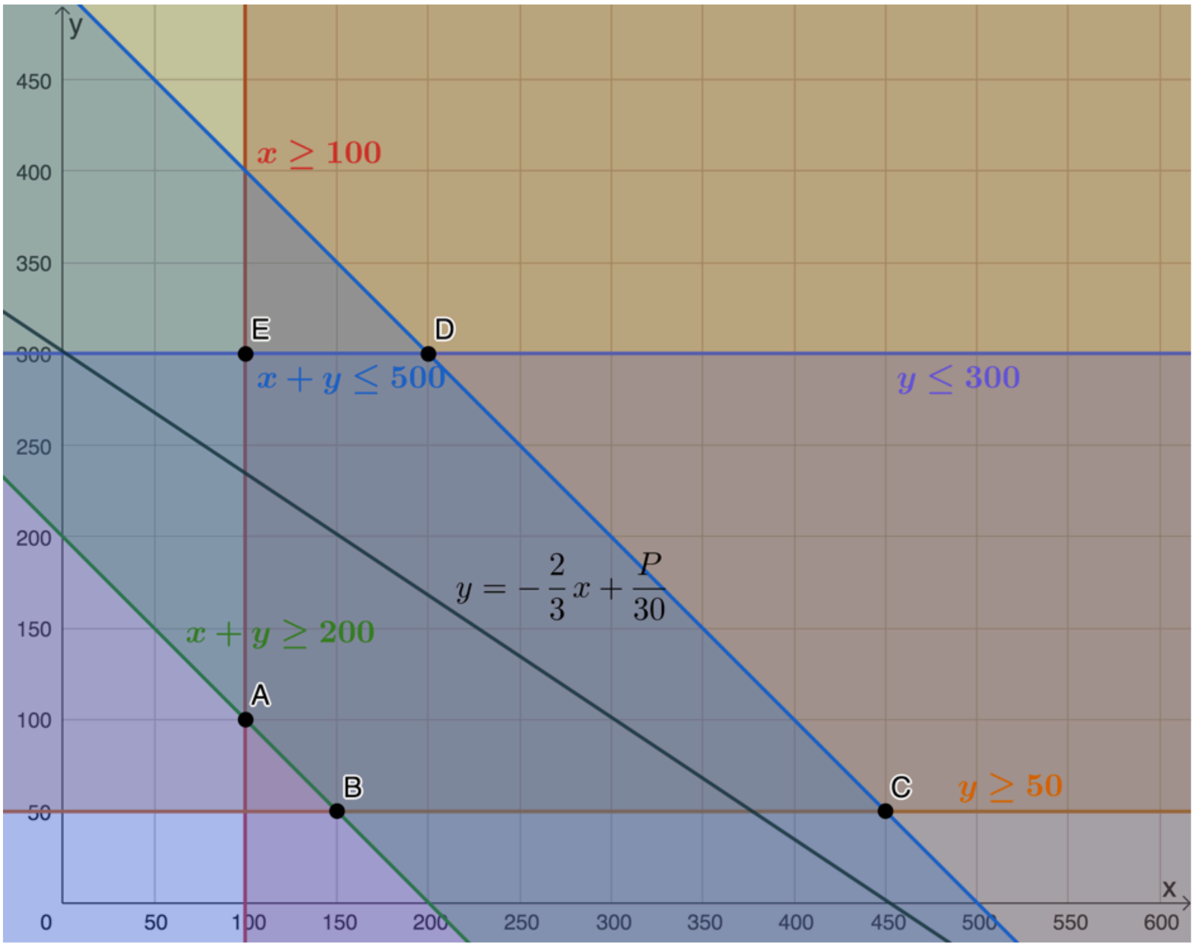
a. Number of boxes of brown sugar is x, $x \in \mathbb{N}$
Number of boxes of white sugar is y, $y \in \mathbb{N}$
Minimum number of boxes of sugar: $x + y \geq 200$
Maximum number of boxes of sugar: $x + y \leq 500$
Minimum number of boxes of brown sugar: $x \geq 100$
Minimum number of boxes of white sugar: $y \geq 50$
Maximum number of boxes of white sugar: $y \leq 300$

b. The feasible region is bound by points A, B, C, D, and E.



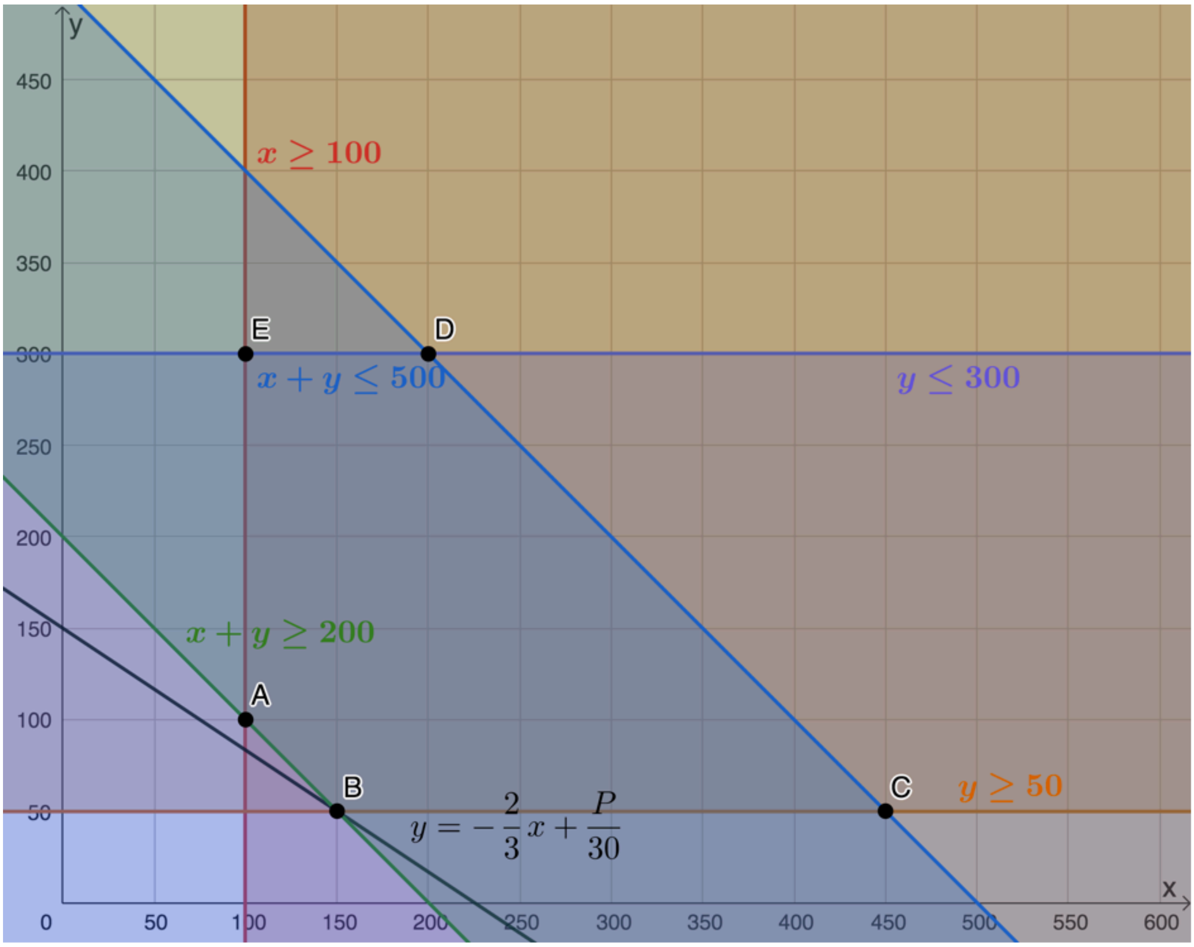
c.

$$P = 20x + 30y$$
$$\therefore 30y = -20x + P$$
$$\therefore y = -\frac{2}{3}x + \frac{P}{30}$$



d. Minimum profit occurs when the objective function or search line passes through $B(150, 50)$. Therefore, the minimum profit is:

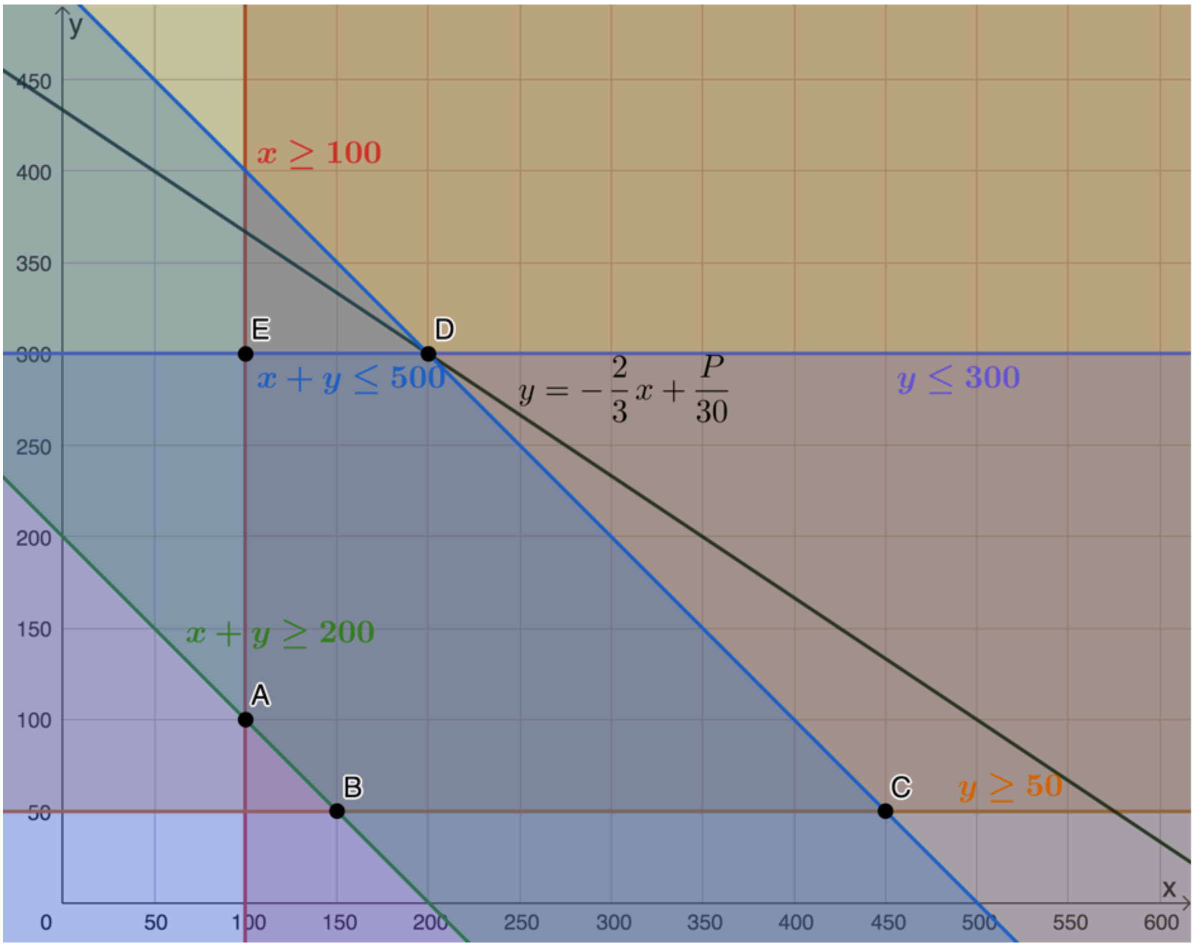
$$P = 20(150) + 30(50)$$
$$= 3\ 000 + 1\ 500$$
$$= 4\ 500$$



Maximum profit occurs when the objective function or search line passes through $D(200, 300)$.

Therefore, maximum profit is:

$$\begin{aligned} P &= 20(200) + 30(300) \\ &= 4\ 000 + 9\ 000 \\ &= 13\ 000 \end{aligned}$$

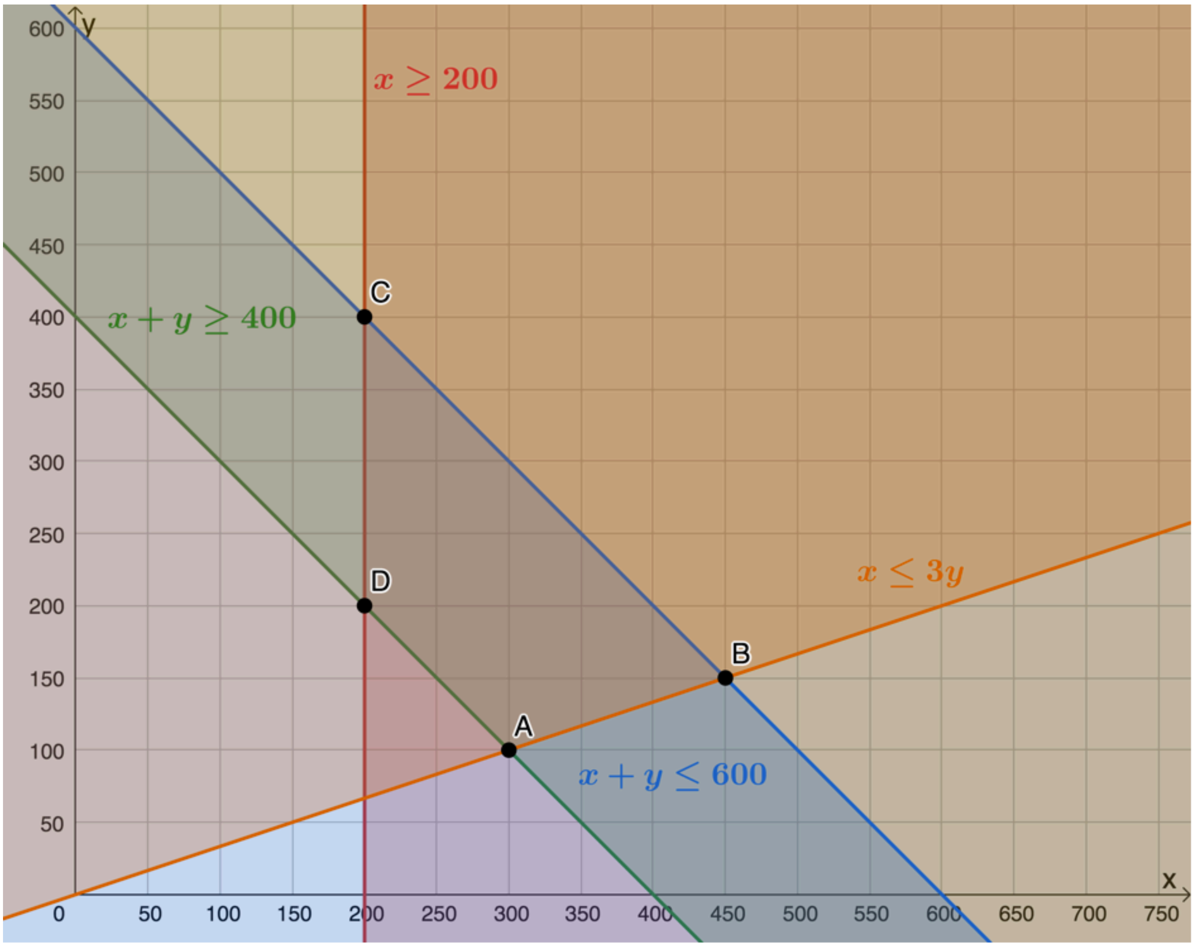


Visit the interactive solution.

2.

a. Passengers transported by bus is x, $x \in \mathbb{N}$
   Passengers transported by minibus is y, $y \in \mathbb{N}$
   Minimum number of passengers per day: $x + y \geq 400$
   Maximum number of passengers per day: $x + y \leq 600$
   Minimum number of bus passengers: $x \geq 200$
   Bus passengers cannot be more than three times minibus passengers: $x \leq 3y$

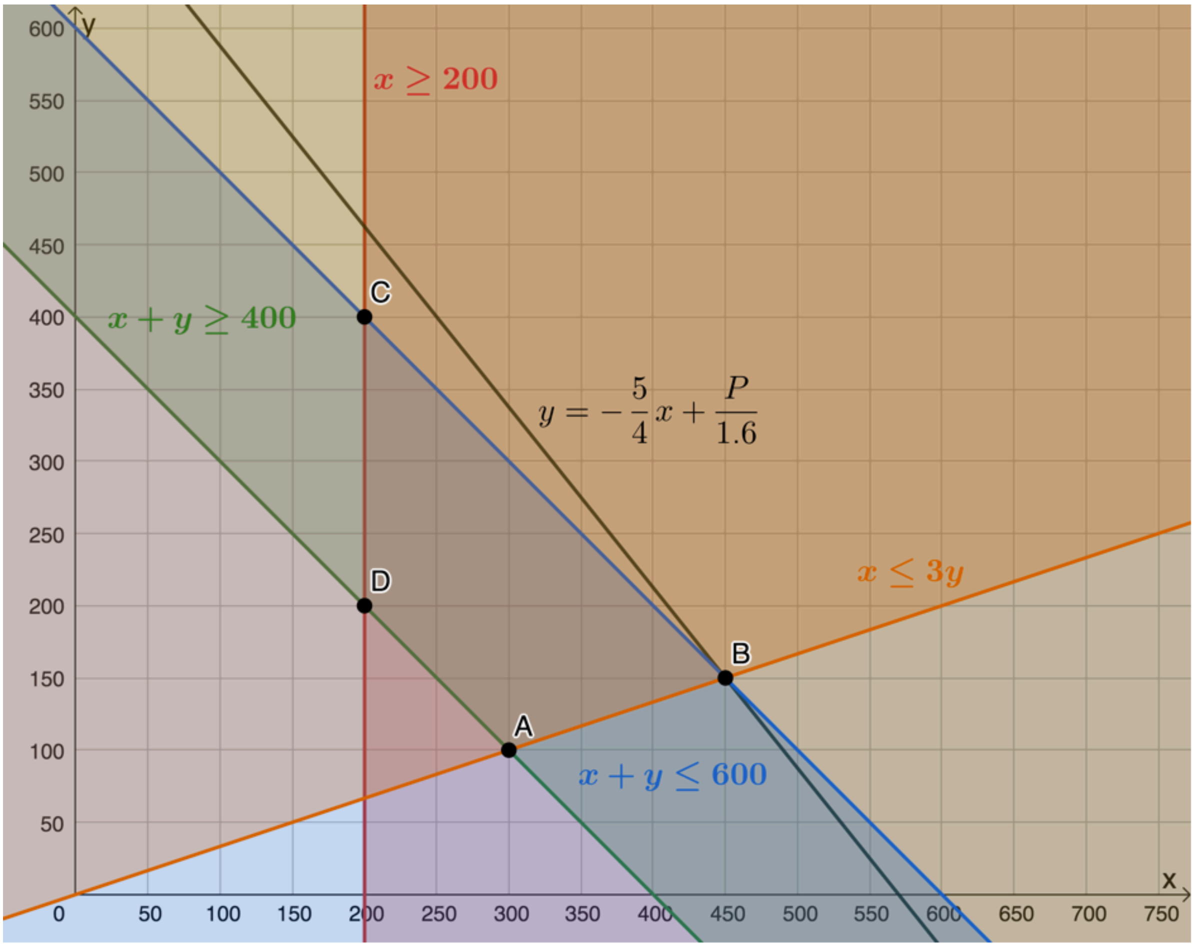
b. Feasible region is bound by points A, B, C and D.



c.

$$P = 2x + 1.6y$$
$$\therefore 1.6y = -2x + P$$
$$\therefore y = -\frac{2}{1.6}x + \frac{P}{1.6}$$
$$\therefore y = -\frac{5}{4}x + \frac{P}{1.6}$$

Maximum profit will occur when the objective function or search lines passes through point $B(450, 150)$.



d. The maximum profit is:

$$\begin{aligned} P &= 2(450) + 1.6(150) \\ &= 900 + 240 \\ &= 1\ 140 \end{aligned}$$

Visit the interactive solution.

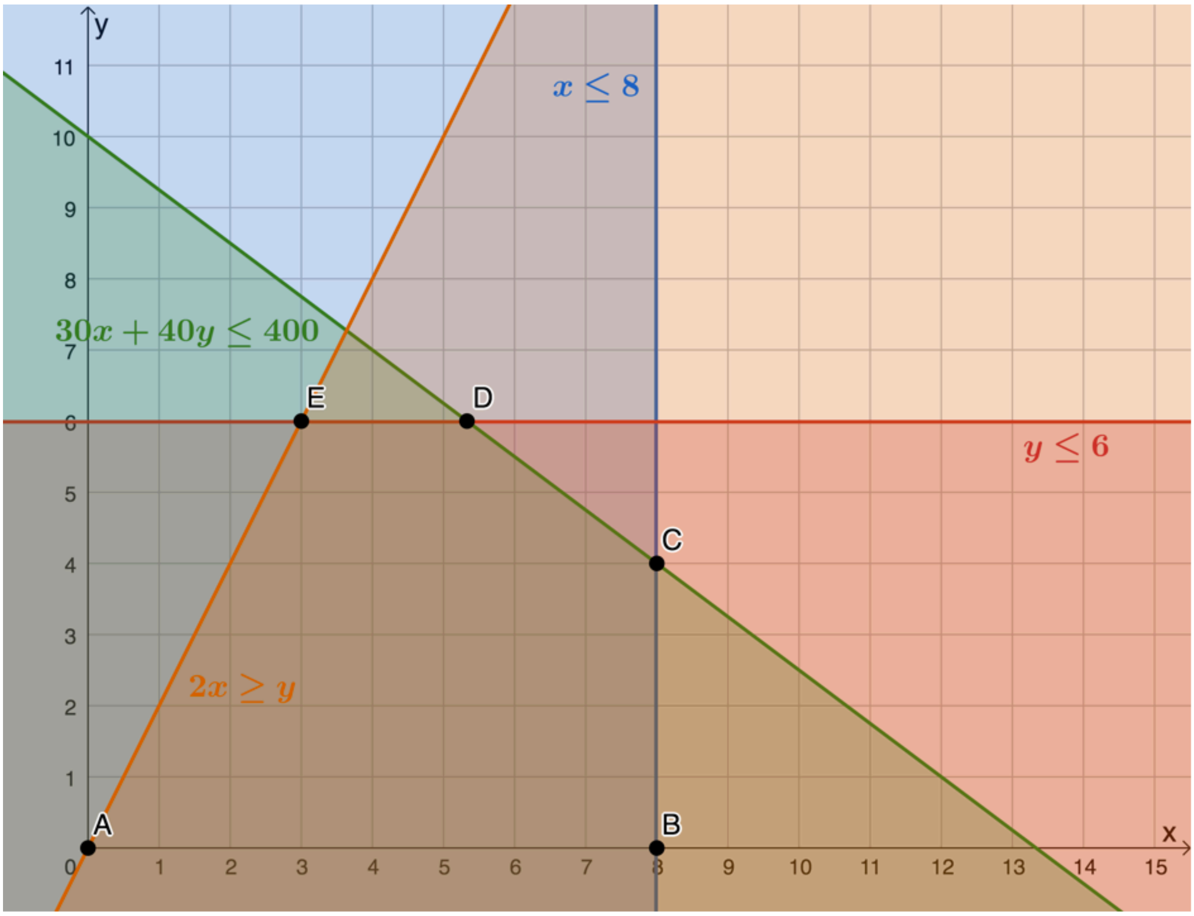
Back to Exercise 1.1

Unit 1: Assessment

1.

a. Table type A is x, $x \in \mathbb{N}$
Table type B is y, $y \in \mathbb{N}$
Floor space available: $30x + 40y \leq 400$
Maximum type A tables: $x \leq 8$
Maximum type B tables: $y \leq 6$
At most two type A tables for each type B table: $2x \geq y$

b. Feasible region is bound by points A, B, C, D and E.



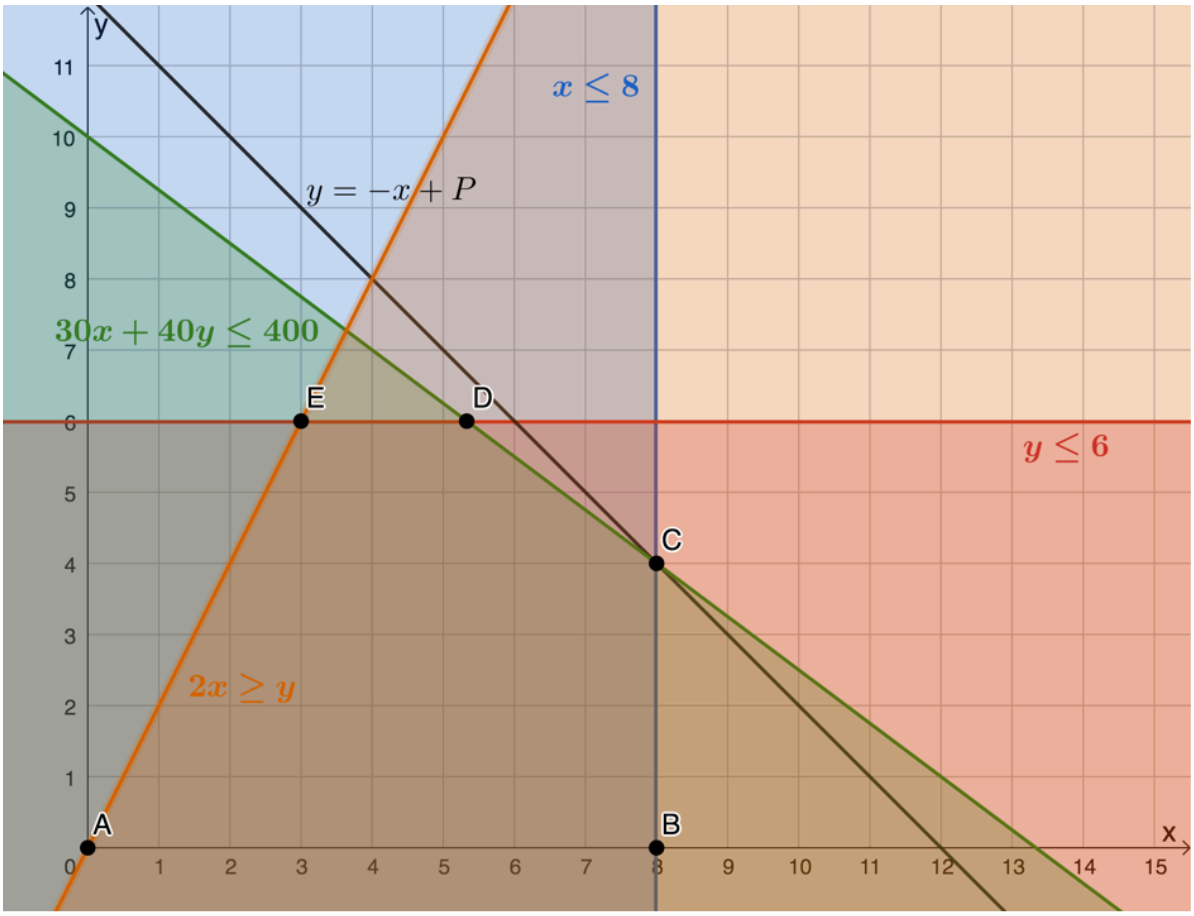
c.

$P = x + y$

$\therefore y = -x + P$

Maximum profit occurs when the objective function or search line passes through $C(8, 4)$.

Therefore, 8 units of table A and 4 units of table B should be made to maximise profit.

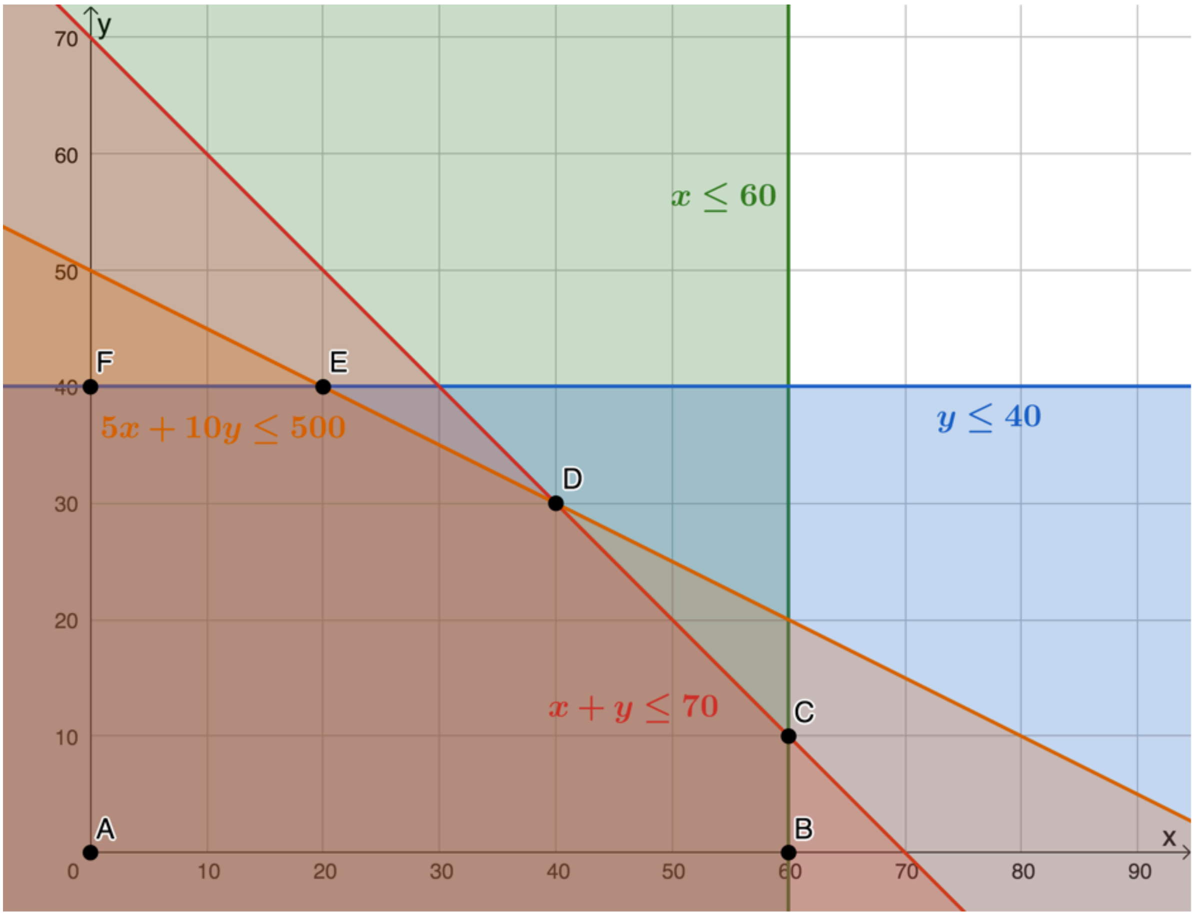


Visit the interactive solution.

2.

a. Number of jackets is x, $x \in \mathbb{N}$
Number of jerseys is y, $y \in \mathbb{N}$
Maximum jackets: $x \leq 60$
Maximum jerseys: $y \leq 40$
Maximum pieces of clothing: $x + y \leq 70$
Working hours available: $5x + 10y \leq 500$

b. Feasible region is bound by points A, B, C, D, E and F.

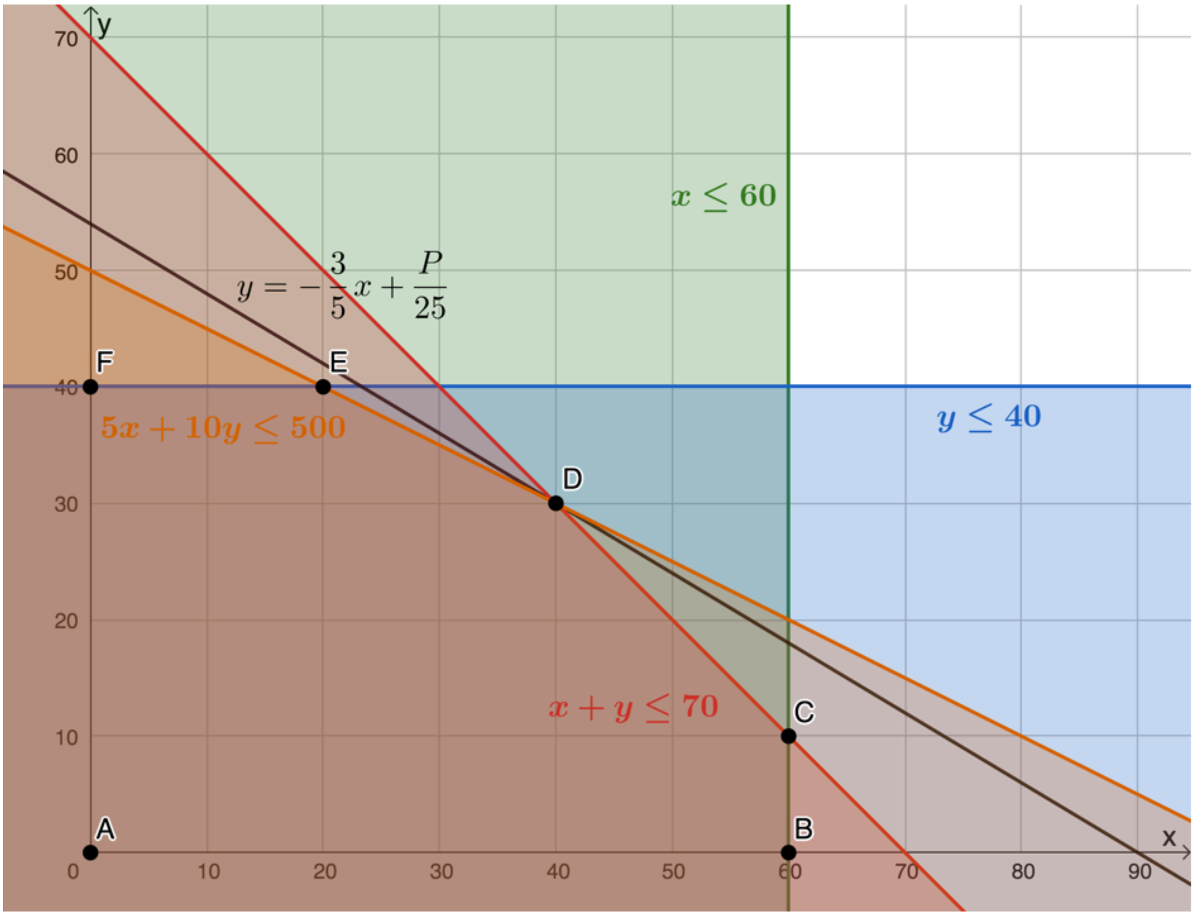


c. $P = 15x + 25y$

d.

$$P = 15x + 25y$$
$$\therefore 25y = -15x + P$$
$$\therefore y = -\frac{3}{5}x + \frac{P}{25}$$

Profit is at a maximum when the objective function of the search line passes through $D(40, 30)$.



e.

$$\begin{aligned} P &= 15(40) + 25(30) \\ &= 600 + 750 \\ &= 1\ 350 \end{aligned}$$

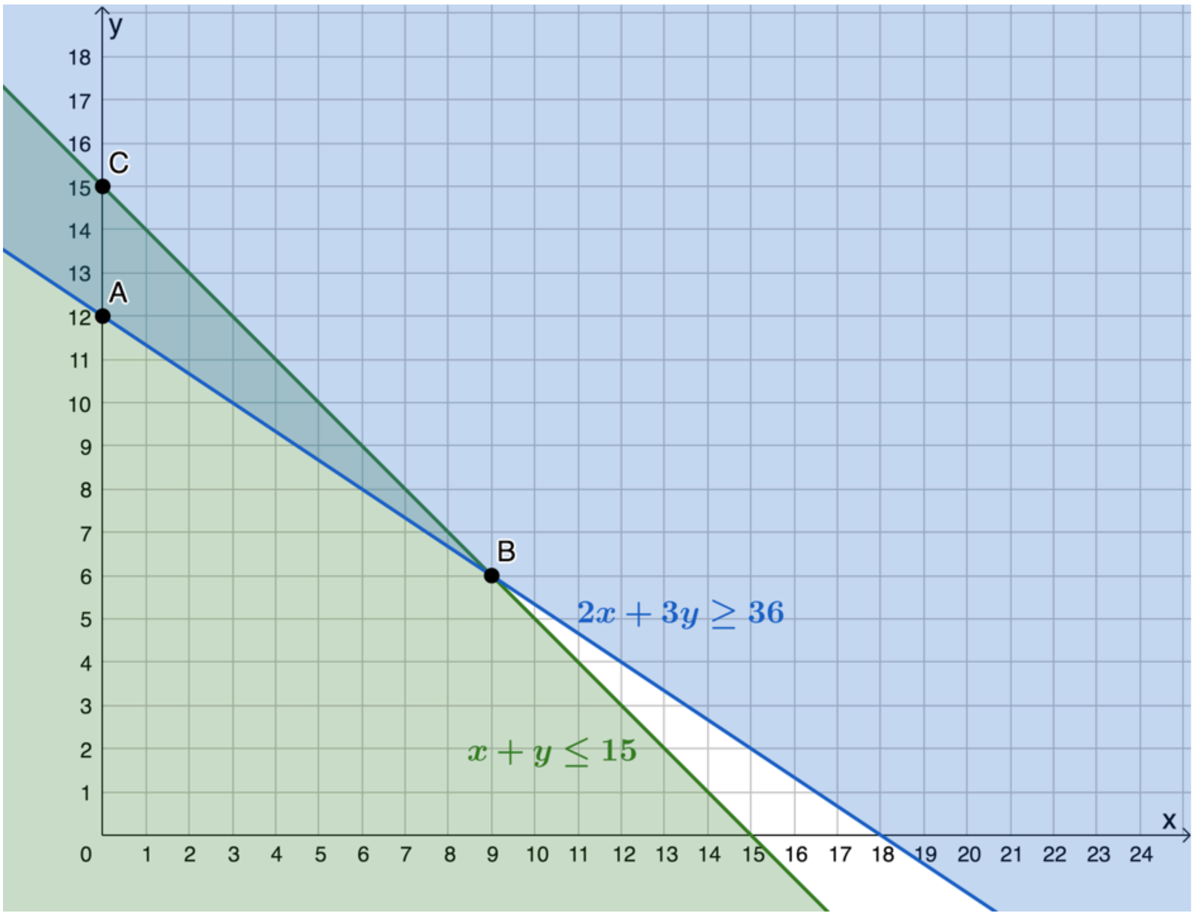
The maximum profit is $\text{R}1\ 350.00$.

Visit the interactive solution.

3.

a. Number of model A trucks is x, $x \in \mathbb{N}$
Number of model B trucks is y, $y \in \mathbb{N}$
Maximum number of trucks: $x + y \leq 15$
Minimum shipping capacity: $2x + 3y \geq 36$

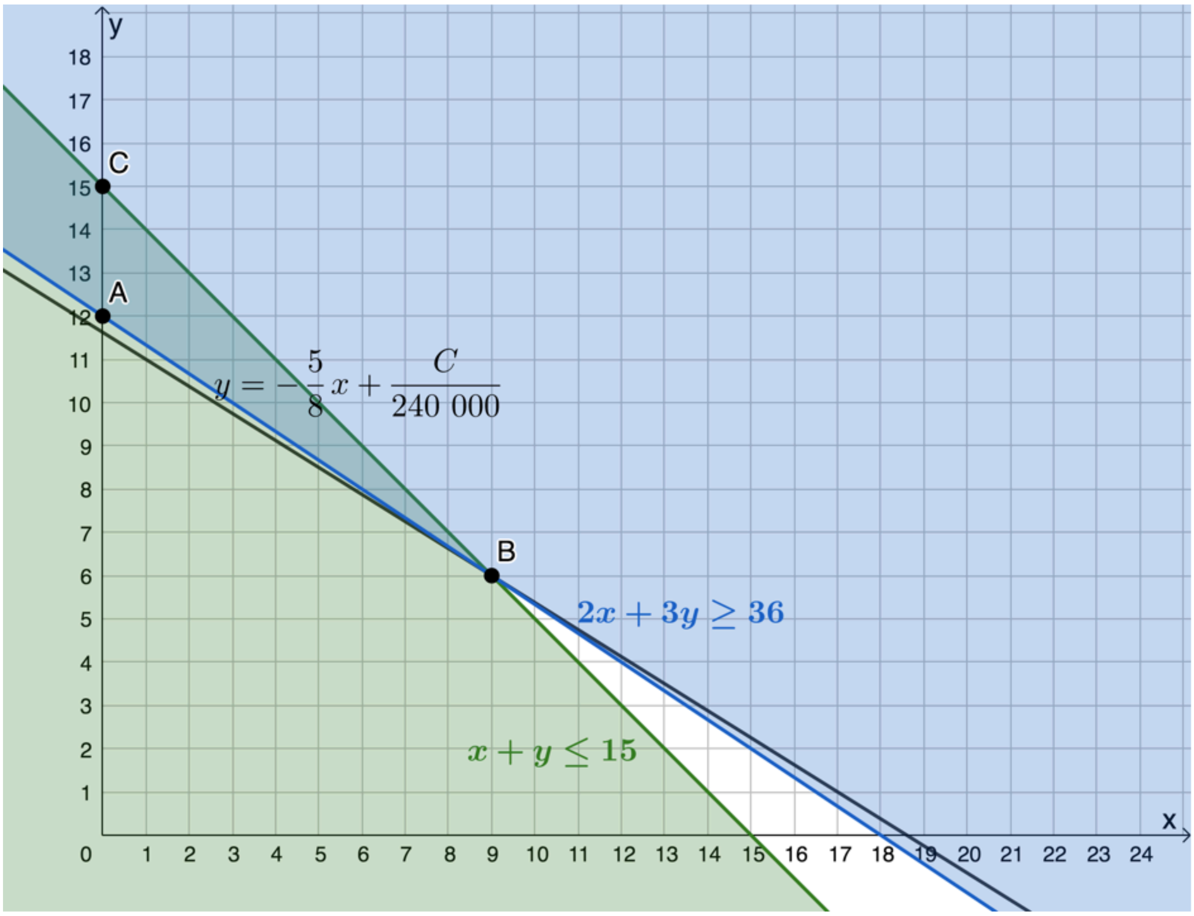
b. Feasible region is bound by points A, B and C.



c. The total cost of the trucks will be $C = 150\ 000x + 240\ 000y$. To plot and minimise this objective function we get it into standard form.

$$C = 150\ 000x + 240\ 000y$$
$$\therefore 240\ 000y = -150\ 000x + C$$
$$\therefore y = -\frac{5}{8}x + \frac{C}{240\ 000}$$

Cost is at a minimum when the objective function passes through $B(9,6)$. 9 model A trucks and 6 model B trucks should be purchased to minimise the cost.



d.

$$\begin{aligned} C &= 150\ 000(9) + 240\ 000(6) \\ &= 1\ 350\ 000 + 1\ 440\ 000 \\ &= 2\ 790\ 000 \end{aligned}$$

The minimum cost will be $\text{R}2\ 790\ 000$.

Visit the interactive solution.

Back to Unit 1: Assessment

Media Attributions

SUBJECT OUTCOME VI

FUNCTIONS AND ALGEBRA: INVESTIGATE AND USE INSTANTANEOUS RATE OF CHANGE OF A VARIABLE WHEN INTERPRETING MODELS BOTH IN MATHEMATICAL AND REAL LIFE SITUATIONS

Subject outcome

Subject outcome 2.4: Investigate and use instantaneous rate of change of a variable when interpreting models both in mathematical and real life situations

Learning outcomes

- Establish the derivatives of the following functions from first principles:
  - $f(x) = b$; $f(x) = ax + b$; $f(x) = ax^2 + b$; $f(x) = x^3$; $f(x) = ax^3$; $f(x) = \frac{1}{x}$; $f(x) = \frac{a}{x}$

    Note: The binomial theorem does not form part of the curriculum.
- Find the derivatives of the functions in the form:
  - $f(x) = ax^n$
  - $f(x) = a \ln kx$
  - $f(x) = ae^{kx}$
  - $f(x) = \text{asin}kx$
  - $f(x) = \text{acos}kx$
  - $f(x) = \text{atan}kx$

  Where:

  - $f(x) = ax^n$ $\quad f'(x) = nax^{n-1}$
  - $f(x) = a \ln kx$ $\quad f'(x) = \frac{k}{x}$
  - $f(x) = ae^{kx}$ $\quad f'(x) = ke^{kx}$
  - $f(x) = \text{asin}kx$ $\quad f'(x) = ka \cos kx$
  - $f(x) = \text{acos}kx$ $\quad f'(x) = -ka \sin kx$

  Examples to include are:

- $3x^2; \frac{3}{x^{-3}}; \quad -\frac{2}{\sqrt[3]{x^2}}; 2\ln 3x; \frac{1}{2}e^{-2x}; 2\sin 3x; \frac{1}{3}\cos\frac{x}{2}; \quad -4\tan x$; etc.

- Use the constant, sum and/or difference, product, quotient and chain rules for differentiation. Note: Combinations of rules in the same problem are excluded.
- Find the equation of the tangent to a graph at a specific point.
- Solve practical problems involving rates of change. Note: velocity and acceleration may be included.
- Draw graphs of cubic functions by determining:
  - y-intercept
  - roots (x-intercepts)
  - turning points using derivatives
- Determine/prove maximum and minimum turning points by making use of second order derivatives (Only: quadratic and cubic functions)
- Determine the point of inflection of cubic graphs by using second order derivatives.

Unit 1 outcomes

By the end of this unit you will be able to:

- Define the derivative using limits.
- Calculate the derivative from first principles.

Unit 2 outcomes

By the end of this unit you will be able to:

- Use various forms of notation to represent the derivative.
- Apply the power rule.
- Find the derivative of a constant.
- Find the derivative of a constant multiplied by a function.
- Find the derivative of a sum/difference.
- Find the derivative of a product.
- Find the derivative of a quotient.
- Apply the chain rule.

Unit 3 outcomes

By the end of this unit you will be able to:

- Define a tangent line.
- Determine the gradient of the tangent at a point.
- Find the equation of a tangent.

Unit 4 outcomes

By the end of this unit you will be able to:

- Solve practical problems involving rates of change.

Unit 5 outcomes

By the end of this unit you will be able to:

- Determine the shape of a cubic function.
- Determine the x and y intercepts.
- Find the turning points of the graph.
- Find the maximum and minimum values of the graph.
- Find the point of inflection and discuss concavity using second derivatives.

Unit 1: Use first principles to find the derivative

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Unit outcomes

By the end of this unit you will be able to:

- Define the derivative using limits.
- Calculate the derivative from first principles.

What you should know

Before you start this unit, make sure you can:

- Find limits of a function as shown in level 3 subject outcome 2.5 unit 1. You may also watch this video to revise limits, "Introduction to limits".

Introduction to limits (Duration: 04.16)

- Find the average gradient of a function. To revise this refer to level 3 subject outcome 2.5 unit 2.

Introduction

Cheetahs are the fastest land animal, reaching speeds of up to 113 km/h. But, cheetahs do not run at their top speed at every instant. So, how do we calculate their speed at any given instant?

Before calculus was invented there was no way to calculate instantaneous speed. Calculus makes the study of the smallest rates of change possible. Key to the concept of calculus is finding the limits of functions.

The type of limit we use to find the slope of the tangent line to a function at a point has many applications. It allows us to get the most or the best, out of any deal. This limit occurs so frequently that we give it a special name; the derivative. The process of finding a derivative is called differentiation.

Limits revised

In level 3 subject outcome 2.5 unit 1, we saw that when a function approaches the same y-value from the left and from the right, the limit exists. The limit of a function is shown with the following notation: $\lim_{x \to a} f(x) = L$
.

Example 1.1

Find $\lim_{x \to 3} \frac{x^2 - 9}{x - 3}$.

Solution

If you substitute $x = 3$ directly into the expression you will get $\frac{0}{0}$, which is undefined. To find the limit we must first simplify the expression.

$$\begin{aligned} \lim_{x \to 3} \frac{x^2 - 9}{x - 3} &= \lim_{x \to 3} \frac{\cancel{(x - 3)}(x + 3)}{\cancel{(x - 3)}} \\ &= \lim_{x \to 3} (x + 3) \\ &= 6 \end{aligned}$$

Example 1.2

Find: $\lim_{x \to \infty} \frac{2x^2 + x}{3x - 2}$.

Solution

If the degree of the numerator is greater than the degree of the denominator, then the limit will either be positive infinity or negative infinity. We have to look at the signs of the terms with the largest exponents in both the numerator and denominator.

$$\lim_{x \to \infty} \frac{2x^2 + x}{3x - 2} = +\infty$$

This is because both x^2 in the numerator and $3x$ in the denominator are positive.

While $\lim_{x \to \infty} \frac{4x^2 - 5x^3}{3x - 2} = -\infty$ because $\frac{-5}{3}$ is negative.

Revise limits by completing the following exercise.

Exercise 1.1

Calculate:

1. $\lim_{h\to 0} \frac{h^2+4h}{h}$
2. $\lim_{h\to 0} \frac{(2+h)^2-4}{h}$
3. $\lim_{x\to \infty} \frac{2x^2+1}{x^2+1}$

The full solutions are at the end of the unit.

Average rate of change

Remember that the gradient of a line passing through the points $\mathrm{A}(x_\mathrm{A};y_\mathrm{A})$ and $\mathrm{B}(x_\mathrm{B};y_\mathrm{B})$ is found using the formula:

$$\begin{aligned} m &= \frac{y_\mathrm{B}-y_\mathrm{A}}{x_\mathrm{B}-x_\mathrm{A}} \\ &= \frac{\Delta y}{\Delta x} \end{aligned}.$$

The gradient of a linear function is constant.

With curved graphs the gradient changes at every point on the curve so we need to work with the average gradient. Remember that the average gradient between two points is equal to the gradient of a straight line, called a secant, drawn between the two points.

To find the average gradient between any two points on a curve we calculate the gradient of the secant line that passes through both points.

Note

The average rate of change tells us whether the new change produced an increase, decrease or no change and how fast the change occurred, on average.

The average rate of change or average gradient of a function between $\mathrm{A}(x_\mathrm{A};y_\mathrm{A})$ and $\mathrm{B}(x_\mathrm{B};y_\mathrm{B})$ is given by:

$$\begin{aligned} \frac{\Delta y}{\Delta x} &= \frac{y_\mathrm{B}-y_\mathrm{A}}{x_\mathrm{B}-x_\mathrm{A}} \\ &= \frac{f(x_\mathrm{B})-f(x_\mathrm{A})}{x_\mathrm{B}-x_\mathrm{A}} \end{aligned}$$

Remember that $f(x_B) = y_B$

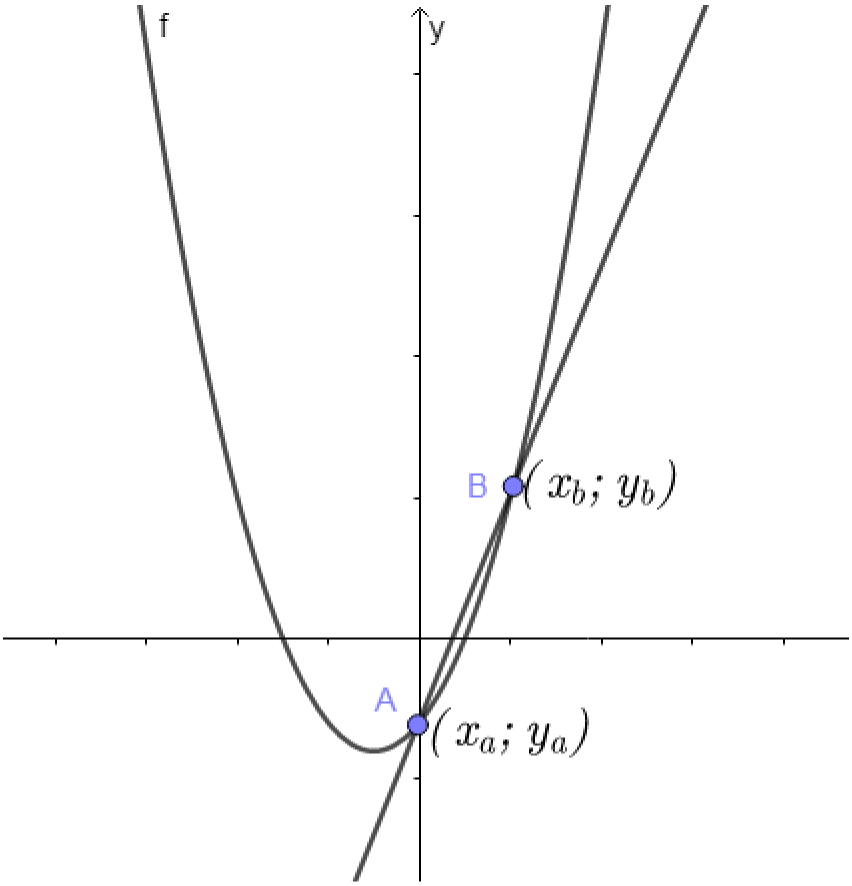

Figure 1: Average gradient

If we let the change in x values equal h then $x_B - x_A = h$. Therefore, $x_B = h + x_A$.

So, the average rate of change of a function between $\mathrm{A}(x_\mathrm{A}; y_\mathrm{A})$ and $\mathrm{B}(x_\mathrm{B}; y_\mathrm{B})$ can be rewritten as $\dfrac{f(x_A + h) - f(x_\mathrm{A})}{h}$.

Example 1.3

Use the formula $\dfrac{f(x_1 + h) - f(x_1)}{h}$ to determine the average gradient of the curve $f(x) = x(x + 3)$ between $x = 5$ and $x = 3$.

Solution

Method 1:

$$
\begin{aligned}
f(x) &= x^2 + 3x \\
f(x_1 + h) &= f(5 + h) \\
&= (5 + h)^2 + 3(5 + h) \\
&= 25 + 10h + h^2 + 15 + 3h \\
&= h^2 + 13h + 40
\end{aligned}
$$

$$f(5) = 5^2 + 3(5)$$
$$= 40$$

$$\begin{aligned}\frac{f(x_1+h)-f(x_1)}{h} &= \frac{h^2+13h+40-40}{h}\\ &= \frac{h^2+13h}{h}\\ &= \frac{h(h+13)}{h}\\ &= h+13\end{aligned}$$

Find h

$$\begin{aligned}h &= x_B - x_A\\ &= 3-5\\ &= -2\end{aligned}$$

Substitute the value of $h=-2$ into the average gradient expression and the average gradient is $-2+13$, which is equal to 11.

Method 2:

We can rewrite the average gradient formula as: $\frac{f(a+h)-f(a)}{(a+h)-a} = \frac{f(a+h)-f(a)}{h}$ where a is the x-value we start from, and $h = x_B - x_A$ as in method 1.

Find an expression for the average gradient.

$$\begin{aligned}f(x) &= x^2+3x\\ \frac{f(a+h)-f(a)}{h} &= \frac{(a+h)^2+3(a+h)-(a^2+3a)}{h}\\ &= \frac{a^2+2ah+h^2+3a+3h-a^2-3a}{h}\\ &= \frac{2ah+h^2+3h}{h}\\ &= \frac{\cancel{h}(2a+h+3)}{\cancel{h}}\\ &= 2a+h+3\end{aligned}$$

Note: we start at $x_1 = a = 5$ therefore substitute $a = 5$.

$$\begin{aligned}h &= x_B - x_A\\ &= 3-5\\ &= -2\end{aligned}$$

$$\begin{aligned}\text{Average gradient} &= 2a+h+3\\ &= 2(5)-2+3\\ &= 11\end{aligned}$$

The average rate of change gives the average or net change of a function over an interval and is useful, for example, in the study of climate change, population growth and economics.

Exercise 1.2

Use the formula $\frac{f(a+h)-f(a)}{h}$ to find the average gradient of $f(x)=x^2-x-2$ between the points with the following x-coordinates:

1. $x_1=2$ and $x_2=5$
2. $x_1=5$ and $x_2=2$
3. $x_1=-2$ and $x_2=-1$

The full solutions are at the end of the unit.

Note

This 'Applet' gives an interactive demonstration of gradient between two points.

Gradient at a point

The way a function changes at a single point has even more uses than the average rate of change between two points. The way a function changes at a particular instant is known as the instantaneous rate of change at the point. Limits are used to calculate the gradient of a function at a specific point on a curve.

Make sure you have revised limits of a function (level 3 subject outcome 2.5 unit 1) where the limit is defined. We saw in level 3 subject outcome 2.5 unit 2 that the gradient of any curve at any point is the same as the gradient of the tangent at that point.

Let's revise what you learnt in level 3 subject outcome 2.5 unit 2 activity 2.1.

Imagine that there is a point B that moves closer and closer to a fixed point $\mathrm{A}(1,6)$ as shown in figure 2.

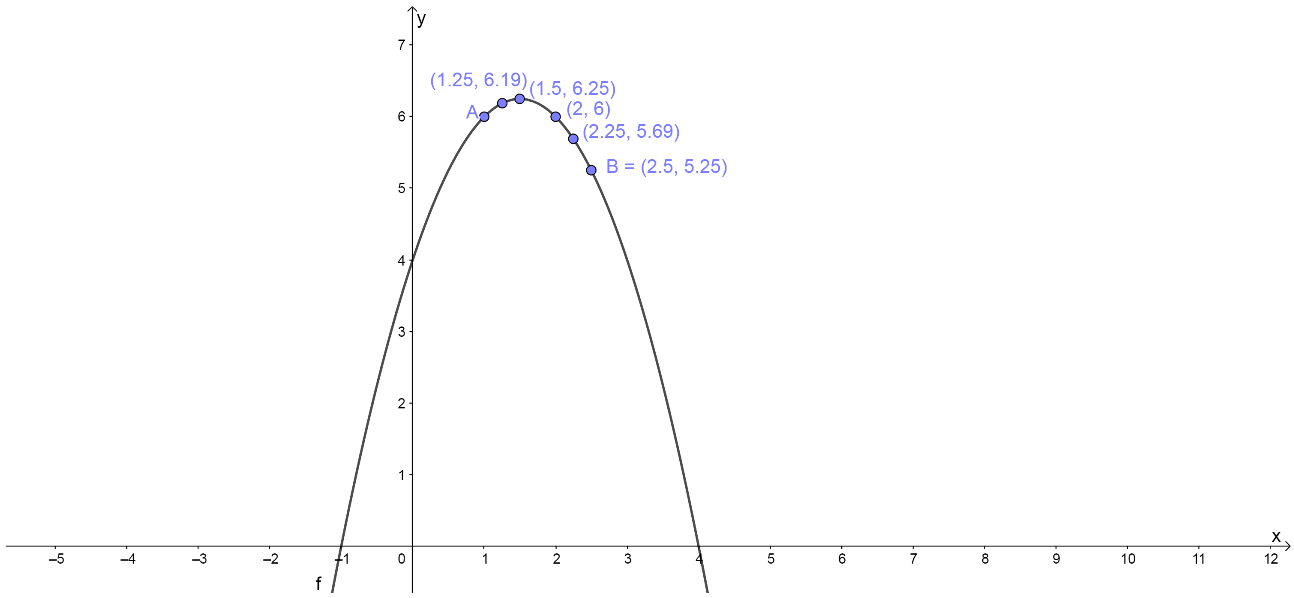

Figure 2: Gradient at a point

We can find the average gradient between any two points as B moves closer to A. This is shown in the following table:

Points	**Average gradient**
$\text{A}(1,6)$ and $(2.5,\ 5.25)$	$m = \dfrac{5.25-6}{2.5-1}$ $= -0.5$
$\text{A}(1,6)$ and $(2.25,\ 5.69)$	$m = \dfrac{5.69-6}{2.25-1}$ $= -0.25$
$\text{A}(1,6)$ and $(2,\ 6)$	$m = \dfrac{6-6}{2-1}$ $= 0$
$\text{A}(1,6)$ and $(1.5,\ 6.25)$	$m = \dfrac{6.25-6}{1.5-1}$ $= 0.5$
$\text{A}(1,6)$ and $(1.25,\ 6.19)$	$m = \dfrac{6.19-6}{1.25-1}$ $= 0.76$

If h is the distance between the x-values at point A and point B then h approaches 0 as B moves closer and closer to A. As point B approaches point A the average gradient changes as shown in the table and graph below:

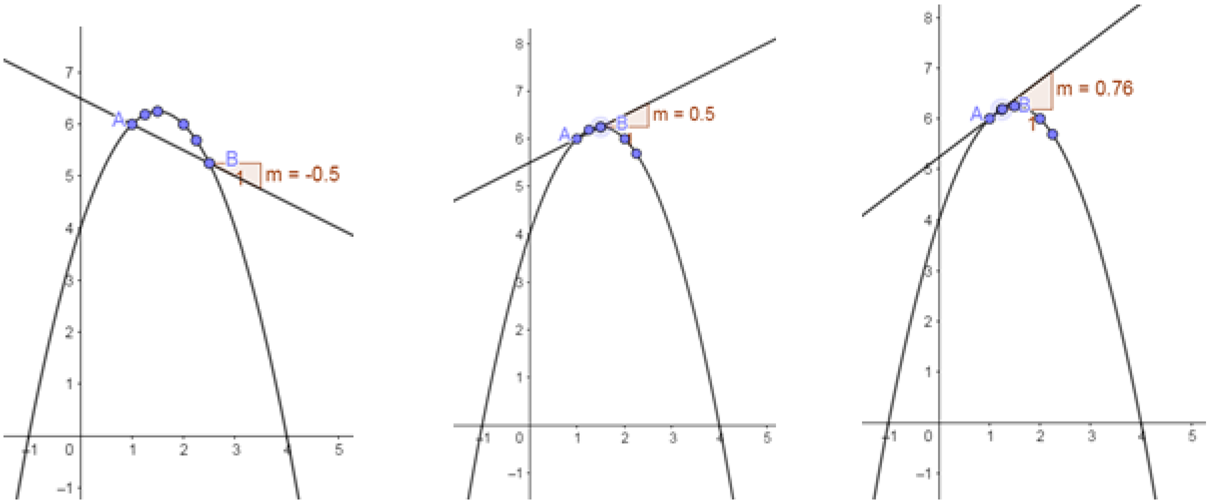

Figure 3: Average gradient changes as B moves closer to A.

At the point where A and B overlap, the straight line passes through only one point on the curve. This line is known as a **tangent** to the curve. Therefore, the gradient at a point on a curve is the same as the gradient of the tangent to the curve at the given point.

We see that as point B approaches point A, h gets closer to 0. If point B lies on point A, then $h = 0$ and the formula for average gradient is undefined.

We use our knowledge of limits to let h tend to 0 and determine the gradient of the curve at point A. The gradient at point A as $\lim_{h \to 0}$ is given by:

$$\begin{aligned} m_A &= \lim_{h \to 0} \frac{f(a+h) - f(a)}{a + h - a} \\ &= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \end{aligned}.$$

Note

When you have access to the internet you can click on this link to view an interactive graph, which shows how the gradient of the straight line through points A and B changes as B moves closer to A.

Example 1.4

Given $f(x) = -2x^2$, determine the gradient of the curve at point A where $x = 3$.

Solution

The expression below is used to find the average gradient.

$$\begin{aligned} \frac{f(a+h)-f(a)}{(a+h)-a} &= \frac{-2(a+h)^2-(-2a)^2}{h} \\ &= \frac{-2a^2-4ah-2h^2+2a^2}{h} \\ &= \frac{-4ah-2h^2}{h} \\ &= \frac{h(-4a-2h)}{h} \\ &= -4a-2h \end{aligned}$$

Now, to find the gradient at point A we must add in the limit.

$m_A = \lim_{h \to 0} (-4a - 2h)$

At point A: $x = 3$ therefore $a = 3$ as this is the point where we want to find the gradient. **At that point on the curve** $h = 0$ so the gradient is:

$-4(3) - 2(0) = -12.$

The gradient of a curve at any point is called the derivative of the function. As we have seen, the derivative of a function at a given point gives us the slope of the tangent line to the function at that point.

Example 1.5

Given the function $f(x) = 3x^2 + x - 2$, determine the gradient of the tangent to the curve at the point $x = 1$.

Solution

Step 1: Write down the formula for the gradient at a point.

$$\lim_{h \to 0} \frac{f(a+h)-f(a)}{(a+h)-a}$$

Step 2: Find $f(a+h)$ and $f(a)$.

We need to find the gradient of the tangent to the curve at $x = 1$, therefore we let $a = 1$:

$$\begin{aligned} f(a+h) &= f(1+h) \\ &= 3(1+h)^2 + (1+h) - 2 \\ &= 3(1+2h+h^2) + h - 1 \\ &= 3 + 6h + 3h^2 + h - 1 \\ &= 3h^2 + 7h + 2 \end{aligned}$$

$$\begin{aligned} f(a) &= f(1) \\ &= 3(1)^2 + 1 - 2 \\ &= 2 \end{aligned}$$

Step 3: Substitute into the formula and simplify.

$$\begin{aligned} \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} &= \lim_{h \to 0} \frac{3h^2 + 7h + 2 - (2)}{h} \\ &= \lim_{h \to 0} \frac{3h^2 + 7h}{h} \\ &= \lim_{h \to 0} \frac{\cancel{h}(3h+7)}{\cancel{h}} \\ &= 7 \end{aligned}$$

Step 4: Write the final answer.

The gradient of the tangent to the curve $f(x) = 3x^2 + x - 2$ at the point $x = 1$ is 7.

Exercise 1.3

1. Given $f(x) = x^2 + x$, determine the gradient of the curve at point A where $x = -1$.
2. Given the function $g(x) = x^2 - 2$, determine the gradient of the tangent to the curve at the point $x = 3$.

The full solutions are at the end of the unit.

Differentiation from first principles

We know that the gradient of the tangent to a curve with equation $y = f(x)$ at $x = a$ can be determined using the formula: $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$.

We can use this formula to determine an expression that describes the gradient of the graph (or the gradient of the tangent to the graph) at any point on the graph.

The derivative of a function $f(x)$ is written as $f'(x)$ and is defined as: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

We use the formula $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ to find the derivative of a function. This method is called differentiation from first principles, or using the definition.

Example 1.6

Calculate the derivative of $f(x) = 4x - 5$ from first principles.

Solution

Step 1: Write down the formula for finding the derivative using first principles.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Step 2: Determine $f(x+h)$.

$$\begin{aligned} f(x+h) &= 4(x+h) - 5 \\ &= 4x + 4h - 5 \end{aligned}$$

Step 3: Substitute into the formula and simplify.

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{4x + 4h - 5 - (4x - 5)}{h} \\ &= \lim_{h \to 0} \frac{4x + 4h - 5 - 4x + 5}{h} \\ &= \lim_{h \to 0} \frac{4h}{h} \\ &= 4 \end{aligned}$$

Step 4: Write the final answer.

$$f'(x) = 4$$

Example 1.7

Given $f(x) = 2x^3$:

1. Find the derivative of $f(x)$.
2. Determine $f'(-1)$ and interpret the answer.

Solutions

Step 1: Write down the formula for finding the derivative from first principles.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Step 2: Substitute into the formula and simplify.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{2(x+h)^3 - 2x^3}{h} \\ &= \lim_{h\to 0} \frac{2(x^3 + 3x^2h + 3xh^2 + h^3) - 2x^3}{h} \\ &= \lim_{h\to 0} \frac{2x^3 + 6x^2h + 6xh^2 + 2h^3 - 2x^3}{h} \\ &= \lim_{h\to 0} \frac{6x^2h + 6xh^2 + 2h^3}{h} \\ &= \lim_{h\to 0} \frac{\cancel{h}(6x^2 + 6xh + 2h^2)}{\cancel{h}} \\ &= 6x^2 \end{aligned}$$

Step 3: Calculate $f'(-1)$ and interpret the answer.

$$\begin{aligned} f'(-1) &= 6(-1)^2 \\ &= 6 \end{aligned}$$

The gradient of the tangent to the curve $f(x)$ at the point $x = -1$ is 6.

OR

The gradient of the function $f(x)$ at $x = -1$ is 6.

Exercise 1.4

1. Find the derivative of $g(x) = \frac{1}{x}$ from first principles.
2. Use the definition to find $f'(x)$ of $f(x) = 2x^2 - 2x + 1$.
3. Determine $f'(2)$ from first principles if $f(x) = -3x^2$.

The full solutions are at the end of the unit.

Summary

In this unit you have learnt the following:

- How to find the limit of a function.
- How limits are applied to average gradient to find the derivative of a function.
- How to find the gradient of a curve at a point.
- How to find the derivative by using first principles.

Unit 1: Assessment

Suggested time to complete: 30 minutes

1. Determine: $\lim\limits_{x \to -3} \dfrac{(x+3)}{x^2-9}$.
2. Determine: $\lim\limits_{x \to 2} \dfrac{\sqrt{x}-\sqrt{2}}{x-2}$.
3. Use first principles to find $f'(x)$ when $f(x) = 9$.
4. Given $g(x) = \dfrac{2}{x}$, find $g'(x)$ using the definition.
5. Given $f(x) = \dfrac{1}{x} - 2$, find $f'(x)$ using the definition of the derivative.
6. If $f(x) = \dfrac{1}{2}x^3$:
    a. Determine $f'(x)$ from first principles.
    b. Calculate the gradient of the tangent to $f(x)$ at the point $x = -2$.

The full solutions are at the end of the unit.

Unit 1: Solutions

Exercise 1.1

1.

$$\begin{aligned} \lim_{h \to 0} \frac{h^2+4h}{h} &= \lim_{h \to 0} \frac{h(h+4)}{h} \\ &= 4 \end{aligned}$$

2.

$$\begin{aligned} \lim_{h \to 0} \frac{(2+h)^2-4}{h} &= \lim_{h \to 0} \frac{4+4h+h^2-4}{h} \\ &= \lim_{h \to 0} \frac{\cancel{h}(4+h)}{\cancel{h}} \\ &= 4 \end{aligned}$$

3.

$$\lim_{x\to\infty}\frac{2x^2+1}{x^2+1}=\lim_{x\to\infty}\frac{\frac{2x^2}{x^2}+\frac{1}{x^2}}{\frac{x^2}{x^2}+\frac{1}{x^2}}$$
$$=\lim_{x\to\infty}\frac{2+\frac{1}{x^2}}{1+\frac{1}{x^2}}$$
$$=\frac{2+0}{1+0}$$
$$=2$$

(If the degree of denominator is equal to the numerator, then divide the coefficients of the terms with the highest powers.)

Back to Exercise 1.1

Exercise 1.2

Find an expression for the average gradient that you will use to answer the questions.

$$\frac{f(a+h)-f(a)}{h}=\frac{(a+h)^2-(a+h)-2}{h}$$
$$=\frac{a^2+2ah+h^2-a-h-2-(a^2-a-2)}{h}$$
$$=\frac{2ah+h^2-h}{h}$$
$$=\frac{\cancel{h}(2a+h-1)}{\cancel{h}}$$
$$=2a+h-1$$

1. $$\begin{aligned}\text{Average gradient}&=2(2)+3-1\\&=6\end{aligned}$$
2. $$\begin{aligned}\text{Average gradient}&=2(5)+(-3)-1\\&=6\end{aligned}$$
3. $$\begin{aligned}\text{Average gradient}&=2(-2)+1-1\\&=-4\end{aligned}$$

Back to Exercise 1.2

Exercise 1.3

1.

$$\begin{aligned}\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} &= \lim_{h\to 0}\frac{(h-1)^2+(h-1)-[(-1)^2+(-1)]}{h}\\ &= \lim_{h\to 0}\frac{h^2-2h+1+h-1-(0)}{h}\\ &= \lim_{h\to 0}\frac{\cancel{h}(h-1)}{\cancel{h}}\\ &= -1\end{aligned}$$

2.

$$\begin{aligned}\lim_{h\to 0}\frac{g(a+h)-g(a)}{h} &= \lim_{h\to 0}\frac{(3+h)^2-2-[(3)^2-2]}{h}\\ &= \lim_{h\to 0}\frac{9+6h+h^2-2-(7)}{h}\\ &= \lim_{h\to 0}\frac{\cancel{h}(6+h)}{\cancel{h}}\\ &= 6\end{aligned}$$

The gradient of the tangent to the curve at the point $x=3$ is 6.

Back to Exercise 1.3

Exercise 1.4

1.

$$\begin{aligned}\lim_{h\to 0}\frac{g(x+h)-g(x)}{h} &= \lim_{h\to 0}\frac{\frac{1}{x+h}-\frac{1}{x}}{h}\\ &= \lim_{h\to 0}\frac{\frac{x-(x+h)}{x(x+h)}}{h}\\ &= \lim_{h\to 0}\frac{x-(x+h)}{xh(x+h)} \quad \text{Rewrite denominator}\\ &= \lim_{h\to 0}\frac{-\cancel{h}}{x\cancel{h}(x+h)}\\ &= \frac{-1}{x^2}\end{aligned}$$

2.

$$\begin{aligned}\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0}\frac{2(x+h)^2-2(x+h)+1-(2x^2-2x+1)}{h}\\ &= \lim_{h\to 0}\frac{2(x^2+2xh+h^2)-2x-2h+1-2x^2+2x-1}{h}\\ &= \lim_{h\to 0}\frac{2x^2+4xh+2h^2-2h-2x^2}{h}\\ &= \lim_{h\to 0}\frac{4xh+2h^2-2h}{h}\\ &= \lim_{h\to 0}\frac{\cancel{h}(4x+2h-2)}{\cancel{h}}\\ &= 4x-2\end{aligned}$$

3.

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{-3(x+h)^2-(-3x^2)}{h}$$
$$= \lim_{h\to 0}\frac{-3x^2-6xh-3h^2+3x^2}{h}$$
$$= \lim_{h\to 0}\frac{-6xh-3h^2}{h}$$
$$= \lim_{h\to 0}\frac{\cancel{h}(-6x-3h)}{\cancel{h}}$$
$$= -6x$$
$$f'(2) = -12$$

Back to Exercise 1.4

Unit 1: Assessment

1.

$$\lim_{x\to -3}\frac{(x+3)}{x^2-9} = \lim_{x\to -3}\frac{\cancel{(x+3)}}{(x-3)\cancel{(x+3)}}$$
$$= \frac{1}{-6}$$

2.

$$\lim_{x\to 2}\frac{\sqrt{x}-\sqrt{2}}{x-2} = \lim_{x\to 2}\frac{\cancel{(\sqrt{x}-\sqrt{2})}}{\cancel{(\sqrt{x}-\sqrt{2})}(\sqrt{x}+\sqrt{2})}$$
$$= \frac{1}{2\sqrt{2}}$$

3.

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$= \frac{9-9}{h}$$
$$= 0$$

4.

$$g'(x) = \lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$
$$= \lim_{h\to 0}\frac{\frac{2}{x+h}-\frac{2}{x}}{h}$$
$$= \lim_{h\to 0}\frac{\frac{2x-2(x+h)}{x(x+h)}}{h}$$
$$= \lim_{h\to 0}\frac{2x-2x-2h}{xh(x+h)}$$
$$= \lim_{h\to 0}\frac{-2}{x(x+h)}$$
$$= \lim_{h\to 0}\frac{-2}{x^2+xh}$$
$$= \frac{-2}{x^2}$$

5.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\frac{1}{x+h} - 2 - \left(\frac{1}{x} - 2\right)}{h} \\
&= \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} \\
&= \lim_{h\to 0} \frac{x-x-h}{xh(x+h)} \\
&= \frac{-1}{x^2}
\end{aligned}$$

6.

a.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\frac{1}{2}(x^3+3x^2h+3xh^2+h^3) - \frac{1}{2}x^3}{h} \\
&= \lim_{h\to 0} \frac{\frac{1}{2}x^3+\frac{3}{2}x^2h+\frac{3}{2}xh^2+\frac{1}{2}h^3-\frac{1}{2}x^3}{h} \\
&= \lim_{h\to 0} \frac{3}{2}x^2+\frac{3}{2}xh+\frac{1}{2}h^2 \\
&= \frac{3}{2}x^2
\end{aligned}$$

b.

The gradient of the tangent to $f(x)$at the point $x=-2$ is $\frac{3}{2}(-2)^2=6$.

Back to Unit 1: Assessment

Media Attributions

Unit 2: Work with rules for differentiation

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Unit outcomes

By the end of this unit you will be able to:

- Use various forms of notation to represent the derivative.
- Apply the power rule.
- Find the derivative of a constant.
- Find the derivative of a constant multiplied by a function.
- Find the derivative of a sum/difference.
- Find the derivative of a product.
- Find the derivative of a quotient.
- Apply the chain rule.

What you should know

Before you start this unit, make sure you can:

- Find the derivative by applying the constant and sum and difference rules taught in level 3 subject outcome 2.5 unit 3.

Try the following questions to see if you are ready for this unit.
Use the rules of differentiation to find the derivative of the following:

1. $g(x) = 3x^{\frac{2}{3}} - 4x + 10$
2. $f(x) = \dfrac{x^2 - 5x + 6}{x - 3}$
3. $y = \sqrt{x^3} + \dfrac{1}{3x^3}$

Solutions

1.

$$\begin{aligned} g(x) &= 3x^{\frac{2}{3}} - 4x + 10 \\ g'(x) &= 3 \times \frac{2}{3} x^{\frac{2}{3} - 1} - 4 \\ &= \frac{2}{x^{\frac{1}{3}}} - 4 \end{aligned}$$

2.

$$
\begin{aligned}
f(x) &= \frac{x^2 - 5x + 6}{x - 3} && \text{Factorise and cancel} \\
&= \frac{(x-3)(x-2)}{(x-3)} \\
&= x - 2 \\
f'(x) &= 1x^{1-1} = 1
\end{aligned}
$$

3.

$$
\begin{aligned}
y &= \sqrt{x^3} + \frac{1}{3x^3} \\
&= x^{\frac{3}{2}} + \frac{1}{3}x^{-3} && \text{Get rid of roots and variables in denominator} \\
y' &= \frac{3}{2}x^{\frac{1}{2}} - x^{-4} \\
&= \frac{3}{2}x^{\frac{1}{2}} - \frac{1}{x^4}
\end{aligned}
$$

Introduction

Finding derivatives of functions by using first principles can be tedious. So, we use rules for differentiation to make the process of differentiating simpler.

As we saw in level 3 there are different notations used for derivatives. If we use the common notation $y = f(x)$ where the dependent variable is y and the independent variable is x, then some alternative notations for the derivative are:

$$f'(x) = y' = \frac{dy}{dx} = \frac{d}{dx}[f(x)] = Df(x) = D_x y$$

The symbols D and $\frac{d}{dx}$ are called differential operators because they tell us that we are differentiating whenever see them.

Here is a reminder of the rules of differentiation that you should know by now.

The power rule:

If $f(x) = x^n$ then $f'(x) = nx^{n-1}$ where $n \in \mathbb{R},\ \mathrm{n} \neq 0.$

The sum/difference rule:

$$\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x)$$

In this unit we build on the differentiation rules you learnt in level 3 to include the product, quotient and chain rules. But, before we look at those there are some derivative functions that need a special mention.

The exponential and natural logarithm functions

The constant e is a special type of exponent, known as Euler's number, which has many real-life applications. $f(x) = e^x$ is called the natural exponential function, and it is its own derivative.

$$\frac{d}{dx}e^x = e^x$$

Note

You can learn more about the uses of the natural exponent, e, by watching the video "e and Compound Interest".

[e and Compound Interest] (Duration: 11.38)

The inverse of an exponential function is a logarithmic function. The inverse of the natural exponential function is the natural logarithmic function $\ln x$. The base of the natural logarithmic function is e, so $\ln x$ is actually $\ln_e x$. The derivative of the natural logarithm function is $\frac{1}{x}$.

$$\frac{d}{dx}\ln x = \frac{1}{x} \text{ where } x > 0, x \neq 0$$

Derivatives of trigonometric functions

The trigonometric functions have the following derivatives:

$$\frac{d}{dx}\sin x = \cos x$$

$$\frac{d}{dx}\cos x = -\sin x$$

$$\frac{d}{dx}tanx = \sec^2 x$$

Example 2.1

Find the derivative of:

1. e^x
2. $y = \ln x$
3. $2\sin x + 2\cos x$
4. $f(x) = \tan x$

Solutions

1. $$\frac{d}{dx}e^x = e^x$$
2. $$y' = \frac{1}{x}$$
3. $$\begin{aligned}\frac{d}{dx}(2\sin x + 2\cos x) &= \frac{d}{dx}2\sin x + \frac{d}{dx}2\cos x \qquad \text{apply the sum rule for differentiation}\\ &= 2\cos x - 2\sin x\end{aligned}$$
4. $$f'(x) = \sec^2 x$$

The product rule

Now, we are ready for some of the more advanced rules of differentiation. Using the **product rule** we can find the derivative of the product of two functions.

$$\frac{d}{dx}[f(x) \cdot g(x)] = f'(x) \cdot g(x) + g'(x) \cdot f(x)$$

OR

If u and v are functions of x, and $y = u \cdot v$, then:

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}.$$

Example 2.2

Differentiate with respect to x:

1. $h(x) = x \cdot \cos x$
2. $y = x^2 \cdot e^x$

3. $y = \ln x \cdot e^x$

Solutions

1. $h(x) = x \cdot \cos x$

   $\therefore f(x) = x \text{ and } g(x) = \cos x$

   $$\begin{aligned}\frac{d}{dx}[h(x)] &= \frac{d}{dx}[f(x) \cdot g(x)] \\ &= f'(x) \cdot g(x) + g'(x) \cdot f(x) && \text{by the product rule} \\ &= (\frac{d}{dx}x) \cos x + (\frac{d}{dx}\cos x)x \\ &= \cos x + (-\sin x)x \\ &= \cos x - x \sin x\end{aligned}$$

2. $y = x^2 \cdot e^x$

   $u = x^2 \text{ and v} = e^x$

   $$\begin{aligned}\frac{dy}{dx} &= \left(\frac{d}{dx}x^2\right) e^x + \left(\frac{d}{dx}e^x\right) x^2 \\ &= 2x \cdot e^x + e^x \cdot x^2\end{aligned}$$

3. $y = \ln x \cdot e^x$

   $u = \ln x \text{ and } v = e^x$

   $$\begin{aligned}y' &= \left(\frac{d}{dx}\ln x\right) e^x + \left(\frac{d}{dx}e^x\right) \ln x \\ &= \frac{1}{x} \cdot e^x + e^x \cdot \ln x\end{aligned}$$

Exercise 2.1

Use the product rule to differentiate with respect to x:

1. $y = x \cdot \sqrt{x}$
2. $y = e^x \sin x$
3. $k(x) = \ln x \cdot \tan x$
4. $h(x) = \dfrac{e^x}{x}$

The full solutions are at the end of the unit.

The quotient rule

Now that you understand the product rule, we will move onto differentiating quotients of functions. Remember that the answer to division of quantities is called a quotient. So the quotient rule is what we use to find the derivative of functions that are divided.

If $h(x)=\dfrac{f(x)}{g(x)}$ then $h'(x)=\dfrac{f'(x)g(x)-g'(x).f(x)}{(g(x))^2}$.

Example 2.3

Use the quotient rule to differentiate:

1. $h(x)=\dfrac{2x^2}{(x-3)}$
2. $y=\dfrac{3x-1}{2x+3}$

Solutions

1. $f(x)=2x^2$ and $g(x)=x-3$

   $$h'(x)=\frac{\frac{d}{dx}(2x^2)\cdot(x-3)-2x^2\cdot\frac{d}{dx}(x-3)}{(x-3)^2}$$

   Since the terms in the numerator are separated by a negative sign, you must be careful and use brackets to avoid mistakes.

   $$\begin{aligned}h'(x)&=\frac{4x(x-3)-2x^2(1)}{(x-3)^2}\\&=\frac{4x^2-12x-2x^2}{(x-3)^2}\\&=\frac{2x^2-12x}{(x-3)^2}\\&=\frac{2x(x-6)}{(x-3)^2}\end{aligned}$$

2. $f(x)=3x-1$ and $g(x)=2x+3$

   $$\begin{aligned}\frac{dy}{dx}&=\frac{(2x+3)\frac{d}{dx}(3x-1)-(3x-1)\frac{d}{dx}(2x+3)}{(2x+3)^2}\\&=\frac{(2x+3)(3)-(3x-1)(2)}{(2x+3)^2}\\&=\frac{6x+9-6x+2}{(2x+3)^2}\\&=\frac{11}{(2x+3)^2}\end{aligned}$$

Exercise 2.2

Use the quotient rule to differentiate:

1. $h(x) = \dfrac{e^x}{(x-1)}$
2. $y = \dfrac{\sin x}{\tan x}$
3. $g(x) = \dfrac{x}{e^x}$

The full solutions are at the end of the unit.

The chain rule

So far we have differentiated functions using the following rules for differentiation: sum and difference rule, constant rule, power rule, product rule and quotient rule. However, these techniques do not allow us to differentiate compositions of functions, such as $y = \sin(x^2)$. To differentiate composite functions we must use the chain rule. But, what are composite functions?

Note

You can watch this animation of the chain rule for an 'intuitive notion of the chain rule'.

$y = f(g(x))$ is a composite function or a function of a function. Consider the function $\sin x^2$. You should be able to see that this expression is different from the normal $\sin x$ function since x is squared.

This is a function of function with $f(x) = \sin x$ and $g(x) = x^2$.

So $f(g(x)) = f(x^2)$.

Substitute x with x^2 and you get:

$f(g(x)) = f(x^2) = \sin x^2$.

We can think of the derivative of this function as the rate of change of $\sin(x^2)$ relative to the change in x. We want to know how $\sin(x^2)$ changes as x changes. Any change in x leads a chain reaction; as x changes, x^2 changes , which leads $\sin(x^2)$ to change too.

To differentiate a composite function $y = f(g(x))$:

1. Substitute u for $g(x)$ to get $y = f(u)$. Soy is a function of u and u is a function of x.
2. Apply the chain rule.

Chain rule:

If y is a function of u and u is a function of x, then $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$.

When applying the chain rule to the composition of functions, we work our way from the outside function in.

Example 2.4

Differentiate $y = \sin(x^2)$.

Solution

Let $u = x^2$ then $y = \sin(u)$.

$\frac{du}{dx} = 2x$ and $\frac{dy}{du} = \cos u$

From the chain rule $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$.

$$\begin{aligned}\frac{dy}{dx} &= \cos u \cdot (2x) \\ &= 2x \cos x^2\end{aligned}$$

In the last step we substituted $u = x^2$.

We can expand on the basic trigonometric derivative rules to include composite functions as shown in the box below.

$$\frac{d}{dx} a \sin kx = k \cdot a \cos kx$$

$$\frac{d}{dx} a \cos kx = -k \cdot a \sin kx$$

$$\frac{d}{dx} a \ \tan nx = k \cdot a\sec^2 kx$$

Example 2.5

Find the derivative of $y = (2x - 1)^5$.

Solution

You may be tempted to say just multiply out the brackets to derive, but to multiply out the brackets would take a long time and it is easy to make a mistake with so many terms. So we will treat this as a function of a function. Let $u = 2x - 1$ then $y = u^5$.

Then $\frac{du}{dx} = 2$ and $\frac{dy}{du} = 5x^4$.

From the chain rule $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$.

So we get $\begin{aligned} \frac{dy}{dx} &= 5x^4 \cdot (2) \\ &= 10x^4 \end{aligned}$.

Here is a summary of the rules of differentiation for simple functions and composite functions.

Simple functions	**Composite functions**
$\frac{d}{dx}k = 0, \text{ where k is a constant.}$	
$\frac{d}{dx}kx^n = n \cdot kx^{n-1}, \text{where k is a constant.}$	$\frac{d}{dx}k[f(x)]^n = k\frac{d}{dx}[f(x)]^n, \quad \text{where k is a constant.}$ $= k \cdot n[f(x)]^{n-1} \cdot f'(x)$
$\frac{d}{dx}e^x = e^x$	$\frac{d}{dx}e^{f(x)} = f'(x) \cdot e^{f(x)}$
$\frac{d}{dx}\ln x = \frac{1}{x}$	$\frac{d}{dx}\ln[f(x)] = \frac{1}{f(x)} \cdot f'(x)$
$\frac{d}{dx}\sin x = \cos x$	$\frac{d}{dx}\sin f(x) = f'(x) \cdot \cos f(x)$
$\frac{d}{dx}\cos x = -\sin x$	$\frac{d}{dx}\cos f(x) = -\sin f(x) \cdot f'(x)$
$\frac{d}{dx}\tan x = \sec^2 x$	$\frac{d}{dx}\tan f(x) = f'(x) \cdot \sec^2 f(x)$

Exercise 2.3

Differentiate the following functions:

1. $f(x) = \sqrt{x^2 + 1}$
2. $g(x) = 3\tan 2x$
3. $y = e^{\sin^2(2x)}$
4. $s = \cos e^{3t}$
5. $h(x) = \ln(\sin x)$

The full solutions are at the end of the unit.

Summary

In this unit you have learnt the following:

- How to apply the product rule for differentiation.
- How to apply the quotient rule for differentiation.
- How to define composite functions.
- How to apply the chain rule.

Unit 2: Assessment

Suggested time to complete: 35 minutes

Differentiate with respect to x:

1. $y = \frac{1}{x}(2x+3)$
2. $f(x) = \cos 2x$
3. $y = \ln 3x$
4. $y = \sqrt{x-1}$
5. $g(x) = e^x \cdot \ln 3x$
6. $h(x) = e^{2x} \cdot x$
7. $y = x \tan 2x$
8. $y = \frac{e^{2x}}{e^x - e^{-x}}$
9. $y = \frac{x}{\sqrt{2x^2+1}}$

The full solutions are at the end of the unit.

Unit 2: Solutions

Exercise 2.1

1.

$$\begin{aligned} y' &= \frac{d}{dx}x \cdot x^{\frac{1}{2}} + \frac{d}{dx}x^{\frac{1}{2}} \cdot x \\ &= x^{\frac{1}{2}} + \frac{1}{2}x^{-\frac{1}{2}} \cdot x \\ &= x^{\frac{1}{2}} + \frac{1}{2}x^{\frac{1}{2}} \\ &= \frac{3}{2}\sqrt{x} \end{aligned}$$

2.

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}e^x \cdot \sin x + \frac{d}{dx}\sin x \cdot e^x \\ &= e^x \cdot \sin x + \cos x \cdot e^x\end{aligned}$$

3.

$$\begin{aligned}k(x) &= \ln x \cdot \tan x \\ k'(x) &= \frac{d}{dx}\ln x \cdot \tan x + \ln x \cdot \frac{d}{dx}\tan x \\ &= \frac{\tan x}{x} + \ln x \sec^2 x\end{aligned}$$

4.

$$\begin{aligned}h(x) &= \frac{e^x}{x} = e^x \cdot x^{-1} \\ h'(x) &= \frac{d}{dx}e^x \cdot x^{-1} + \frac{d}{dx}x^{-1} \cdot e^x \, [] \\ &= \frac{e^x}{x} + (-x^{-2})e^x \\ &= \frac{e^x}{x} - \frac{e^x}{x^2}\end{aligned}$$

Back to Exercise 2.1

Exercise 2.2

1.

$$\begin{aligned}h'(x) &= \frac{(x-1)\frac{d}{dx}e^x - e^x\frac{d}{dx}(x-1)}{(x-1)^2} \\ &= \frac{(x-1)e^x - e^x(1)}{(x-1)^2} \\ &= \frac{xe^x - e^x - e^x}{(x-1)^2} \\ &= \frac{xe^x - 2e^x}{(x-1)^2} \\ &= \frac{e^x(x-2)}{(x-1)^2}\end{aligned}$$

2.

$$
\begin{aligned}
y' &= \frac{\tan x \dfrac{d}{dx}\sin x - \sin x \dfrac{d}{dx}\tan x}{(\tan x)^2} \\
&= \frac{\tan x(\cos x) - \sin x(\sec^2 x)}{(\tan x)^2} && \text{simplify using trig identities} \\
&= \frac{\dfrac{\sin x}{\cos x}\dfrac{(\cos x)}{1} - \dfrac{\sin x}{\cos^2 x}}{(\dfrac{\sin x}{\cos x})^2} && \text{find an LCD in the numerator and subtract terms} \\
&= \frac{\sin x \cos^2 x - \sin x}{\cancel{\cos^2 x}} \cdot \frac{\cancel{\cos^2 x}}{\sin^2 x} \\
&= \frac{-\sin x(1-\cos^2 x)}{\sin^2 x} \\
&= \frac{-\sin^3 x}{\sin^2 x} \\
&= -\sin x
\end{aligned}
$$

3.

$$
\begin{aligned}
g'(x) &= \frac{e^x \dfrac{d}{dx}x - x\dfrac{d}{dx}e^x}{(e^x)^2} \\
&= \frac{e^x(1) - x(e^x)}{(e^x)^2} \\
&= \frac{e^x(1-x)}{e^{2x}} \\
&= \frac{1-x}{e^x}
\end{aligned}
$$

Back to Exercise 2.2

Exercise 2.3

Determine $\dfrac{dy}{dx}$ for the following functions.

1.

$$
\begin{aligned}
f(x) &= (x^2+1)^{\frac{1}{2}} \\
f'(x) &= \frac{1}{2}(x^2+1)^{-\frac{1}{2}} \cdot \frac{d}{dx}(x^2+1) \\
&= \frac{1}{2}(x^2+1)^{-\frac{1}{2}} \cdot (2x) \\
&= \frac{x}{\sqrt{x^2+1}}
\end{aligned}
$$

2.

$$
\begin{aligned}
g'(x) &= 3(2\sec^2 2x) \\
&= 6\sec^2 2x
\end{aligned}
$$

3.

$$
\begin{aligned}
y' &= e^{\sin^2(2x)} \cdot \frac{d}{dx}[\sin(2x)]^2 \\
&= e^{\sin^2(2x)} \cdot 2(\sin 2x)^1 \cdot \frac{d}{dx}\sin 2x \\
&= e^{\sin^2(2x)} \cdot 2(\sin 2x) \cdot (\cos 2x) \cdot \frac{d}{dx}(2x) \\
&= e^{\sin^2(2x)} \cdot 2(\sin 2x) \cdot (\cos 2x) \cdot 2 \\
&= 2e^{\sin^2(2x)} \cdot 2\sin 2x \cos 2x \text{ Note: } \sin 2A = 2\sin A \cos A \\
&= 2e^{\sin^2(2x)} \sin 4x
\end{aligned}
$$

4.

$$
\begin{aligned}
s' &= -\sin e^{3t} \cdot \frac{d}{dx}e^{3t} \\
&= -\sin e^{3t} \cdot 3e^{3t} \\
&= -3e^{3t} \sin e^{3t}
\end{aligned}
$$

5.

$$
\begin{aligned}
h'(x) &= \frac{1}{\sin x}(\cos x) \\
&= \frac{\cos x}{\sin x}
\end{aligned}
$$

Back to Exercise 2.3

Unit 2: Assessment

1.

$$
\begin{aligned}
y' &= \frac{d}{dx}x^{-1} \cdot (2x+3) + \frac{d}{dx}(2x+3) \cdot \frac{1}{x} \\
&= \frac{-(2x+3)}{x^2} + \frac{2}{x} \\
&= \frac{-2x-3+2x}{x^2} \\
&= \frac{-3}{x^2}
\end{aligned}
$$

2.

$$
f'(x) = -2\sin 2x
$$

3.

$$
\begin{aligned}
y' &= \frac{3}{3x} \\
&= \frac{1}{x}
\end{aligned}
$$

4.

$$
\begin{aligned}
y &= \sqrt{x-1} \\
\text{Let } u &= x-1 \\
\frac{du}{dx} &= 1 \\
y &= u^{\frac{1}{2}} \\
\frac{dy}{du} &= \frac{1}{2}u^{\frac{-1}{2}} \\
\therefore y' &= \frac{1}{2\sqrt{x-1}}
\end{aligned}
$$

5.

$$
\begin{aligned}
g'(x) &= e^x \cdot \ln 3x + \frac{3}{3x} \cdot e^x \\
&= e^x \cdot \ln 3x + \frac{e^x}{x}
\end{aligned}
$$

6.

$$
\begin{aligned}
h'(x) &= 2e^{2x} \cdot x + (1)e^{2x} \\
&= 2xe^{2x} + e^{2x}
\end{aligned}
$$

7.

$$
y' = \tan 2x + 2x\sec^2 2x
$$

8.

$$
\begin{aligned}
y' &= \frac{\frac{d}{dx}e^{2x}(e^x - e^{-x}) - \frac{d}{dx}(e^x - e^{-x})e^{2x}}{(e^x - e^{-x})^2} \\
&= \frac{2e^{2x}(e^x - e^{-x}) - e^{2x}(e^x + e^{-x})}{(e^x - e^{-x})^2} \\
&= \frac{2e^{3x} - 2e^x - e^{3x} - e^x}{(e^x - e^{-x})^2} \\
&= \frac{e^{3x} - 3e^x}{(e^x - e^{-x})^2}
\end{aligned}
$$

9.

$$
\begin{aligned}
y' &= \frac{\frac{d}{dx}x(\sqrt{2x^2+1}) - x\frac{d}{dx}(2x^2+1)^{\frac{1}{2}}}{(\sqrt{2x^2+1})^2} \\
&= \frac{\sqrt{2x^2+1} - x[\frac{1}{2}(2x^2+1)^{-\frac{1}{2}}(4x)]}{2x^2+1} \\
&= \frac{\sqrt{2x^2+1} - \frac{2x^2}{\sqrt{2x^2+1}}}{2x^2+1} \\
&= \frac{\left(\sqrt{2x^2+1}\right)^2 - 2x^2}{\sqrt{2x^2+1}} \cdot \frac{1}{2x^2+1} \\
&= \frac{-2x^2}{\sqrt{2x^2+1}}
\end{aligned}
$$

Back to Unit 2: Assessment

Unit 3: Find the equation of a tangent to a curve

NATASHIA BEARAM-EDMUNDS

Unit outcomes

By the end of this unit you will be able to:

- Define a tangent line.
- Determine the gradient of the tangent at a point.
- Find the equation of a tangent.

What you should know

Before you start this unit, make sure you can:

- Use the rules of differentiation to find the derivative. Revise differentiation rules in unit 2 of this subject outcome and level 3 subject outcome 2.5 unit 3.

Introduction

We have seen that the straight line that touches a curve at one point is a tangent to the curve at that point. In unit 1 we also saw that the gradient at a point on a curve is the same as the gradient of the tangent to the curve at the given point.

Gradients and tangents

If we zoom in on a tangent drawn to a curve, as shown in figure 1, we can see that the tangent touches the curve at a point A.

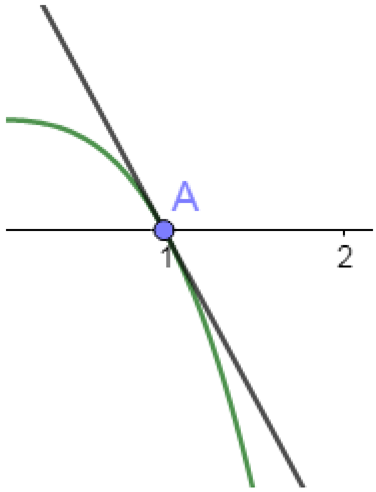

Figure 1: Tangent drawn to a curve

However, further away from point A it is possible for the straight line to cut the curve again, as shown in figure 2.

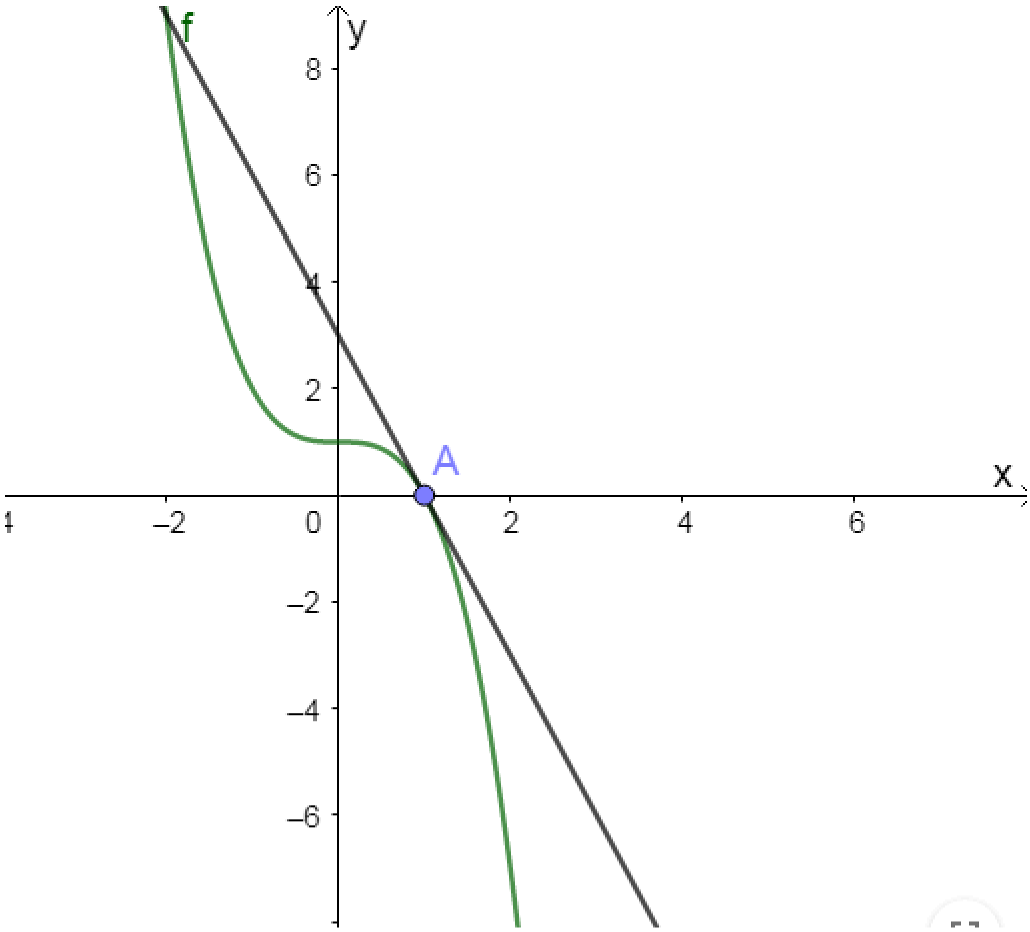

Figure 2 Tangent to a curve zoomed out

As we can see it is possible for the straight line to be tangential at point A on the curve but intersect the curve again at another point.

Take note!

The tangent to the curve $f(x)$ at the point $\mathrm{A}(x_1; y_1)$ is the straight line through A with gradient $m = f'(x_1)$.

Note

For a visual exploration of the tangent line to a curve at a point you can play around with the points found here when you have access to the internet.

Example 3.1

Find the gradient of the tangent to the curve $f(x) = -x^3 + 1$ at the point $\mathrm{A}(1; 0)$.

Solution

The gradient of the tangent at point A is the same as the gradient of the curve at that point.

So, we must find the gradient of $f(x)$ at point A and that will give us the gradient of the tangent to the curve.

To find an expression for the gradient of the curve we find the derivative.

$f'(x) = -3x^2$

Next, substitute the x-value of point A into the derivative and this gives the gradient of the curve at point A.

$$\begin{aligned} f'(1) &= -3(1)^2 \\ &= -3 \end{aligned}$$

Therefore, the gradient of the tangent to $f(x)$ at point $\mathrm{A}(1; 0)$ is also -3.

Example 3.2

Find the point B on the curve $f(x) = x^2 - 4$ at which the gradient of the tangent to the curve is 6.

Solution

The gradient of the tangent at point B is the same as the gradient of the curve. Therefore, the gradient of the curve is also 6 at point B.

$$f'(x) = 6$$
$$\therefore 2x = 6$$
$$x = 3$$

To find the y-value at B:

$$f(3) = (3)^2 - 4$$
$$= 5$$

Therefore, point B is $(3; 5)$.

Exercise 3.1

1. Find the gradient of the tangent to the curve $y = -x^3 + x - 2$ at the point $(1; -2)$.
2. Determine the point where the gradient of the tangent to the curve:
   a. $f(x) = 3x^2 - 2x - 1$ is equal to 3.
   b. $y = x^3 - x^2 - 5x + 2$ is equal to 0.
3. Determine the point(s) on the curve $f(x) = (2x - 1)^2$ where the tangent is parallel to the line $y = 4x - 2$.

The full solutions are at the end of the unit.

Equation of a tangent to a curve

A tangent line is a straight line, therefore its equation will be of the form $y = mx + c$. Remember that to find the equation of a straight line, you need its gradient m and the y-intercept c.

To determine the equation of a tangent to a curve:

1. Find the derivative using the rules of differentiation.
2. Substitute the x-coordinate of the given point into the derivative to calculate the gradient of the tangent.
3. Substitute the gradient of the tangent and the coordinates of the given point into an appropriate form of the straight line equation.
4. Make y the subject of the formula to write the equation in standard form.

Example 3.3

Find the equation of the tangent to the curve $f(x) = -x^3 + 1$ at the point $\mathrm{A}(1; 0)$.

Solution

Step 1: Find the derivative.

$f'(x) = -3x^2$

Step 2: Find the gradient of the tangent.

Substitute the x-coordinate of the given point into the derivative to calculate the gradient of the tangent.

$$\begin{aligned} f'(1) &= -3(1)^2 \\ &= -3 \end{aligned}$$

Step 3: Determine the equation of the tangent.

Substitute the gradient of the tangent and the coordinates of the given point into an appropriate form of the straight line equation.

$y = mx + c$

$m = -3$

Find the y-intercept c by substituting $\mathrm{A}(1; 0)$.

$$\begin{aligned} y &= -3x + c \\ 0 &= -3(1) + c \\ \therefore c &= 3 \end{aligned}$$

Step 4: Write the equation in standard form.

Make y the subject of the formula.

$y = -3x + 3$

Example 3.4

Given $g(x) = (x + 2)(2x + 1)^2$, determine the equation of the tangent to the curve at $x = -1$.

Solution

Step 1: Determine the y-coordinate of the point.

$$\begin{aligned} g(-1) &= (-1 + 2)(2(-1) + 1)^2 \\ &= 1 \end{aligned}$$

Therefore, the tangent to the curve passes through the point $(-1; 1)$.

Step 2: Expand and simplify the given function.

$$\begin{aligned} g(x) &= (x+2)(2x+1)^2 \\ &= (x+2)(4x^2+4x+1) \\ &= 4x^3+4x^2+x+8x^2+8x+2 \\ &= 4x^3+12x^2+9x+2 \end{aligned}$$

Step 3: Find the derivative.

$$g'(x) = 12x^2+24x+9$$

Step 4: Calculate the gradient of the tangent.

$$\begin{aligned} g'(x) &= 12(-1)^2+24(-1)+9 \\ &= -3 \\ \therefore m &= -3 \end{aligned}$$

Step 5: Determine the equation of the tangent.

You can substitute the gradient of the tangent and the coordinates of the point into the gradient point form of the straight line equation.

$$\begin{aligned} y-y_1 &= m(x-x_1) \\ y-1 &= -3(x-(-1)) \\ y &= -3x-3+1 \\ y &= -3x-2 \end{aligned}$$

Exercise 3.2

1. Find the equation of the tangent to the curve $xy=9$ at $(3;3)$.
2. Given the function $f(x)=-x^2+4x-3$, find the equation of the tangent at:
   a. the y-intercept of f.
   b. the turning point of f.

The full solutions are at the end of the unit.

Summary

In this unit you have learnt the following:

- How to find the gradient of a tangent to a curve at a point.
- How to find the point where the tangent touches the curve.
- How to find the equation of the tangent to a curve.

Unit 3: Assessment

Suggested time to complete: 20 minutes

1. Determine the point where the tangent to the curve $g(x) = x^2 - 4x - 5$:
    a. is parallel to the line $y = 2x - 1$.
    b. is perpendicular to the line $y = 2x - 1$.
2. Find the equation of the tangent to the curve $f(x) = x^3 + 2x^2 + 7x + 1$ at $x = 2$.
3. Find the equation of the line perpendicular to the tangent to the curve $f(x) = 2x^2 - x + 3$ at the point $(2; 9)$.

The full solutions are at the end of the unit.

Unit 3: Solutions

Exercise 3.1

1.
$y' = -3x^2 + 1$ is the gradient of the tangent to the curve at any point.
To find the gradient of the tangent at $(1; -2)$ substitute the x-value into the gradient equation.

$$\begin{aligned} y'(1) &= -3(1)^2 + 1 \\ &= -2 \end{aligned}$$

2.
    a.
$$\begin{aligned} f'(x) &= 6x - 2 \\ f'(x) &= 3 \\ \therefore 6x - 2 &= 3 \\ x &= \frac{5}{6} \end{aligned}$$

$$\text{Substitute back into } f(x) \text{ to find the y-value}$$

$$\begin{aligned} f\left(\frac{5}{6}\right) &= 3\left(\frac{5}{6}\right)^2 - 2\left(\frac{5}{6}\right) - 1 \\ &= \frac{-7}{12} \end{aligned}$$

The point at which the gradient is 3 is $\left(\frac{5}{6}; -\frac{7}{12}\right)$.

    b.
$$\begin{aligned} y' &= 3x^2 - 2x - 5 \\ 3x^2 - 2x - 5 &= 0 \\ (3x - 5)(x + 1) &= 0 \\ x &= \frac{5}{3} \text{ or } x = -1 \end{aligned}$$

$$f\left(\frac{5}{3}\right) = \left(\frac{5}{3}\right)^3 - \left(\frac{5}{3}\right)^2 - 5\left(\frac{5}{3}\right) + 2$$
$$= \frac{-121}{27}$$
$$f(-1) = (-1)^3 - (-1)^2 - 5(-1) + 2$$
$$= 5$$

The gradient of the tangent is equal to 0 at $(\frac{5}{3}; -\frac{121}{27})$ and $(-1; 5)$.

3. Parallel lines have equal gradients.
Therefore, the gradient of the tangent is 4.

$$f(x) = 4x^2 - 4x + 1$$
$$f'(x) = 8x - 4$$
$$8x - 4 = 4$$
$$x = 1$$
$$f(1) = 4(1)^2 - 4(1) + 1$$
$$= 1$$

The gradient of the tangent is equal to 4 at $(1; 1)$.

Back to Exercise 3.1

Exercise 3.2

1.
$$y = \frac{9}{x}$$
$$= 9x^{-1}$$
$$y' = -9x^{-2}$$
$$= \frac{-9}{x^2}$$
$$y'(3) = \frac{-9}{(3)^2}$$
$$= -1$$

Equation of the tangent:

$$y - 3 = -1(x - 3)$$
$$y = -x + 6$$

2.
   a. the y-intercept of f is $(0; -3)$

$$f'(x) = -2x + 4$$
$$f'(-3) = -2(-3) + 4$$
$$= 10$$

Equation of the tangent:

$$y - (-3) = 10(x - 0)$$
$$y = 10x - 3$$

   b. the turning point of f:

$$f'(x) = -2x + 4 = 0$$
$$x = 2$$
$$f(2) = -(2)^2 + 4(2) - 3$$
$$= 1$$

The gradient of the tangent at the turning point is zero:

$f'(2) = -2(2) + 4$
$= 0$
The equation of the tangent:
$y - 1 = 0(x - 2)$
$y = 1$

Back to Exercise 3.2

Unit 3: Assessment

1.
   a. Gradients of parallel lines are equal.
   $g'(x) = 2$
   $2x - 4 = 2$
   $x = 3$
   Find the y-value.
   $g(3) = (3)^2 - 4(3) - 5$
   $= -8$
   There is a tangent at the point $(3; 8)$.
   b. The product of the gradients of perpendicular lines is equal to -1.
   $m_1 \times m_2 = -1$
   Therefore, the gradient of the tangent is $-\frac{1}{2}$ since $-\frac{1}{2} \times 2 = -1$.
   $g'(x) = \frac{-1}{2}$
   $2x - 4 = \frac{-1}{2}$
   $x = \frac{7}{4}$
   Find the y-value.
   $g\left(\frac{7}{4}\right) = \left(\frac{7}{4}\right)^2 - 4\left(\frac{7}{4}\right) - 5$
   $= \frac{-143}{16}$
   There is a tangent at the point $(\frac{7}{4}; \frac{-143}{16})$.
2. Find the equation of the tangent to the curve $f(x) = x^3 + 2x^2 + 7x + 1$ at $x = 2$.
   $f(2) = (2)^3 + 2(2)^2 + 7(2) + 1$
   $= 31$
   $f'(x) = 3x^2 + 4x + 7$
   $f'(2) = 3(2)^2 + 4(2) + 7$
   $= 27$
   Equation of the tangent:
   $y - 31 = 27(x - 2)$
   $y = 27x - 54 + 31$
   $y = 27x - 23$
3. The product of the gradients of perpendicular lines equals -1. So, the gradient of the tangent will be the negative reciprocal of the perpendicular line.
   $f'(x) = 4x - 1$
   $f'(2) = 4(2) - 1$
   $= 7$
   Gradient of perpendicular line is $-\frac{1}{7}$. Use the point $(2; 9)$to find the equation.

$$y - 9 = -\frac{1}{7}(x - 2)$$
$$y = -\frac{1}{7}x + \frac{65}{7}$$

Back to Unit 3: Assessment

Media Attributions

Unit 4: Derivatives as rates of change

NATASHIA BEARAM-EDMUNDS

Unit outcomes

By the end of this unit you will be able to:

- Solve practical problems involving rates of change.

What you should know

Before you start this unit, make sure you can:

- Find the derivative by using the rules of differentiation. To revise the rules of differentiation refer to level 3 subject outcome 2.5 and unit 2 of this subject outcome.

Introduction

What makes calculus so useful is the application of the derivative to determine how fast (the rate) things are changing in relation to each other. We have seen that the derivative may be thought of as the instantaneous rate of change of a function.

From graphs we can see the rate at which the function values change as the independent (input) variable changes. This rate of change is described by the gradient of the graph, and is found by calculating the derivative.

We have learnt how to find the average gradient of a curve and how to determine the gradient of a curve at a given point. These concepts are also referred to as the average rate of change and the instantaneous rate of change.

$$\text{Average rate of change} = \frac{f(x+h)-f(x)}{h}$$

$$\text{Instantaneous rate of change} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

When we mention rate of change, the instantaneous rate of change (the derivative) is implied. When average rate of change is required, it will be specifically referred to as the average rate of change.

Applications of rate of change

Velocity is the rate of change of distance over a corresponding change in time. If we take the derivative of the velocity, we can find the acceleration, or the rate of change of velocity. Velocity is one of the most common forms of rate of change.

If the function $s(t)$ gives the position of an object at time t then:

1. The velocity of the object at time t is given by $v(t) = \frac{ds}{dt} = s'(t)$.
2. The acceleration of the object at t is given by $a(t) = v'(t) = s''(t)$.

Example 4.1

A ball is hit up into the air. Its height (in metres) t seconds later is given by $s(t) = -5t^2 + 20t$. Determine:

1. The average velocity of the ball during the first two seconds.
2. The velocity of the ball after 1.5 seconds.
3. The time at which the velocity is zero.
4. The velocity with which the ball hits the ground.
5. The acceleration of the ball.

Solutions

1. $$\begin{aligned} v_{\text{ave}} &= \frac{s(2) - s(0)}{2 - 0} \\ &= \frac{[-5(2)^2 + 20(2)] - [-5(0)^2 + 20(0)]}{2} \\ &= \frac{20}{2} \\ &= 10 \text{ m.s}^{-1} \end{aligned}$$
2. Calculate the instantaneous velocity.
$$\begin{aligned} v(t) &= s'(t) \\ &= -10t + 20 \\ s'(1.5) &= -10(1.5) + 20 \\ &= 5 \text{ m.s}^{-1} \end{aligned}$$
3. $$\begin{aligned} v(t) &= 0 \\ -10t + 20 &= 0 \\ t &= 2 \end{aligned}$$
Therefore, the velocity is zero after 2 seconds.
4. The ball hits the ground when $s(t) = 0$.
$$\begin{aligned} s(t) &= 0 \\ -5t^2 + 20t &= 0 \\ -5t(t-4) &= 0 \\ t &= 0 \text{ or } t = 4 \end{aligned}$$
The balls hits the ground after 4 seconds. The velocity after 4 seconds will be:
$$\begin{aligned} v(4) &= s'(4) \\ &= -10(4) + 20 \\ &= -20 \text{ m.s}^{-1} \end{aligned}$$
The ball hits the ground at a speed of -20 m.s^{-1}. The sign of the velocity is negative, which means that the ball is moving downward (a positive velocity is used for upwards motion).

5. Acceleration is the derivative of velocity.

$$a(t) = v'(t) = s''(t)$$
$$= -10 \text{ m.s}^{-2}$$

Example 4.2

A particle moves along a coordinate axis in the positive direction to the right. Its position at time t is given by $s(t) = t^3 - 4t + 2$. Find $v(1)$ and $a(1)$, and use these values to answer the following questions:

1. Is the particle moving from left to right or from right to left at time $t = 1$?
2. Is the particle speeding up or slowing down at time $t = 1$?

Solutions

Begin by finding $v(t)$ and $a(t)$.

$$v(t) = s'(t)$$
$$= 3t^2 - 4$$

$$a(t) = v'(t) = s''(t)$$
$$= 6t$$

Calculate $v(1)$ and $a(1)$.

$$v(1) = 3(1)^2 - 4 = -1 \quad \text{and} \quad a(1) = 6(1) = 6$$

1. Since $v(1) < 0$, the particle is moving from right to left.
2. Since $a(1) > 0$ and $v(1) < 0$, velocity and acceleration are acting in opposite directions. In other words, the particle is decelerating or slowing down.

In addition to velocity, speed, acceleration and position, we can use derivatives to analyse various types of populations, including bacteria colonies and cities. We can use a current population, together with a growth rate, to estimate the size of a population in the future. The population growth rate is the rate of change of a population and can therefore be represented by the derivative of the size of the population.

Example 4.3

In a small town the rate of change of the town's population (measured in thousands of people) can be modelled by the function $P(t) = -\frac{1}{3}t^3 + 64t + 3000$, where t is measured in years.

1. Find the rate of change of population function.

2. Find $P'(1),\ P'(2),\ P'(3)$ and $P'(4)$. Interpret what the results mean for the town's population.

Solutions

1. $P'(t) = -t^2 + 64$
2. $P'(1) = 63$
   $P'(2) = 60$
   $P'(3) = 55$
   $P'(4) = 48$
   The town's population is decreasing.

Exercise 4.1

1. A soccer ball is kicked vertically into the air and its motion is represented by the equation $\mathrm{D}(t) = 1 + 18t - 3t^2$, where D is the distance in metres and t is the time elapsed in seconds.
   a. Determine the initial height of the ball at the moment it is being kicked.
   b. Find the initial velocity of the ball.
   c. Determine the velocity of the ball after $1.5\ \mathrm{s}$.
   d. Calculate the maximum height of the ball.
   e. Determine the acceleration of the ball after 1 second and explain the meaning of the answer.
2. If the displacement s (in metres) of a particle at time t (in seconds) is given by the equation $s(t) = \frac{1}{2}t^3 - 2t$, find its acceleration after two seconds.
3. During an experiment the temperature T (in degrees Celsius) varies with time t (in hours) according to the formula $T(t) = 30 + 4t - \frac{1}{2}t^2,\ t \in [1; 10]$.
   a. Determine an expression for the rate of change of temperature with time.
   b. During which time interval was the temperature dropping?

The full solutions are at the end of the unit.

Summary

In this unit you have learnt the following:

- How to find the rate of change of different functions.

Unit 4: Assessment

Suggested time to complete: 45 minutes

1. A rocket is fired vertically upward from the ground. The distance s in metres that the rocket travels from the ground after t seconds is given by $s(t) = -16t^2 + 560t$.
   a. Find the velocity of the rocket 3 seconds after being fired.
   b. Find the acceleration of the rocket 3 seconds after being fired.
2. A particle moves along a coordinate axis in such a way that its position at time t is given by for $0 \leq t \leq 360°$. At what times is the particle at rest?
3. A water reservoir has both an inlet and an outlet pipe to regulate the depth of the water in the reservoir. The depth is given by the function $\mathrm{D}(h) = 3 + \frac{1}{2}h - \frac{1}{4}h^3$, where D is the depth in metres and h is the hours after $06\mathrm{h}00$.
   a. Determine the rate at which the depth of the water is changing at $10\mathrm{h}00$.
   b. Is the depth of the water increasing or decreasing?
   c. At what time will the inflow of water be the same as the outflow?

The full solutions are at the end of the unit.

Unit 4: Solutions

Exercise 4.1

1. $D(t) = 1 + 18t - 3t^2$
   a. The initial height of the ball is found when $t = 0$.
      $D(0) = 1 + 18(0) - 3(0)$
      $= 1 \text{ m}$
   b. Initial velocity is found when $t = 0$.
      $D'(t) = 18 - 6t$
      $D'(0) = 18 \text{ m.s}^{-1}$
   c. $D'(1.5) = 9 \text{ m.s}^{-1}$
   d. The maximum height of the ball is found at the turning point where $D'(t) = 0$.
      $18 - 6t = 0$
      $t = 3$
      $D(3) = 1 + 18(3) - 3(3)^2$
      $= 28 \text{ m}$
   e. After 1 second:
      $D''(t) = -6 \text{ m.s}^{-2}$
      This means the ball is decelerating.
2.

$$s'(t) = \frac{3}{2}t^2 - 2$$
$$s''(t) = 3t$$
$$a(2) = 3(2)$$
$$= 6 \text{ m.s}^{-2}$$

3.

a. $T'(t) = 4 - t$

b. The temperature drops when $T'(t) < 0$

$$4 - t < 0$$
$$-t < -4$$
$$\therefore t > 4$$

The temperature was dropping during $t \in (4; 10]$.

Back to Exercise 4.1

Unit 4: Assessment

1.

a.

$$s'(t) = -32t + 560$$
$$s'(3) = -32(3) + 560$$
$$= 464 \text{ m.s}^{-1}$$

b.

$$s''(t) = -32t$$
$$s''(3) = -32(3)$$
$$= -96 \text{ m.s}^{-2}$$

2. The particle is at rest when $s'(t) = v(t) = 0$.

So we must solve $2\cos t - 1 = 0$ for $0 \leq t \leq 360^\circ$.

$$\cos t = \frac{1}{2}$$
$$t = 60^\circ \text{ and } t = 300^\circ$$

Thus the particle is at rest at times $t = 60^\circ$ and $t = 300^\circ$.

3.

a. $D(h) = 3 + \frac{1}{2}h - \frac{1}{4}h^3$

$$D'(h) = \frac{1}{2} - \frac{3}{4}h^2$$
$$D'(4) = \frac{1}{2} - \frac{3}{4}(4)^2$$
$$= -11.5 \text{ m per hour}$$

b. The water is decreasing as $D'(h) < 0$.

c.

$$D'(h) = \frac{1}{2} - \frac{3}{4}h^2 = 0$$
$$\frac{3}{4}h^2 = \frac{1}{2}$$
$$h^2 = \frac{2}{3}$$
$$h = \sqrt{\frac{2}{3}}$$

There are 60 minutes in an hour, so find the number of minutes in order to find the time after $06\text{h}00$.

$\sqrt{\frac{2}{3}} \times 60 \approx 49\,\text{min}$

Therefore, at $06\text{h}49$ the inflow will equal the outflow.

Back to Unit 4: Assessment

Unit 5: Sketch cubic functions

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Unit outcomes

By the end of this unit you will be able to:

- Determine the shape of a cubic function.
- Determine the x and y intercepts.
- Find the turning points of the graph.
- Find the maximum and minimum values of the graph.
- Find the point of inflection and discuss concavity using second derivatives.

What you should know

Before you start this unit, make sure you can:

- Find the derivative by using the rules of differentiation.
- Sketch and interpret information from graphs of functions. Revise level 3 subject outcome 2.1 for a refresher on functions.

Introduction

We have worked with cubic polynomials in this subject outcome. Cubic functions are of the form $y = ax^3 + bx^2 + cx + d$. The highest power or degree of a cubic function is three. This also tells us that the graph will have at most three x-intercepts.

There are two general shapes for cubic functions depending on the value of a the co-efficient of x^3.

When $a > 0$, the graph starts out concave down (sad ∩) and ends up concave up (happy ∪).

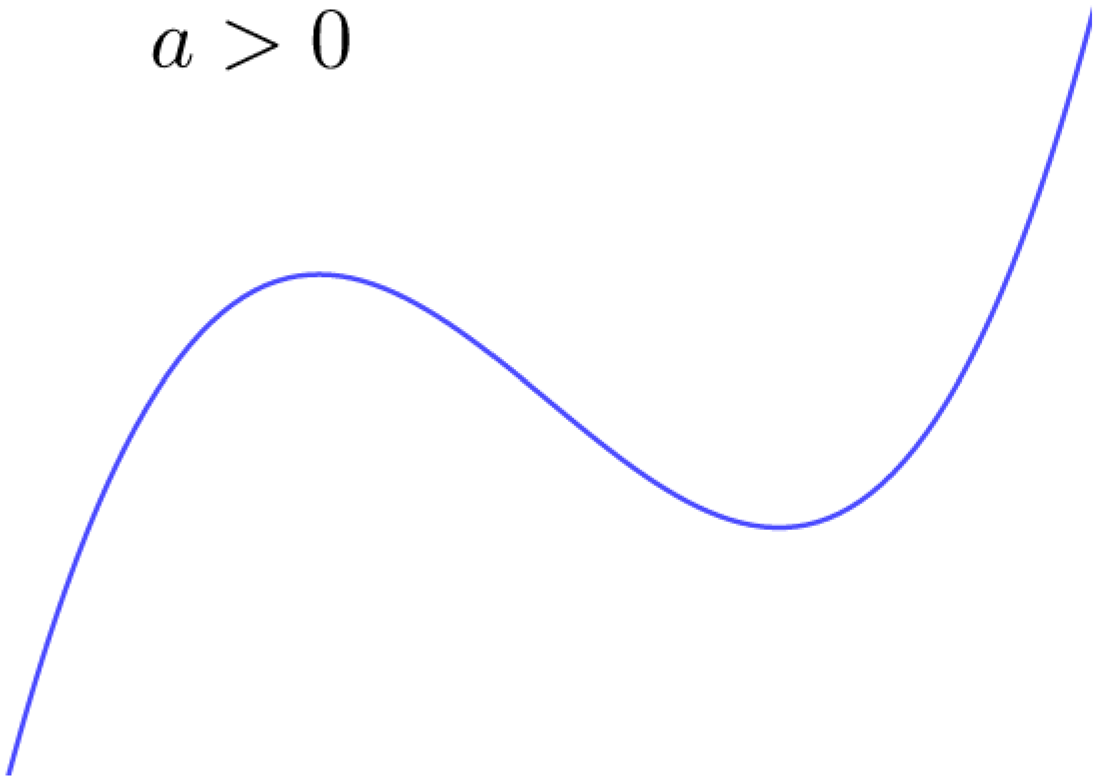


Figure 1: Graph of a cubic function when $a > 0$

When $a < 0$ the graph starts out concave up (happy) and ends up concave down (sad).

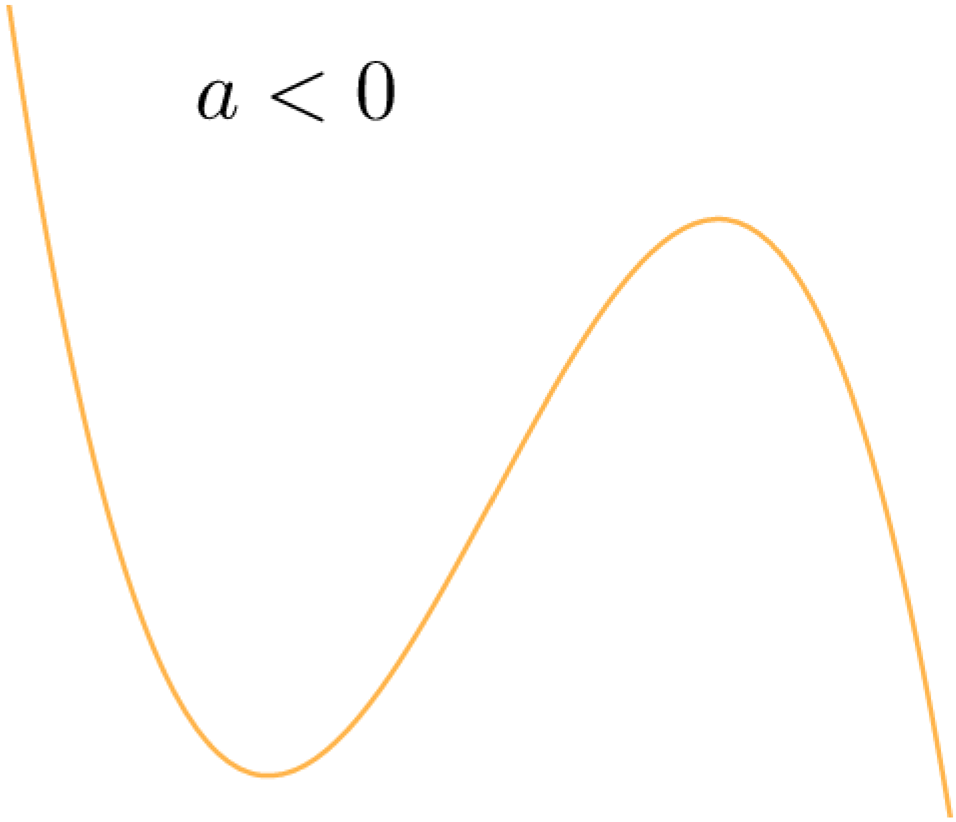


Figure 2: Graph of a cubic function when $a < 0$

Sketching a cubic function

To sketch a cubic function you need the:

- shape
- x- intercepts
- y-intercept
- turning points.

To find the x-intercepts:

1. let $y = 0$
2. factorise by using the factor theorem
3. solve.

To find the y-intercept:

1. let $x = 0$ and solve for y.

Example 5.1

Find the x and y-intercepts of $f(x) = -x^3 + 4x^2 + x - 4$.

Solution

To find the y-intercept let $x = 0$ and solve for y.

$$\begin{aligned} f(0) &= -(0)^3 + 4(0)^2 + (0) - 4 \\ &= -4 \end{aligned}$$

The coordinates of the y-intercept are $(0; -4)$.

To find the x-intercepts you will need to use the factor theorem since we must factorise a cubic expression.

$f(x) = -x^3 + 4x^2 + x - 4$

We use trial and error to find factors of $f(x)$. Remember that a factor will leave no remainder when it divides into an expression.

$$\begin{aligned} f(1) &= -(1)^3 + 4(1)^2 + (1) - 4 \\ &= 0 \end{aligned}$$

$x = 1$ leaves no remainder so one factor of the expression is $(x - 1)$.

Factorise further by inspection (or other methods as shown in level 4 subject outcome 2.1).

$$\begin{aligned} f(x) &= -x^3 + 4x^2 + x - 4 \\ &= -(x^3 - 4x^2 - x + 4) \\ &= -(x - 1)(x^2 - 3x - 4) \\ &= -(x - 1)(x - 4)(x + 1) \end{aligned}$$

To find the x-intercepts let $f(x) = 0$ and solve for x.

$0 = -(x-1)(x-4)(x+1)$
$x = 1,\ x = 4 \text{ or } x = -1$

The coordinates of the x-intercepts are $(-1;0),\ (1;0) \text{ and } (4;0)$.

Exercise 5.1

Determine the x and y-intercepts of the following functions:

1. $f(x) = -x^3 - 5x^2 + 9x + 45$
2. $y = x^3 + 3x^2 - 10x$
3. $y = 2x^3 - 32x$

The full solutions are at the end of the unit.

Stationary points

The turning points of a graph are called the stationary points. A cubic graph will have at most two turning points. At the stationary points a tangent drawn to graph will have gradient of zero, therefore the gradient of the graph is zero and the derivative also will be zero. You can think of a stationary point as a point where the function stops increasing or decreasing.

There are three types of stationary points; local maximum (maxima), local minimum (minima) and horizontal points of inflection.

Local maxima or minima are called relative minimum and maximum values, as there are other points on the graph with lower and higher function values.

The gradient of the function changes on either side of local maxima or minima values as shown in figure 3. The function changes from decreasing to increasing at the local minima and changes from increasing to decreasing at the local maxima.

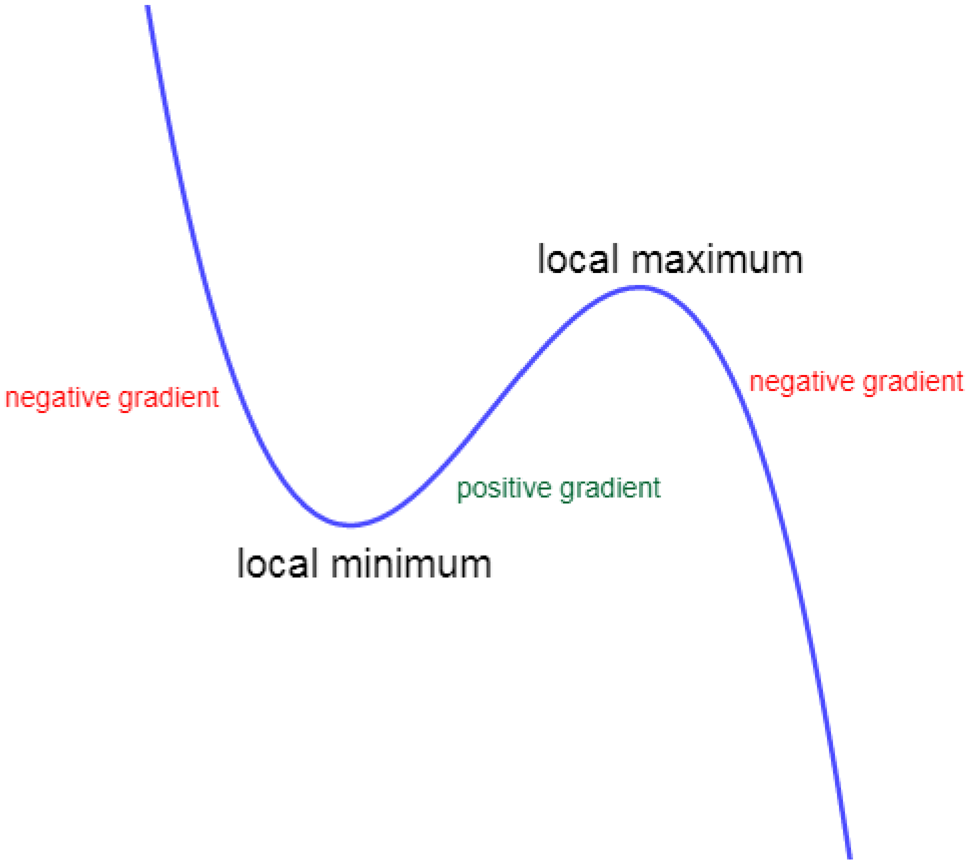


Figure 3: Relative maxima and minima

The gradient on either side of a horizontal point of inflection stays constant as shown in figure 4. The graph is increasing on either side of the point of inflection.

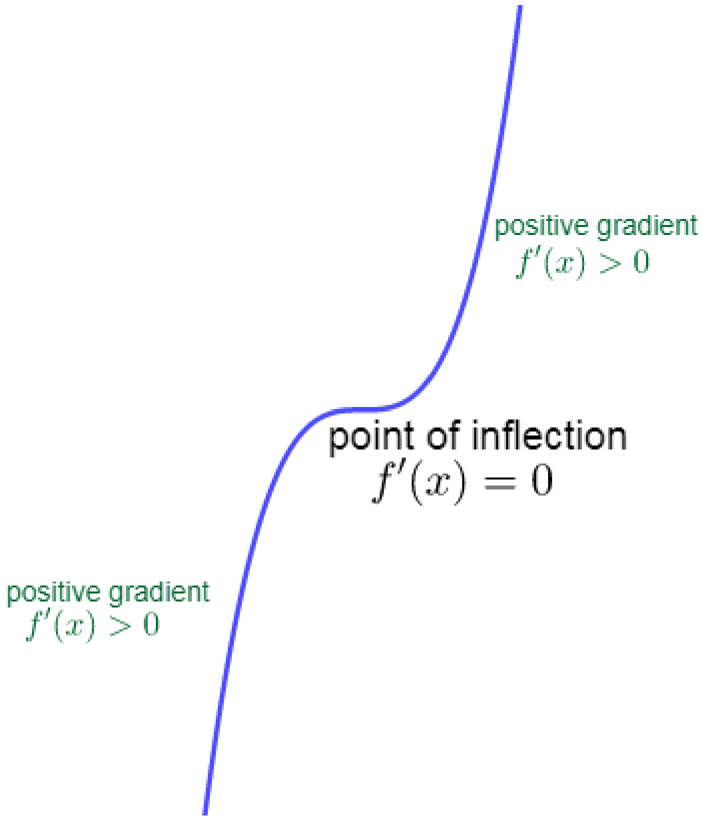


Figure 4: Horizontal point of inflection

To determine the coordinates of the stationary point(s) of $f(x)$:

1. Find the derivative.
2. Let $f'(x) = 0$ and solve for the x-coordinate(s) of the stationary point(s).
3. Substitute value(s) of x into $f(x)$ to calculate the y-coordinate(s) of the stationary point(s).

Example 5.2

Calculate the stationary point(s) of the graph $f(x) = -x^3 + 4x^2 + 3x - 4$.

Solution

Step 1: Determine the derivative of $f(x)$.

$f'(x) = -3x^2 + 8x + 3$

Step 2: Let $f'(x) = 0$ and solve for the x-values of the turning point(s).

$$\begin{aligned} -3x^2 + 8x + 3 &= 0 \\ 3x^2 - 8x - 3 &= 0 \\ (3x + 1)(x - 3) &= 0 \\ x = -\frac{1}{3} &\text{ and } x = 3 \end{aligned}$$

Step 3: Substitute the x-values into $f(x)$ to calculate the corresponding y-coordinates of the stationary points.

$$\begin{aligned} f(-\frac{1}{3}) &= -(-\frac{1}{3})^3 + 4(-\frac{1}{3})^2 + 3(-\frac{1}{3}) - 4 \\ &= -\frac{122}{27} \end{aligned}$$

$$\begin{aligned} f(3) &= -(3)^3 + 4(3)^2 + 3(3) - 4 \\ &= 14 \end{aligned}$$

Step 4: Write the final answer.

The stationary points are $(-\frac{1}{3};-\frac{122}{27})$ and $(3;14)$.

Exercise 5.2

Find the turning points of the following functions:

1. $f(x) = -x^3 - 3x^2 + 9x - 10$
2. $y = x^3 + 3x^2 - 10x$
3. $y = 2x^3 - 54x + 40$

The full solutions are at the end of the unit.

Now, we are ready to sketch a cubic function.

General method for sketching cubic graphs:

1. Use the sign of a to determine the general shape of the graph.
2. Determine the y-intercept by letting $x = 0$.
3. Determine the x-intercepts by letting $y = 0$ and solving for x.
4. Find the x-coordinates of the turning points by letting $f'(x) = 0$ and solving for x.
5. Determine the y-coordinates of the turning points by substituting their x-values into $f(x)$.
6. Plot the points and join as a smooth curve.

Example 5.3

Sketch the graph $f(x) = -x^3 - 3x^2 + 9x + 27$.

Solution

Step 1: Determine the shape of the graph.

The coefficient of the x^3 term is less than zero, therefore the graph will have the following shape:

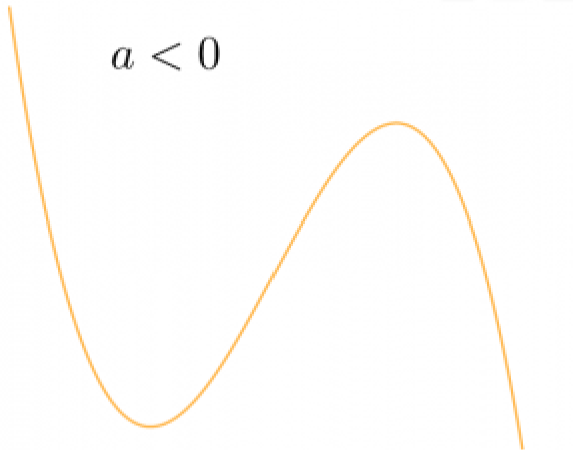


Figure 2: Graph of a cubic function when $a < 0$

Step 2: Determine the intercepts.

To find the y-intercept let $x = 0$ and solve for y.

$$\begin{aligned} f(0) &= -x^3 - 3x^2 + 9x + 27 \\ &= 27 \end{aligned}$$

y-intercept: $(0; 27)$

Find the x-intercepts by letting $f(x) = 0$ and solving for x:

$$\begin{aligned} -x^3 - 3x^2 + 9x + 27 &= 0 \\ x^3 + 3x^2 - 9x - 27 &= 0 \\ (x+3)(x^2 - 9) &= 0 \\ (x+3)(x+3)(x-3) &= 0 \\ x &= -3 \text{ or } x = 3 \end{aligned}$$

x-intercepts: $(-3; 0)$and $(3; 0)$

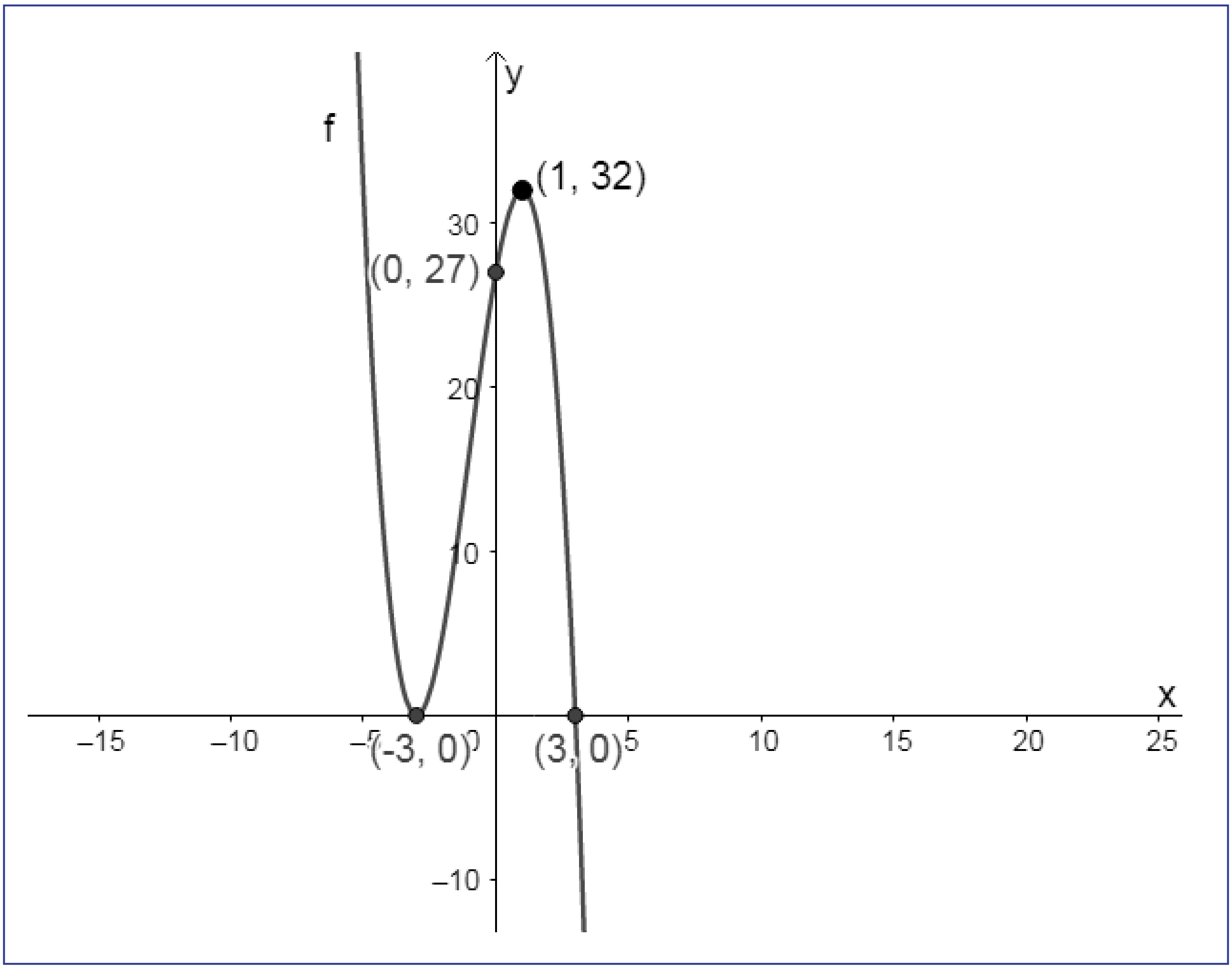
Step 3: Calculate the stationary/turning points.

$$\begin{aligned} f'(x) &= -3x^2 - 6x + 9 \\ -3x^2 - 6x + 9 &= 0 \\ x^2 + 2x - 3 &= 0 \\ (x+3)(x-1) &= 0 \\ x &= -3, \ \ x = 1 \\ f(-3) &= -(-3)^3 - 3(-3)^2 + 9(-3) + 27 \\ &= 0 \\ f(1) &= -(1)^3 - 3(1)^2 + 9(1) + 27 \\ &= 32 \end{aligned}$$

Turning points: $(-3; 0)$ and $(1; 32)$

Step 4: Draw a neat sketch (not necessarily drawn to scale).

Show all key points on the graph.



Exercise 5.3

1. Given $f(x) = x^3 + x^2 - 5x + 3$:
   a. Show that $(x - 1)$ is a factor of $f(x)$ and hence find the x-intercepts.
   b. Determine the y-intercept and turning points.
   c. Sketch the graph.
2. Sketch the graph of $f(x) = -x^3 + 4x^2 + 11x - 30$. Show all the turning points and intercepts with the axes.

The full solutions are at the end of the unit.

Second derivative and concavity

A change in the sign of the second derivative shows that there is a change in the direction of the gradient of the original function. To find the second derivative of a function you take the derivative of the first derivative.

The second derivative tells us about the **concavity** of a graph. Concavity relates to the rate of change of a function's derivative. Concavity indicates whether the gradient of a curve is increasing, decreasing or constant.

Take note!

If $f''(x) < 0$, the graph is concave down and there is a local maximum. Tangent lines drawn to the graph lie above the graph. The gradient of the curve decreases as x increases.

If $f''(x) > 0$, the graph is concave up and there is a local minimum. Tangent lines lie below the graph. The gradient of the curve increases as x increases.

If $f''(x) = 0$, the graph *may have* a point of inflection. If $f''(x)$ changes sign from one side of the point of inflection to the other, the concavity changes. For a cubic function, the graph will have a point of inflection when $f''(x) = 0$.

Figure 5 shows how the concavity tells us about the shape of a graph.

Concave down: $f''(x) < 0$.

Concave up: $f''(x) > 0$.

Point of inflection: $f''(x) = 0$ and changes sign at this point.

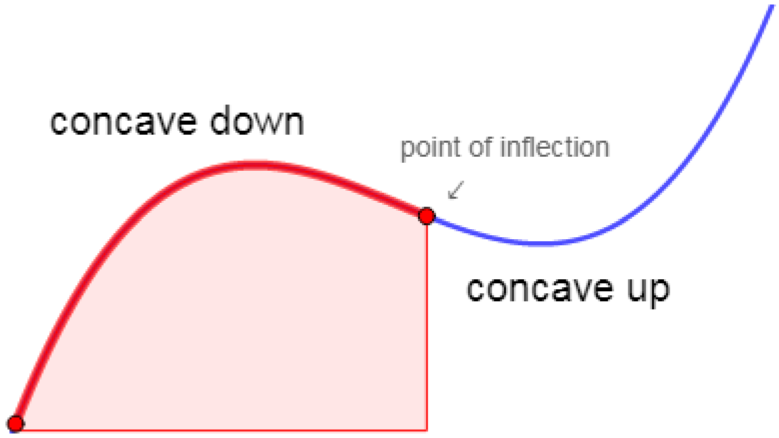


Figure 5: Concavity and shape of a curve

Exercise 5.4

Determine if the following graphs have a local maximum, minimum or point of inflection at the given points, by finding the second derivative:

1. $f(x) = -x^3 - 3x^2 + 9x - 10$ at $x = 1$
2. $y = x^3 + 3x^2 - 10x$ at $x = -1$
3. $y = 2x^3 - 54x + 40$ at $x = 3$

The full solutions are at the end of the unit.

Summary

In this unit you have learnt the following:

- How to identify the shape of a cubic function.
- How to find the x and y-intercepts of a cubic function.
- How to find the stationary points.
- How to find the point of inflection of a cubic function.
- How to test concavity.
- How to sketch a cubic function.

Unit 5: Assessment

Suggested time to complete: 45 minutes

1. Given $g(x) = x^3 - 2x^2 - 3x + 6$:
   a. Determine the intercepts with the axes.
   b. Find the stationary points.
   c. Calculate the point of inflection.
   d. Sketch the graph.
2. Answer the following questions for $f(x) = (x-2)^3$:
   a. Determine the concavity of the graph.
   b. Find the inflection point.
   c. Sketch the graph.

The full solutions are at the end of the unit.

Unit 5: Solutions

Exercise 5.1

1.

$$\begin{aligned} f(x) &= -x^3 - 5x^2 + 9x + 45 \\ f(3) &= 0 \\ &\therefore (x-3) \text{ is a factor} \\ f(x) &= -(x^3 + 5x^2 - 9x - 45) \\ &= -(x-3)(x^2 + 8x + 15) \\ &= -(x-3)(x+3)(x+5) \end{aligned}$$

Find the x-intercepts:

$$-(x-3)(x+3)(x+5) = 0$$
$$\therefore x = 3, \ -3 \text{ or } -5$$
$$\text{x-intercepts :}$$
$$(-5;0),\ (-3;0) \text{ and } (3;0)$$

y-intercept: $(0;45)$.

2.

$$\begin{aligned} y &= x^3 + 3x^2 - 10x \\ &= x(x^2 + 3x - 10) \\ &= x(x+5)(x-2) \end{aligned}$$

x-intercepts:

$(0;0), (2;0) \text{ and } (-5;0)$

y-intercept: $(0;0)$.

3.

$$\begin{aligned} y &= 2x^3 - 32x \\ &= 2x(x^2 - 16) \\ &= 2x(x-4)(x+4) \end{aligned}$$

x-intercepts:

$(0;0), (-4;0) \text{ and } (4;0)$

y-intercept: $(0;0)$.

Back to Exercise 5.1

Exercise 5.2

1.

$$\begin{aligned} f'(x) &= -3x^2 - 6x + 9 \\ -3x^2 - 6x + 9 &= 0 \\ x^2 + 2x - 3 &= 0 \\ (x+3)(x-1) &= 0 \\ x &= -3,\ \ x = 1 \\ f(-3) &= -(-3)^3 - 3(-3)^2 + 9(-3) - 10 \\ &= -37 \\ f(1) &= -(1)^3 - 3(1)^2 + 9(1) - 10 \\ &= -5 \end{aligned}$$

$\text{Turning points: } (-3;-37) \text{ and } (1;-5)$

2.

$$\begin{aligned} y' &= 3x^2 + 6x - 10 \\ 3x^2 + 6x - 10 &= 0 \\ x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-6 \pm \sqrt{(6)^2 - 4(3)(-10)}}{2(3)} \\ &= \frac{-3 \pm \sqrt{39}}{3} \\ x &= \frac{-3 + \sqrt{39}}{3} \text{ or } x = \frac{-3 - \sqrt{39}}{3} \\ x &= 0.180... \text{ or } x = -3.081... \\ f(0.180...) &= -6.04 \\ f(-3.081...) &= 30.04 \end{aligned}$$

$\text{Turning points: } (0.18;-6.04) \text{ and } (-3.08;30.04)$

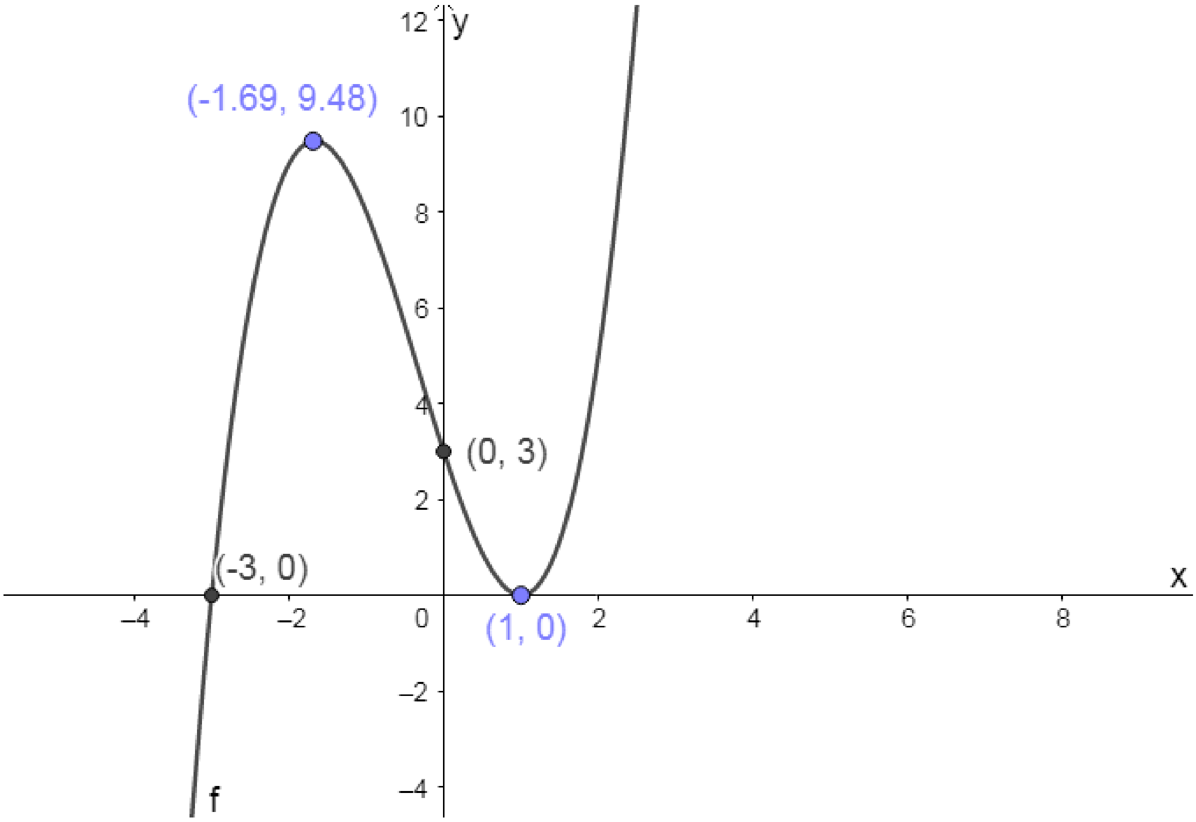
3.

$$y' = 6x^2 - 54$$
$$x^2 - 9 = 0$$
$$x = 3, \; x = -3$$
$$f(3) = 2(3)^3 - 54(3) + 40$$
$$= -68$$
$$f(-3) = 2(-3)^3 - 54(-3) + 40$$
$$= 148$$
$$\text{Turning points: } (3; -68) \text{ and } (-3; 148)$$

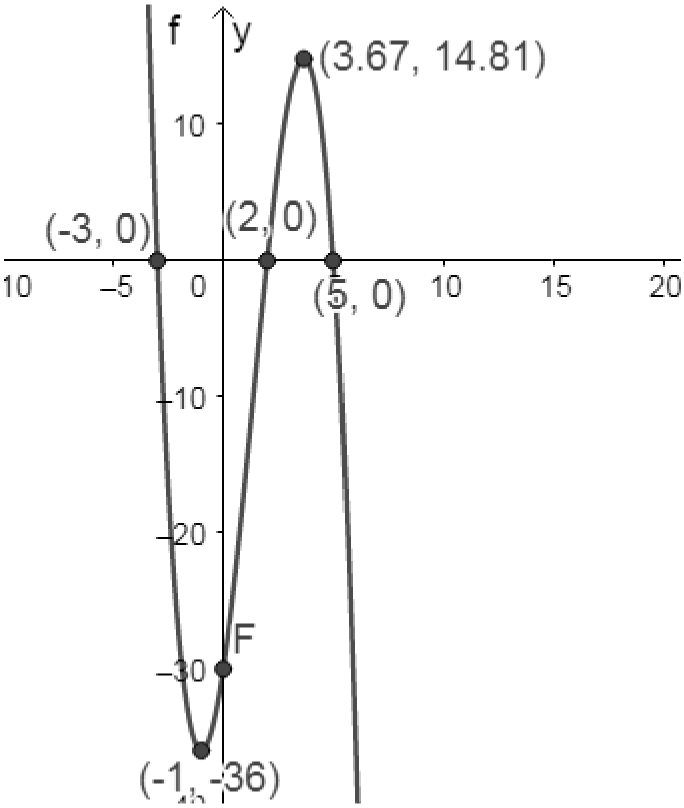
Back to Exercise 5.2

Exercise 5.3

1. $f(x) = x^3 + x^2 - 5x + 3$
   a.
   $$f(1) = (1)^3 + (1)^2 - 5(1) + 3$$
   $$= 0$$
   $$\therefore (x-1) \text{ is a factor}$$
   $$x^3 + x^2 - 5x + 3 = (x-1)(x^2 + 2x - 3)$$
   $$(x-1)(x^2 + 2x - 3) = 0$$
   $$(x-1)^2(x+3) = 0$$
   $$x = 1 \text{ or } x = -3$$
   $$\text{x-intercepts: } (1; 0) \text{ and } (-3; 0)$$
   b. $\text{y-intercept } (0; 3)$
   $$f'(x) = 3x^2 + 2x - 5$$
   $$(3x+5)(x-1) = 0$$
   $$x = -\frac{5}{3} \text{ and } x = 1$$
   $$f(-\frac{5}{3}) = 9.48$$
   $$f(1) = 0$$
   $$\text{Turning points: } (-\frac{5}{3}; 9.48) \text{ and } (1; 0)$$
   c.



2. $f(x) = -x^3 + 4x^2 + 11x - 30$



Back to Exercise 5.3

Exercise 5.4

1. $f'(x) = -3x^2 - 6x + 9$

$$f''(x) = -6x - 6$$
$$f''(1) = -6(1) - 6$$
$$= -12$$
$$f''(1) < 0$$
$$\therefore \text{ local maximum at } x = 1$$

2. $y' = 3x^2 + 6x - 10$
$$y'' = 6x + 6$$
$$y''(-1) = 0$$
$$\therefore \text{ there is a point of inflection at } x = -1$$
Note: we can make the above statement since y is a cubic function.

3. $y' = 6x^2 - 54$
$$y'' = 12x$$
$$y''(3) = 36$$
$$y''(3) > 0$$
$$\therefore \text{ local minimum at } x = 3$$

Back to Exercise 5.4

Unit 5: Assessment

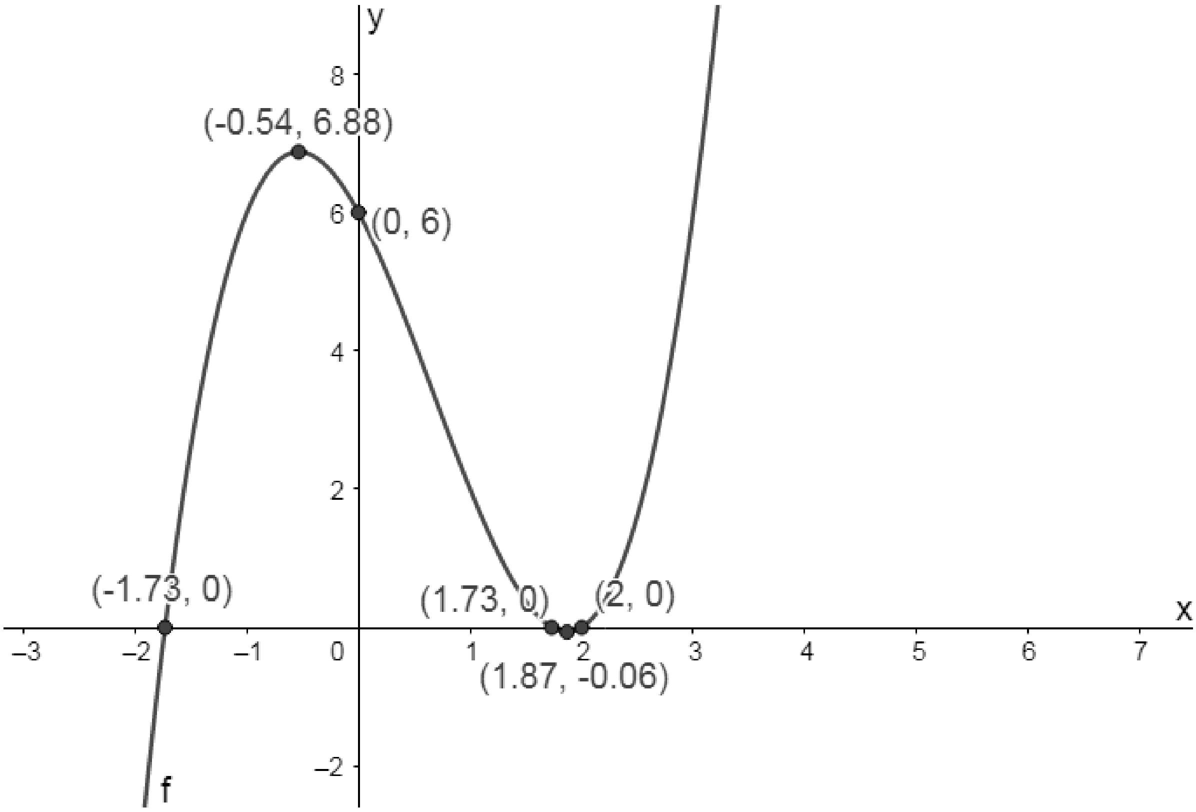
1.

a.
$$g(0) = 6$$
$$\text{y-intercept } (0;6)$$
$$g(2) = 0$$
$$\therefore (x-2) \text{ is a factor}$$
$$x^3 - 2x^2 - 4x + 6 = (x-2)(x^2-3)$$
$$x = 2 \text{ or } x = \pm\sqrt{3}$$
$$\text{x-intercepts: } (2;0),\ (1.73;0) \text{ and } (-1.73;0)$$

b.
$$g'(x) = 3x^2 - 4x - 3$$
$$3x^2 - 4x - 3 = 0 \qquad \text{Use the quadratic formula}$$
$$x = -0.54 \text{ or } 1.87$$
$$g(-0.54) = 6.88$$
$$g(1.87) = -0.06$$
$$\text{Stationary points: } (-0.54;6.88) \text{ and } (1.87;-0.06)$$

c.
$$g'(x) = 3x^2 - 4x - 4$$
$$g''(x) = 6x - 4$$
Note: since g is a cubic function we can let $g''(x) = 0$ to find the point of inflection.
$$g''(x) = 0$$
$$6x - 4 = 0$$
$$x = \frac{2}{3}$$
Point of inflection:
$$(\frac{2}{3};3.41)$$

d.



2.

a.

$$f'(x) = 3(x-2)^2(1)$$
$$= 3(x^2 - 4x + 4)$$
$$= 3x^2 - 12x + 12$$
$$f''(x) = 6x - 12$$

For $x < 2$, the graph is concave down.

For $x > 2$, the graph is concave up.

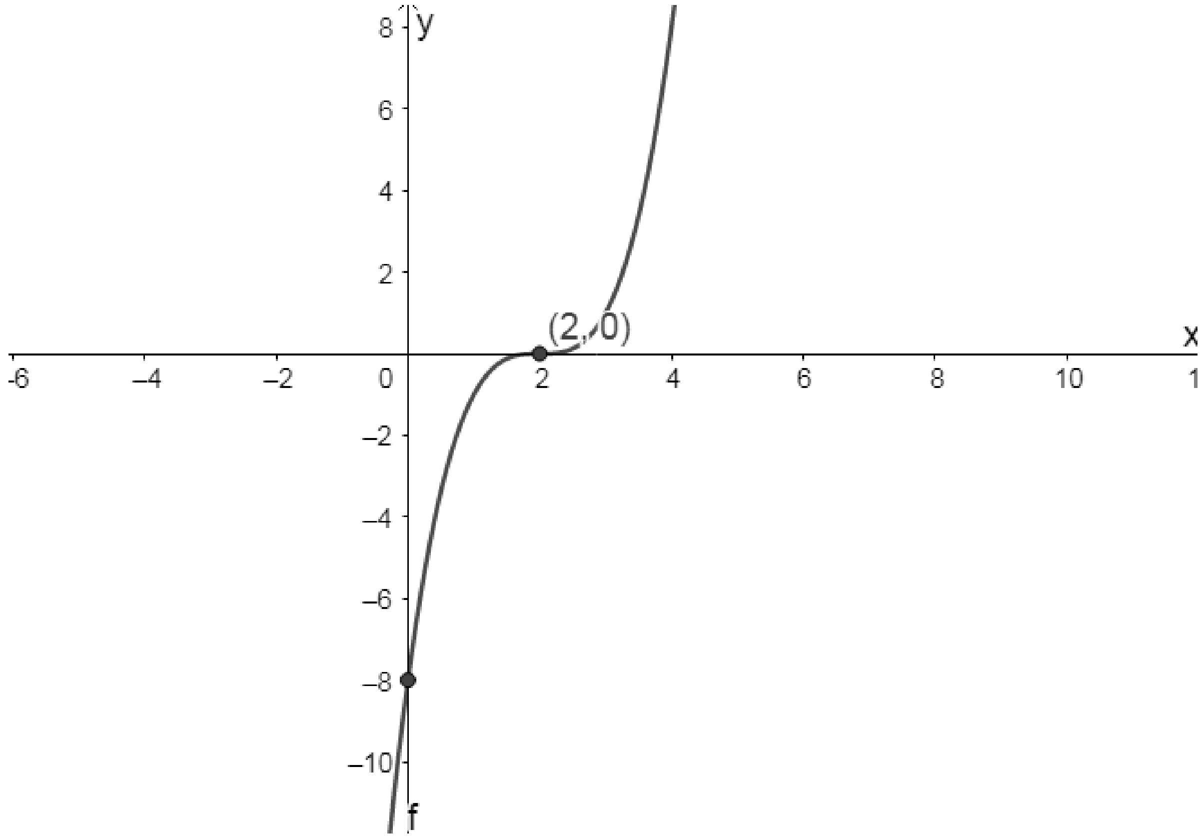
b.

Note that the function is a cubic:

$$f''(x) = 6x - 12$$
$$= 0$$
$$\therefore x = 2$$
$$f(2) = 0$$

point of inflection: $(2; 0)$

c.



[Back to Unit 5: Assessment](#)

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SUBJECT OUTCOME VII

FUNCTIONS AND ALGEBRA: ANALYSE AND REPRESENT MATHEMATICAL AND CONTEXTUAL SITUATIONS USING INTEGRALS AND FIND AREAS UNDER CURVES BY USING INTEGRATION RULES

Subject outcome

Subject outcome 2.5: Analyse and represent mathematical and contextual situations using integrals and find areas under curves by using integration rules

Learning outcomes

- Find the integrals of the following:

  $\int ax^n\,dx$; $\int \frac{a}{x}dx$; $\int ae^{kx}\,dx$; $\int a\sin kx\,dx$; $\int a\cos kx\,dx$; $\int a\sec^2 kx\,dx$

  Where:

  $$\int ax^n\,dx = \frac{ax^{n+1}}{n+1} + c$$

  $$\int \frac{a}{x}dx = a\ln x + c$$

  $$\int ae^{kx}\,dx = \frac{ae^{kx}}{k} + c$$

  $$\int a\sin kx\,dx = \frac{-a\cos kx}{k} + c$$

  $$\int a\cos kx\,dx = \frac{a\sin kx}{k} + c$$

  Note:

  - Simplifications may be required where necessary.
  - Integrals of polynomials may be assessed.
  - Integration by parts is excluded.
- Use the upper and lower limits to calculate definite integrals.
- Determine the area under a curve by:
  - Working from a given graph or by sketching a graph.
  - Working with an area bounded by a curve, the x-axis, an upper and a lower limit.
  - Splitting the area into two intervals when the graph crosses the x-axis.

    Note:

    - Integrals with respect to the x-axis only.
    - Areas between two curves are excluded.
    - The y-axis (x = 0) may be used as an upper or lower limit.

Unit 1 outcomes

By the end of this unit you will be able to:

- Understand the relationship between differentiation and integration (anti-derivative).
- Define integration as the approximate area under curves.

Unit 2 outcomes

By the end of this unit you will be able to:

- Apply the rules of integration to various functions.
- Calculate the definite integral.

Unit 3 outcomes

By the end of this unit you will be able to:

- Find the area under a curve between two points and bound by the x-axis.

Unit 1: Introduction to integration

NATASHIA BEARAM-EDMUNDS

Unit outcomes

By the end of this unit you will be able to:

- Understand the relationship between differentiation and integration (anti-derivative).
- Define integration as the approximate area under curves.

What you should know

Before you start this unit, make sure you can:

- Use the various techniques for differentiation as shown in level 4 subject outcome 2.4.
- Use sigma notation to calculate the sum of a series.

Try these questions to make sure you are ready to start integration.

1. Find $\lim\limits_{x \to \infty} \dfrac{2}{x}$
2. Differentiate with respect to x:
   a. $y = \ln 2x$
   b. $y = \sin 2x$
3. Calculate $\sum\limits_{i=1}^{12} (4 + 2i)$.

Solutions

1. $\lim\limits_{x \to \infty} \dfrac{2}{x} = 0$
2.
   a.
   $$\begin{aligned} \frac{d}{dx}\ln 2x &= \frac{1}{2x} \cdot \frac{d}{dx}2x \\ &= \frac{1}{2x} \cdot 2 \\ &= \frac{1}{x} \end{aligned}$$
   b.

$$y' = \frac{d}{dx}(\sin 2x) \cdot \frac{d}{dx}2x$$

$$\frac{d}{dx}(\sin 2x) = \cos 2x$$

$$\frac{d}{dx}2x = 2$$

$$\therefore y' = 2\cos 2x$$

3. $\sum_{i=1}^{12}(4+2i)$

$4+2(1);\ 4+2(2);\ 4+2(3)\ldots$

$6; 8; 10\ldots$

$a=6;\ d=2;\ n=12$

$$\sum_{i=1}^{12}(4+2i) = \tfrac{12}{2}(2(6)+(12-1)2)$$

$$= 6(34)$$

$$= 204$$

Introduction

We live in a world that is in constant motion and changing every instant. It is not static so most 'real world' problems need mathematical techniques that can keep up with these **instantaneous rates of change**. Calculus is the mathematical language that was invented to describe the dynamic nature of our universe.

Calculus is divided into two branches; **differential calculus and integral calculus.**

In differential calculus you find instantaneous rates of change by dividing something up into infinitely many small slices. In integral calculus, you glue together all these little slices to get back to the bigger picture. The word 'integration' means to combine or bring things together. Integration as a mathematical concept has similar meaning.

How do you think integration and differentiation are related?

Mathematically, integration can be thought of in two ways:

- Finding the area under curves.
- Finding the anti-derivative (i.e. undoing differentiation).

In this unit we will discuss both of these approaches to integration.

Estimating areas

Suppose we want to find the area of a circle but do not know the formula. How could we find its area?

One way we could find the area is to place a grid of squares over the circle. Then we can calculate the area of each square and add all of these areas together to estimate the area of the circle. We say it is an estimate of the area because some of the squares will not be completely inside the circle. These squares are highlighted with the blue line in figure 1. Should we count these squares or leave them out?

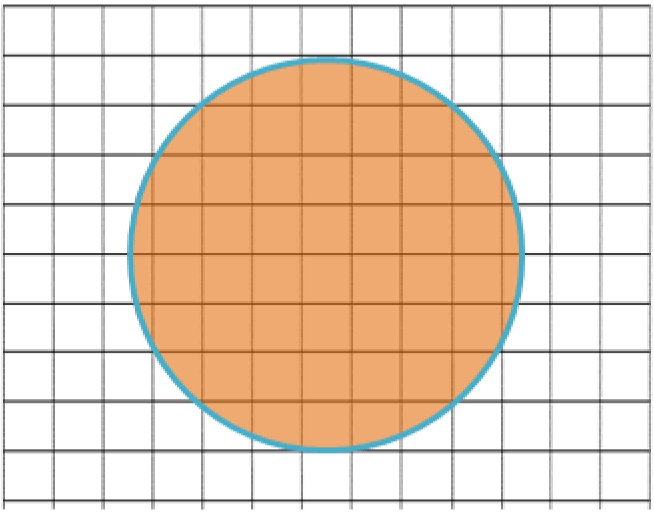


Figure 1: Estimating the area of a circle with a grid

If we add together all the squares that contain any small part of the circle, we will arrive at an overestimate of the area. If we ignore all the squares that do not completely overlap with the circle, then we will underestimate the area.

However, we now have two limits on the area of the circle – an upper one and a lower one. To 'squeeze' the area of the circle ever tighter between an upper limit and a lower limit, we would need to make the squares in the grid smaller and smaller. Then the parts that were included or discarded would be, in total, smaller and smaller. The smaller the squares on the grid paper, the better the estimate of the area of the circle. This is called the method of **exhaustion**.

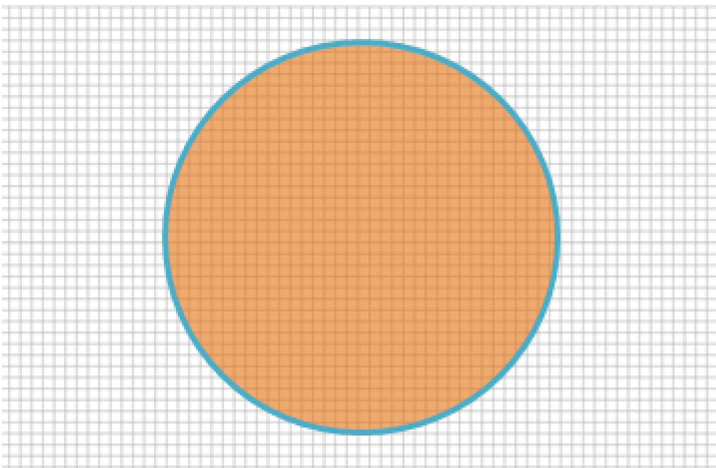


Figure 2: Estimating the area of a circle with a grid of smaller squares

As you can see in figure 2, by dividing a region into many small shapes that have a known area, we can sum these areas and obtain a reasonable estimate of the true area of the circle.

Integration and area under curves

If we know how fast a vehicle is moving, we can use integration to determine how far it travels. Finding distance from velocity is just one of many applications of integration. In fact, integrals are used in a variety of mechanical and physical applications.

In this section, we will learn how to estimate the area under curves. We will come back to the actual calculation of area under curves using integrals in a later unit.

By now you are familiar with finding the area of regular polygons such as squares, rectangles, parallelograms and triangles. Even when these shapes are combined into irregular shapes, it is possible to find the total area by finding the area of the individual shapes and adding those areas together.

For example, you could calculate the area of the 'tangram man' in figure 3 by dividing him into several regular shapes as shown in figure 4.

Figure 3: The tangram man

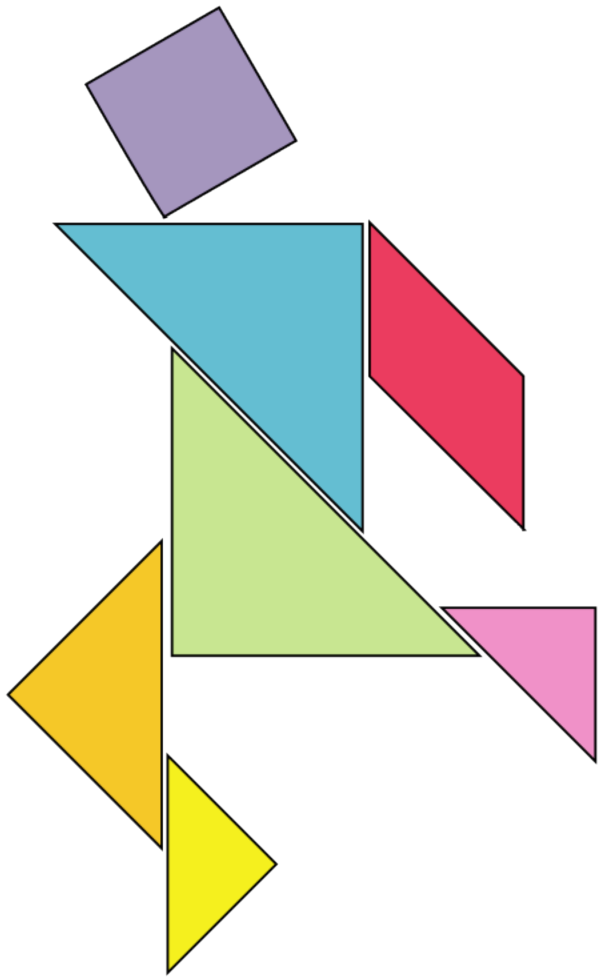


Figure 4: The tangram man split into regular shapes

We can extend that idea to find the area bound by the x-axis and any function between two values as seen in figure 5.

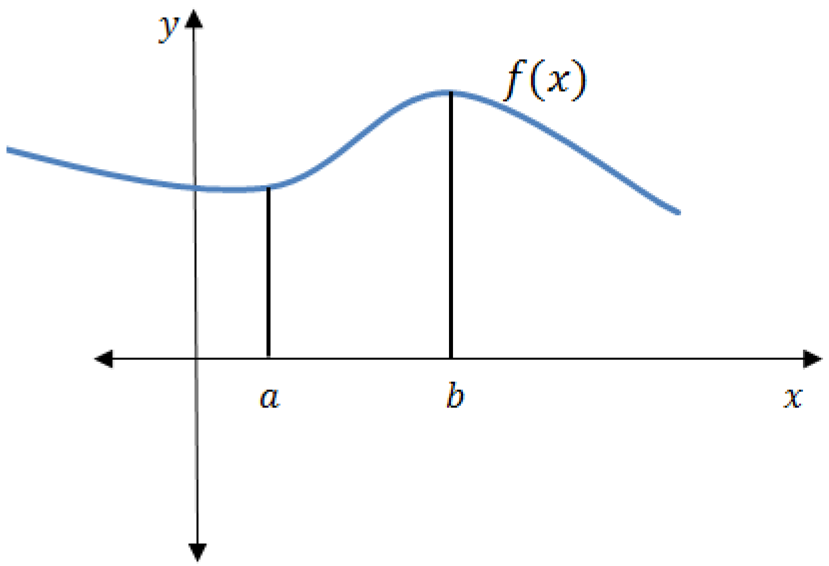


Figure 5: The area bound by $f(x)$ and the x-axis between two points

Say we have a function, $f(x)$, and we want to find the area under this curve over a given interval. That means we need to find the area enclosed by the curve $f(x)$, the x-axis and the vertical lines $x = a$ and $x = b$ (a shown in figure 5). Is there a formula we can use to calculate this area?

The curve produces a very irregular shape and the height of the curve is constantly changing as you move between $x = a$ and $x = b$. Since this shape is not a polygon (a closed shape formed with straight edges) there is no area formula we can use. We need a new technique to find the area under the curve. Integral calculus provides this new technique.

Did you know?

There is a long-standing debate over who invented calculus. It is generally accepted that both Isaac Newton and Gottfried Leibniz invented this revolutionary mathematical idea. For a crash course on the history and development of calculus you can watch this video, Newton and Leibniz: Crash Course History of Science #17.

Newton and Leibniz: Crash Course History of Science #17 (Duration: 13.49)

Think back to how we estimated the area of a circle by adding together the area of all the squares that covered the shape. We can approximate the area of this curved shape in a similar way, using rectangles.

As we did with the circle and squares, we can add rectangles over the area of the curve we want to estimate.

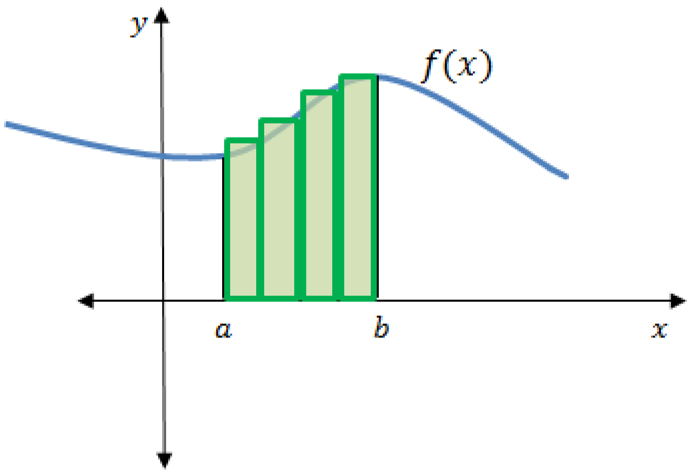


Figure 6: Estimating the area under the curve using rectangles

As you can see in figure 6, the big rectangles we have placed do not give the best approximation and over-estimate the area under the curve as they stick out above the curve. But we can make the rectangles 'narrower' and place more of them on the diagram. More rectangles mean a better approximation of the area.

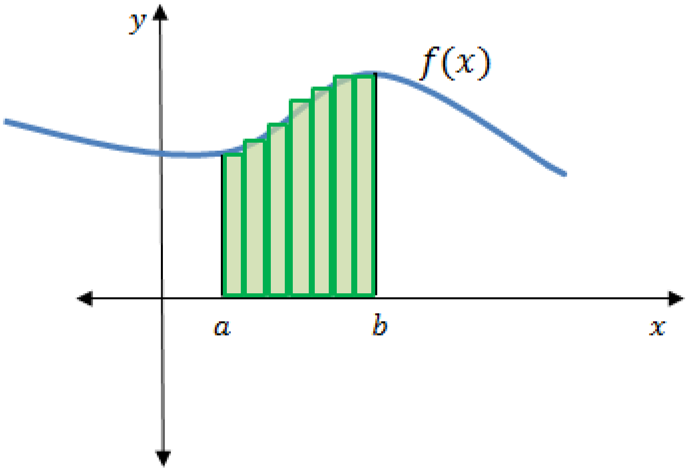


Figure 7: Estimating the area under the curve using narrow rectangles

The ‘narrower’ the rectangles, the less they stick out above the curve and the better they approximate with the area under the curve.

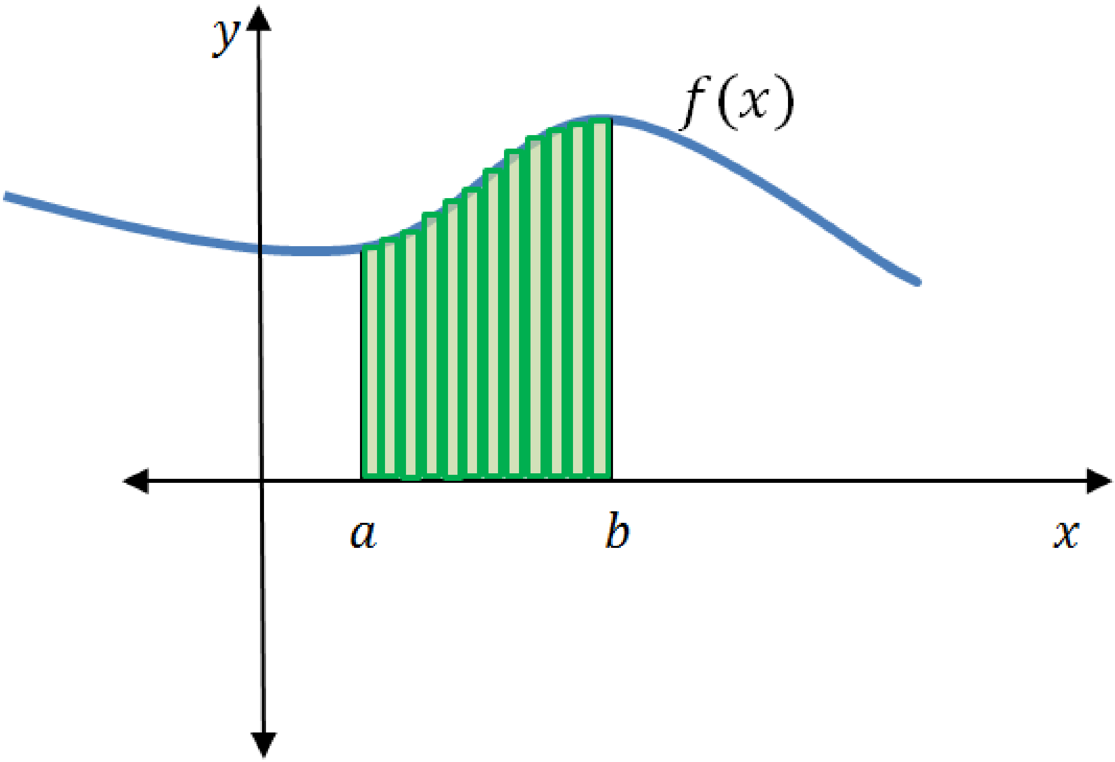


Figure 8: Estimating the area under the curve using narrower rectangles

If we take the limit of an infinite number of really narrow rectangles we will be able to get the precise area under the curve. Let us explore this idea quantitatively with the function $f(x) = x^2$.

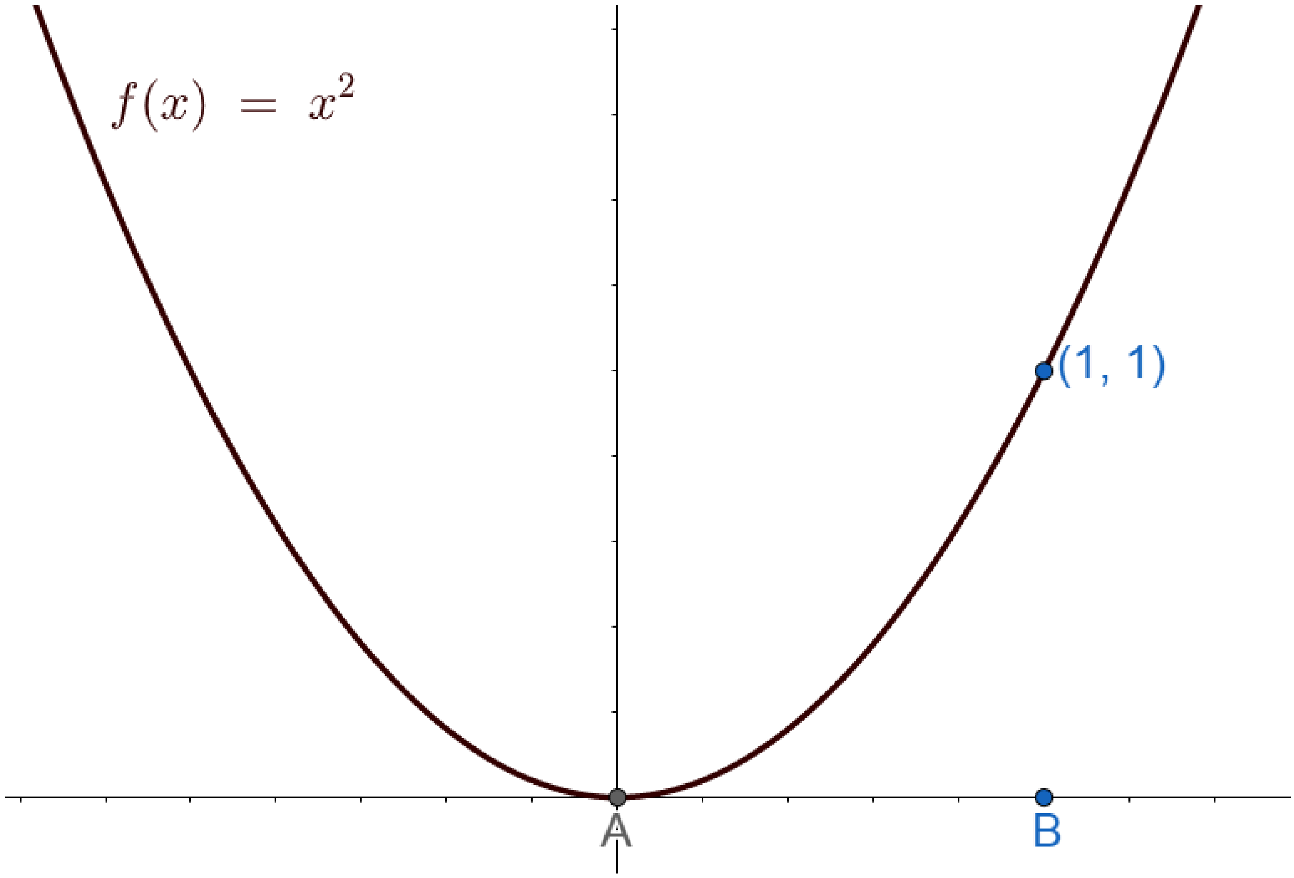


Figure 9: The function $f(x) = x^2$

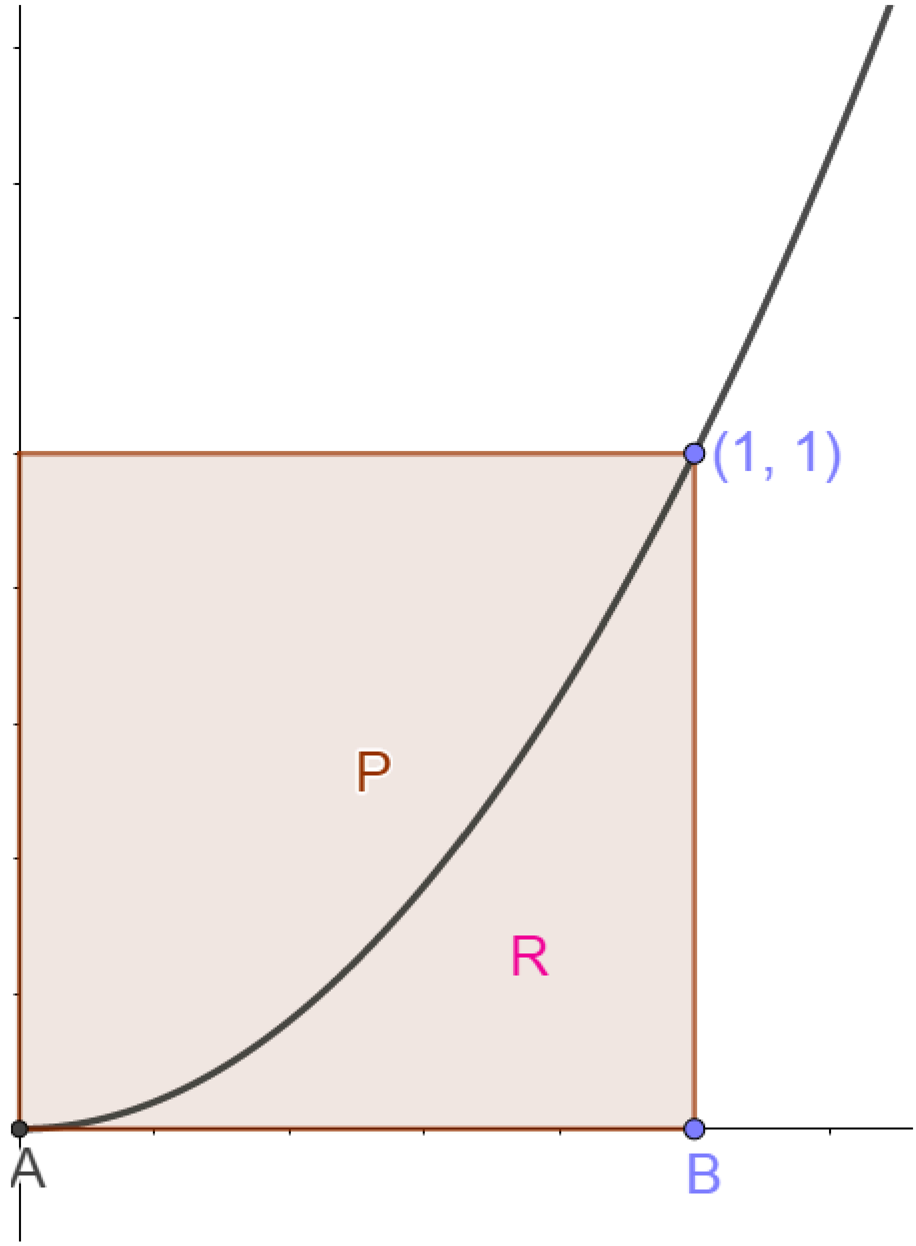


Figure 10: Calculating the area under $f(x) = x^2$ between $(0;0)$ and $(1,1)$

Say we would like to find the area under the curve $f(x) = x^2$ between the points $(0;0)$ and $(1,1)$ as shown in figure 10.

We label the area that we are interested in, R. We know that area R will have some value between zero and one. This is because a square, P, in figure 10, with sides of length one unit will have an area of one. Therefore, we can say that $0 < R < 1$. We will add rectangles to the diagram to see what estimates we get for area R.

First, we divide area R using four rectangles. The width of each of the four rectangles will be 0.25 or $\frac{1}{4}$ units. We can easily calculate the area of each of these rectangles by finding the height of the rectangles from the function. Every point on the curve has co-ordinates (x, x^2).

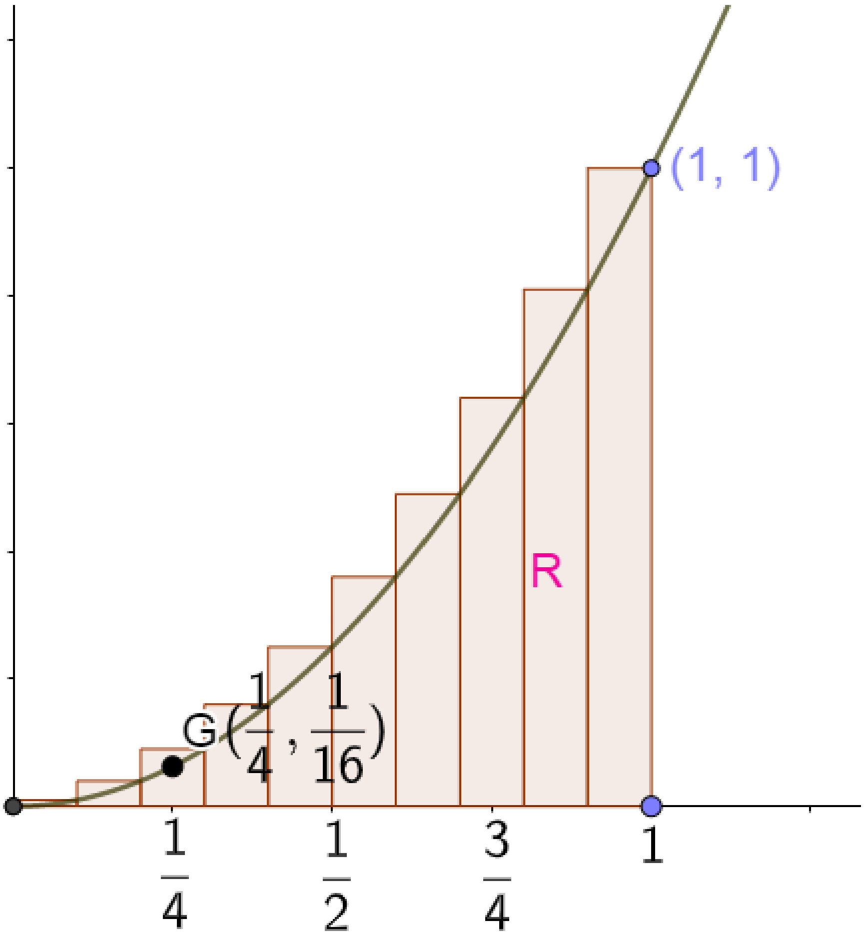


Figure 11: Calculating the area under $f(x) = x^2$ using rectangles

For example, the point G (in figure 11) will have co-ordinates $(\frac{1}{4}, (\frac{1}{4})^2)$. You substitute the x-value into the function to find the y-value. So G is the point$(\frac{1}{4}, \frac{1}{16})$. The height of the first rectangle is $\frac{1}{16}$ and the width is $\frac{1}{4}$. Therefore, the first rectangle's area is $\frac{1}{4} \times \frac{1}{16} = \frac{1}{64}$. We continue this way to find the height and area of the remaining three rectangles.

Try finding the areas of each of the other three rectangles yourself.

To find the total area under the curve we add all of the areas together.

Number of rectangles	**Area under the curve**
4	$(\frac{1}{4} \times \frac{1}{16}) + (\frac{1}{4} \times \frac{1}{4}) + (\frac{1}{4} \times \frac{9}{16}) + (\frac{1}{4} \times 1) = 0.46875$

We know this first approximation is too high because the rectangles stick out above the curve.

Next, we use 10 rectangles. These will each have a width of 0.1 or $\frac{1}{10}$. This is narrower than the width of the four rectangles used previously. Once again, we can calculate the heights of each rectangle using the function.

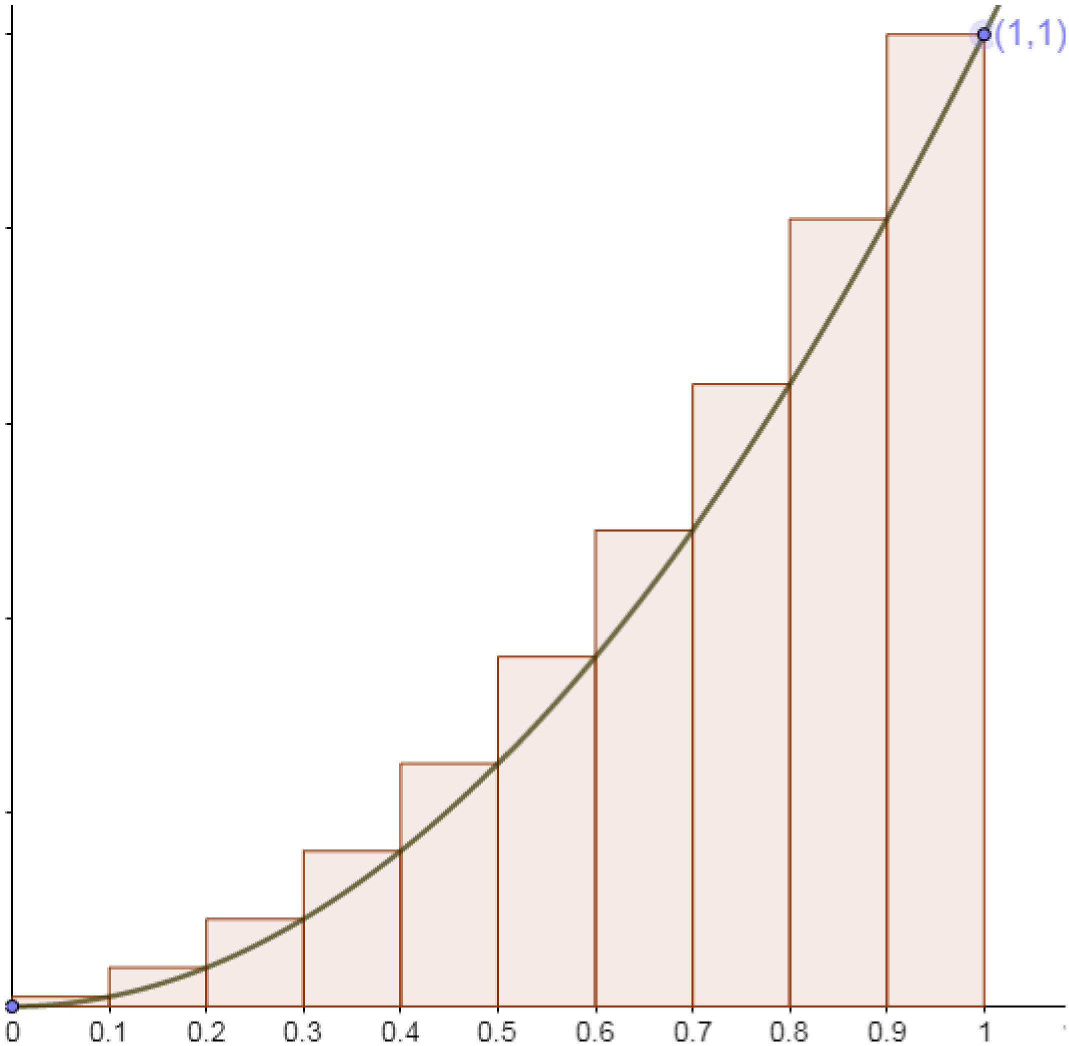
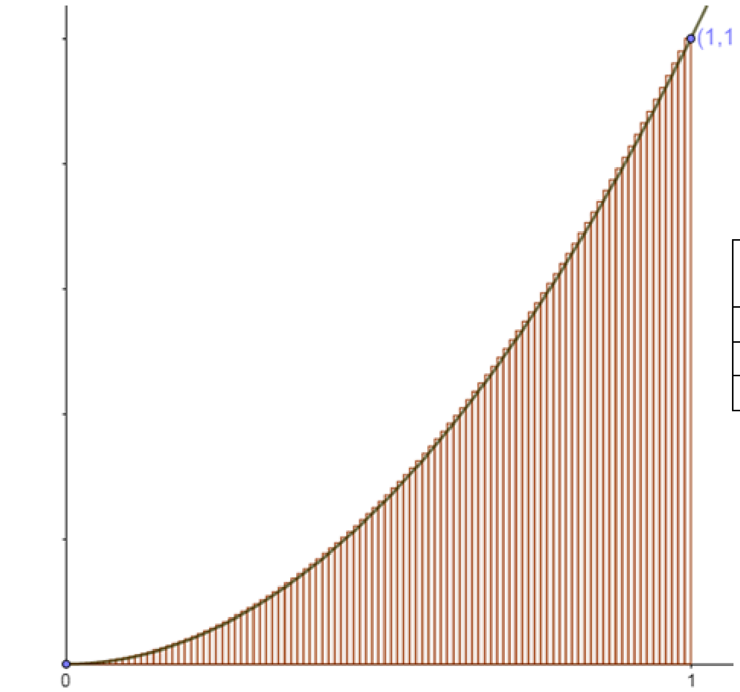


Figure 12: Calculating the area under $f(x) = x^2$ using 10 rectangles

Number of rectangles	Add the area of each rectangle	Approximate area under the curve
4	$(\frac{1}{4} \times \frac{1}{16}) + (\frac{1}{4} \times \frac{1}{4}) + (\frac{1}{4} \times \frac{9}{16}) + (\frac{1}{4} \times 1)$	0.46875
10	$(0.1 \times 0.01) + (0.1 \times 0.04) + (0.1 \times 0.09)$ $+(0.1 \times 0.16) + (0.1 \times 0.25) + (0.1 \times 0.36)$ $+(0.1 \times 0.49) + (0.1 \times 0.64) + (0.1 \times 0.81)$ $+(0.1 \times 1)$	0.385

The rectangles do not stick out as much above the curve so this must be much closer to the actual area but this is still an over-estimate. If we use 100 rectangles the rectangles will be much narrower at 0.01 units in width. The narrower the rectangles, the closer we will get to the precise area.



Number of rectangles	Approximate area under the curve
4	0.46875
10	0.385
100	0.33835

Figure 13: Calculating the area under $f(x) = x^2$ using 100 rectangles

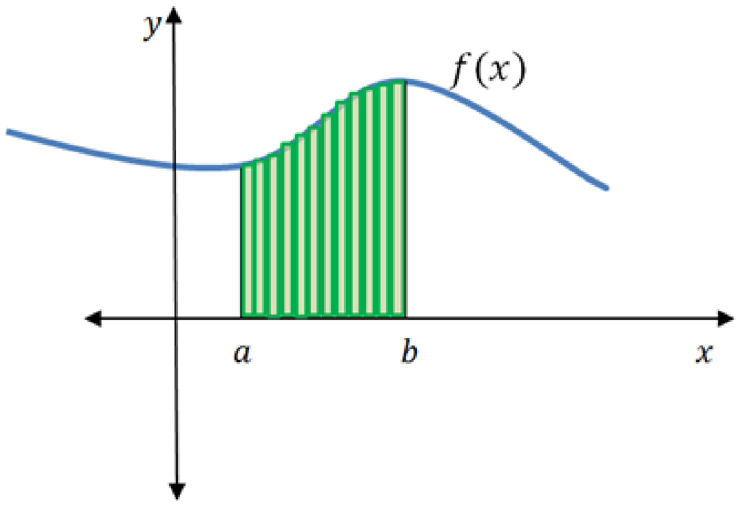
We will not show all of the divisions and calculations, but the table below shows what happens as we increase the number of rectangles all the way to $1\ 000$ rectangles. What number is the area under the curve approaching?

Number of rectangles	**Approximate area under the curve**
4	0.46875
10	0.385
100	0.33835
1000	0.33383

We can clearly see that area is getting closer and closer to $\frac{1}{3}$. You will remember from differential calculus that the word we use to indicate 'getting closer and closer to a specific number' is, **limit**.

Actually we can show, and do show in the next unit, that if we take the limit of an infinite number of rectangles, the sum of the areas of the tiny rectangles will be exactly $\frac{1}{3}$. While we will not show the working here, the area R under the curve $f(x) = x^2$ between the points $(0;0)$ and $(1,1)$ is in fact equal to $\frac{1}{3}$.

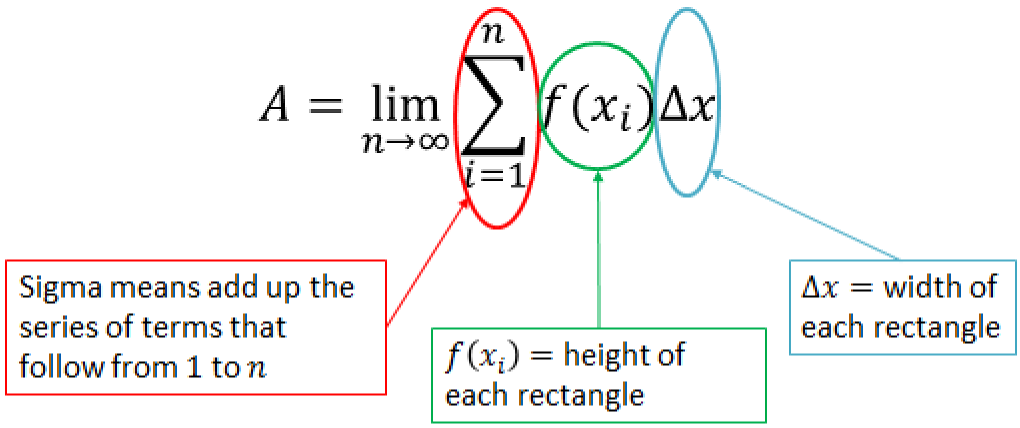
We can apply this technique to find the area between two values under any function. We can represent the sum of the areas of rectangles using sigma notation.



$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

Figure 14: The sum of the areas of rectangles using sigma notation

Take note!



This tells us that we are finding the sum of areas of rectangles.

Taking the $\lim\limits_{n\to\infty}$ of the sum of an infinite number of rectangles gives the exact area under the curve.

Integration allows us to find the area under a curve. We introduced this idea using sigma notation, but integral notation is much more compact to use when we refer to areas under curves.

Did you know?

Calculus is the foundation of modern science. This animated video explains how integral calculus relates to velocity and time (Duration: 19.03).
[link: https://www.youtube.com/watch?v=rjLJIVoQxz4.]

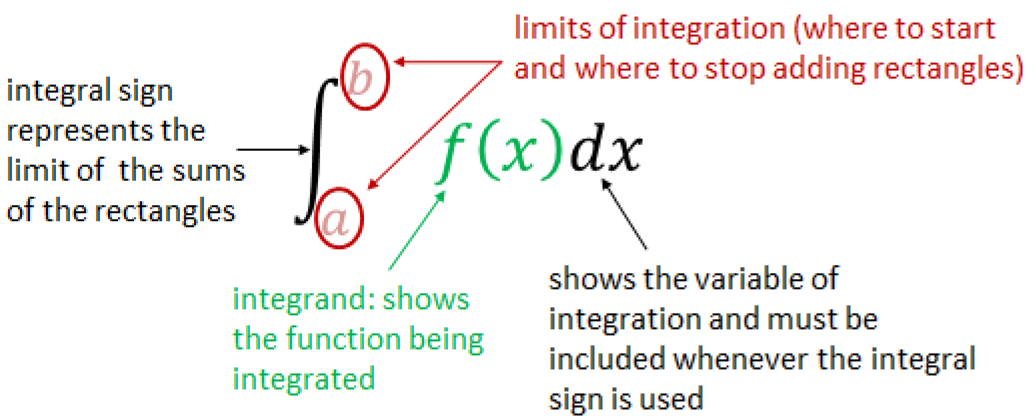
Animated video (Duration: 19.03)

Integral notation

In differentiation there are special notations and instructional symbols used to indicate the derivative. Similarly, we use special notation and symbols when referring to integrals.

The integral sign $\int$ which looks like a long 's' is called 'summa', the Latin word for sum. When we integrate a function, we use special notation $\int f(x)dx$. Each part of integral notation has a name.

Take note!



This is called the **definite integral** and represents the area under a curve for a given interval.

Let us compare sigma notation to integral notation. Which notation would be quicker to write down and easier to work with?

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

VS

$$\int_a^b f(x)dx$$

As you can see it is definitely quicker to use integral notation. While both give the area under the curve it is important to note that these notations are only equivalent when we include $n \to \infty$ in the sigma notation, as shown above. For all other values of n the sigma notation only approximates the value of the definite integral. We use $\int_a^b$ instead of $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n}$ to show that we are calculating the area under a curve.

In the next unit we will show that $\int_0^1 x^2 dx = \frac{1}{3}$. This is the same as showing that the area under the curve $f(x) = x^2$ between the points $(0;0)$ and $(1,1)$ is equal to $\frac{1}{3}$.

Activity 1.1: Explore the area under a curve

Time required: 5 minutes

What you need:

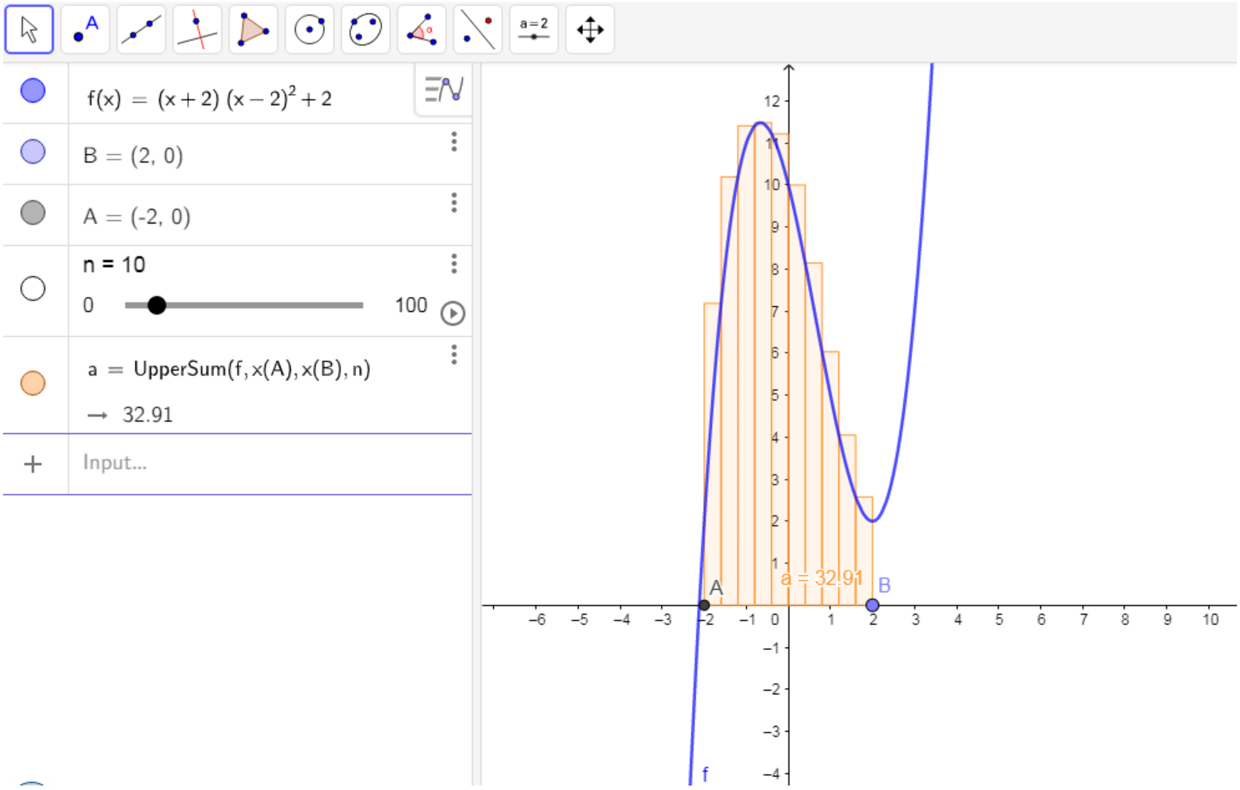
- an internet connection

What to do:

Use the limit of the upper sum of a function to explore the integral.

Click on this link to go to the "Exploring the integral activity".

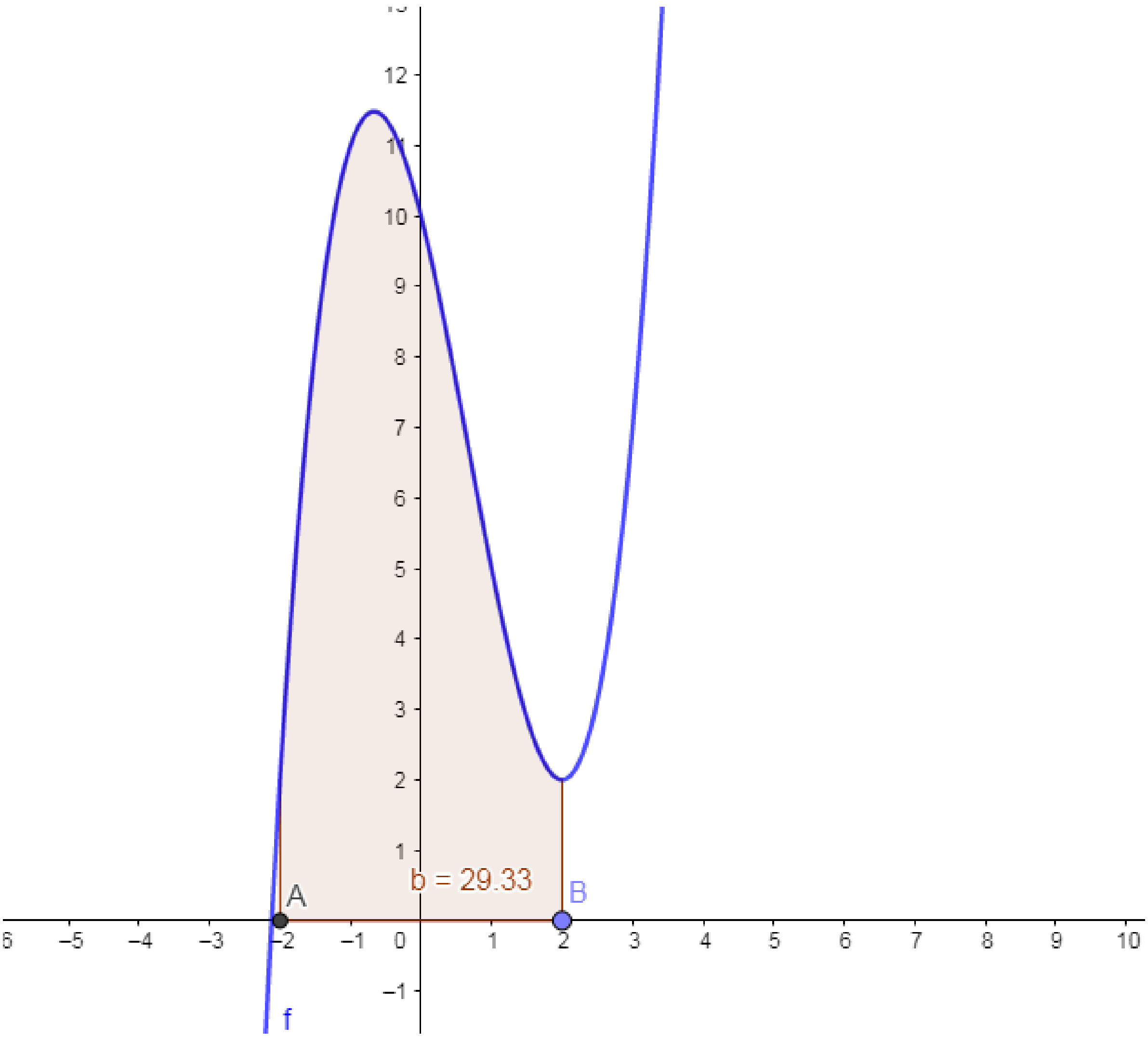
The following screen should appear:



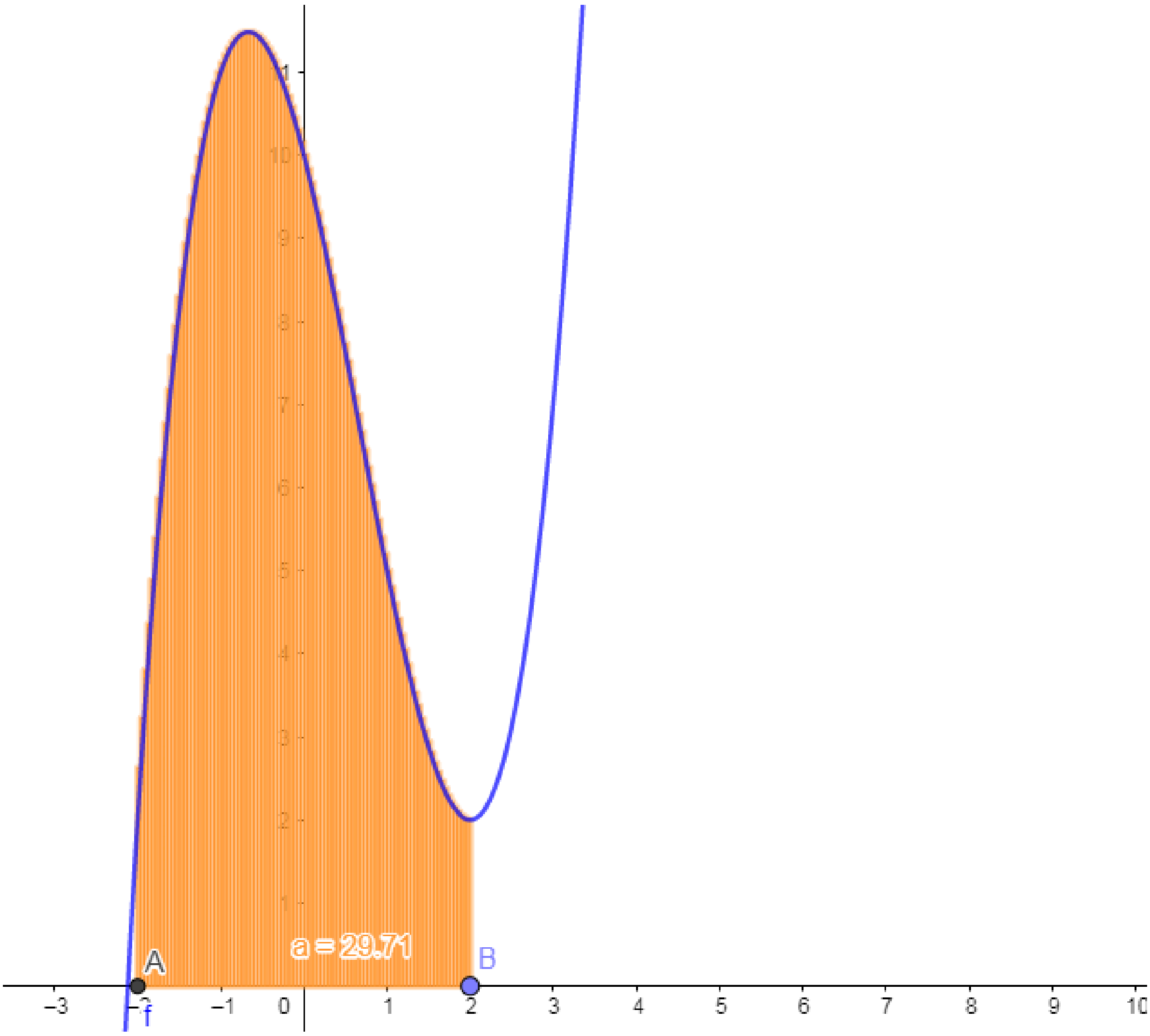
- Press play on the slider n to increase and decrease the number of rectangles over the given interval. Note how the sum of the areas of the rectangles, represented by '$a =$' on the graph, change. You can change the speed of the rectangles appearing by clicking $\ll$ or $\gg$ above the play button.
- Click the circle next to 'UpperSum' to hide the rectangles from the visual. Note: The 'UpperSum' gives you the estimate of the area under the curve for different values of n .
- Next, enter the command 'Integral(f, x(A) ,x(B))' in the box with the + (input) sign and press enter to display the actual area between $f(x)$ and the x-axis over the given interval. The area will be displayed as '$b =$' on the graph. This gives the definite integral under the curve over the interval. If you have entered the command correctly it will display a solid shaded region under the curve between A and B. Make sure that the tiny rectangles are hidden or you will not see the area of the definite integral.
- To display the rectangles again, click the circle to the left of 'UpperSum'. As you change the values of n, zoom in on the graph to see how much the rectangles stick out above the curve.
- Drag and hold the points A and B to move the values along the x-axis.

What did you find?

- As n increases to 100 what value does the sum of the areas of the rectangles approach?
- Is the area b displayed using the integral command, Integral(f, x(A) ,x(B)) , close to the sum of the area a of the rectangles as $n \to 100$?
- Did you get the area of the definite integral $b = 29.33$?



- Did you get the sum of the areas of the tiny rectangles $a = 29.71$ when $n = 100$?



The area using summation and the area using the definite integral are close in value ($a = 29.71$ and $b = 29.33$). The summation of rectangles helps to approximate the definite integral. As the value of n approaches infinity the area found using the sum of the tiny rectangles will be equal to the area found using the definite integral.

- Did you see that the more rectangles you use the less the rectangles stick out above the curve?

Integration reverses differentiation

For a known function $f(x)$ we differentiate to find its derivative $f'(x)$. The reverse of this process is to find the original function $F(x)$ from its derivative. In the 17th century the founders of calculus, Isaac Newton and Gottfried Leibniz discovered that integration is the **inverse** process of differentiation. They proved that evaluating the **anti-derivative** is equivalent to finding the sum of the areas of an infinite number of rectangles under a curve. So, rather than finding the sum of the areas of an infinite number of tiny rectangles all we have to do to evaluate an integral is invert the process of differentiation.

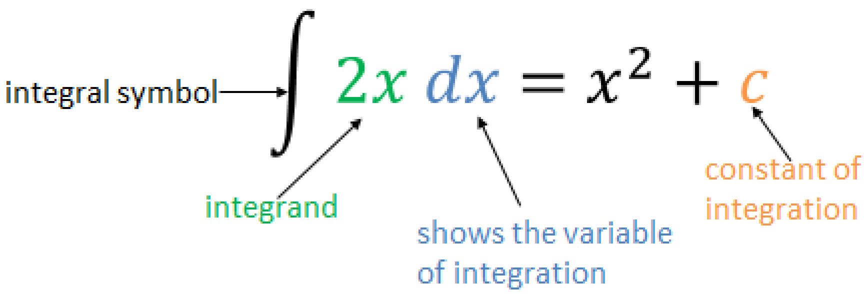
If we differentiate the function $f(x) = x^2$ then its derivative $f'(x) = 2x$. So, the integral of $2x$ takes us back to the original function x^2. We can represent this as shown in figure 15.

Figure 15: Integration reverses differentiation.

Are there other functions that will give us the same derivative of $2x$?

There are lots of functions that will have a derivative of $2x$. For example, $x^2+2,\ x^2+\frac{1}{4},\ x^2-5$ all have the same derivative $2x$. This is because when you differentiate the functions, the derivative of the constant terms will be zero. When we reverse the process of differentiation, we have no idea what the constant term may have been. So we include in the answer a **constant of integration**, this is an unknown constant, which we call c. Therefore, we say the integral or anti-derivative of $2x$ is x^2+c and this accounts for all the possible values of the constant term of the function.

Take note!



To complete the integral sign there is a term of the form dx, which must always be included. It shows the variable involved, in this case x. We read $\int 2x\ dx$ as the integral of $2x$ with respect to x.

The function being integrated is called the integrand. Integrals of this form are called **indefinite integrals,** to distinguish them from **definite integrals**, which we will discuss in the next unit.

Indefinite integrals do not have the upper and lower limits of integration as we are finding a function and not the area under a curve. When you find indefinite integrals, your answer must contain a constant of integration.

Take note!

Integration is simply the evaluation of anti-derivatives. This video called "Fundamental theorem of calculus", explains how integration and differentiation are fundamentally linked.

Fundamental theorem of calculu (Duration: 00.00)

Let's look at an example to explore this further.

Example 1.1

Find the derivatives of $f(x) = x^3$; $f(x) = x^3 + 2$; $f(x) = x^3 - 4$. From there, deduce that $\int 3x^2 \, dx = x^3 + c$.

Solutions

$f(x) = x^3$ and $f'(x) = 3x^2$

$f(x) = x^3 + 2$ and $f'(x) = 3x^2$

$f(x) = x^3 - 4$ and $f'(x) = 3x^2$

The derivatives of all three functions are equal.

$\therefore \int 3x^2 \, dx = x^3 + c$

Any constant will become zero when differentiated so the constant of integration must be included to cover all possible indefinite integrals.

Rules for integration

There are a number of different rules that are used to make the process of integration simple. In this unit we will focus on the following two rules; the power rule for integrals and the constant coefficient rule.

Power rule for integrals $n \neq -1$

$$\int x^n \; dx = \frac{x^{n+1}}{n+1} + c$$

Remember from the power rule for differentiation that the derivative of x^n is $n \cdot x^{n-1}$. To get the antiderivative we must reverse this process by increasing the exponent by one and then divide the result by the new exponent.

Constant coefficient rule:

$$\int kf(x)\ dx = k\int f(x)\ dx$$

A constant in an integral can be moved outside the integral sign.

Example 1.2

Find:

1. $\int 4\ dx$
2. $\int x\ dx$
3. $\int 2x^2\ dx$

Solutions

1. Here we integrate a constant – the variable has a power of zero.

$$\begin{aligned}\int 4\ dx &= \int 4x^0\ dx \\ &= \frac{4x^{0+1}}{0+1} + c \\ &= 4x + c\end{aligned}$$

2. $$\begin{aligned}\int x\ dx &= \frac{x^{1+1}}{1+1} + c \\ &= \frac{x^2}{2} + c\end{aligned}$$

By using the power rule for integration, we increase the power by one and then divide the result by the new power.

3. $$\begin{aligned}\int 2x^2\ dx &= 2\int x^2\ \mathrm{dx} \\ &= 2\left(\frac{x^{2+1}}{2+1}\right) + c \\ &= \frac{2x^3}{3} + c\end{aligned}$$

Keep the constant coefficient.

Note: Since integration is the inverse of differentiation, a way to quickly check your solutions is to derive and see that you arrive at the original expressions.

Exercise 1.1

Find the indefinite integrals of:

1. $\int 5x \ dx$
2. $\int 7 \ dt$
3. $\int y^3 \ dy$
4. $\int \frac{1}{x^4} \ dx$
5. $\int \sqrt{x} \ dx$

The full solutions are at the end of the unit.

Summary

In this unit you have learnt the following:

- How to define the integral as the area under a curve.
- The notation used to represent the definite and indefinite integral.
- The relationship between integration and differentiation.
- How to defined integration as the anti-derivative.
- How to use the power rule for integrals.
- How to use the constant coefficient rule for integrals.

Unit 1: Assessment

Suggested time to complete: 20 minutes

1. Use five rectangles to estimate the area under the curve $f(x) = x^2$ between the points $(0;0)$ and $(1,1)$ (Hint: the width of each rectangle is $\frac{1}{5}$).
2. Find the following integrals:
   a. $\int 3y^2 dy$
   b. $\int \frac{t}{2} \ dt$

c. $\int \sqrt{5x}\ dx$

d. $\int \sqrt[3]{8x}\ dx$

e. $\int \frac{1}{\sqrt[4]{t}}\ dt$ Write the answer in root form.

The full solutions are at the end of the unit.

Unit 1: Solutions

Exercise 1.1

1.

$$\begin{aligned}\int 5x\ dx &= \frac{5x^{1+1}}{1+1}+c\\ &= \frac{5x^2}{2}+c\end{aligned}$$

2.

$$\begin{aligned}\int 7\ dt &= \frac{7t^{0+1}}{0+1}+c \quad \text{Integrate with respect to } t\\ &= 7t+c\end{aligned}$$

3.

$$\int y^3\ dy = \frac{y^4}{4}+c$$

4.

$$\begin{aligned}\int \frac{1}{x^4}\ dx &= \int x^{-4}\ dx \quad \text{Rewrite the integrand using exponents as } x^{-4}\\ &= \frac{x^{-4+1}}{-4+1}+c\\ &= -\frac{x^{-3}}{3}+c\end{aligned}$$

5.

$$\begin{aligned}\int \sqrt{x}\ dx &= \int x^{\frac{1}{2}}\ dx\\ &= \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}+c\\ &= \frac{x^{\frac{3}{2}}}{\frac{3}{2}}+c\\ &= \frac{2}{3}\sqrt{x^3}+c\end{aligned}$$

Back to Exercise 1.1

Unit 1: Assessment

1.

Number of rectangles	Coordinates	Area under the curve
5	$(\frac{1}{5}, \frac{1}{25})$ $(\frac{2}{5}, \frac{4}{25})$ $(\frac{3}{5}, \frac{9}{25})$ $(\frac{4}{5}, \frac{16}{25})$ $(1, 1)$	$(\frac{1}{5} \times \frac{1}{25}) + (\frac{1}{5} \times \frac{4}{25}) + (\frac{1}{5} \times \frac{9}{25}) + (\frac{1}{5} \times \frac{16}{25}) + (\frac{1}{5} \times 1) = 0.44$

2.

a.

$$\int 3y^2\,dy = y^3 + c$$

b.

$$\begin{aligned}\int \frac{t}{2}\;dt &= \int \frac{1}{2}t\;dt \\ &= \frac{1}{2}\left(\frac{t^2}{2}\right) \\ &= \frac{t^2}{4} + c\end{aligned}$$

c.

$$\begin{aligned}\int \sqrt{5x}\;dx &= \int \sqrt{5}x^{\frac{1}{2}}\;dx \\ &= \sqrt{5}\left(\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right) \\ &= \frac{2\sqrt{5}x^{\frac{3}{2}}}{3} + c\end{aligned}$$

d.

$$\begin{aligned}\int \sqrt[3]{8x}\;dx &= \int 2x^{\frac{1}{3}}\;dx \\ &= 2\left(\frac{x^{\frac{4}{3}}}{\frac{4}{3}}\right) \\ &= \frac{3x^{\frac{4}{3}}}{2} + c\end{aligned}$$

e.

$$\int \frac{1}{\sqrt[4]{t}}\, dt = \int t^{\frac{-1}{4}}\, dt$$
$$= \frac{t^{\frac{3}{4}}}{\frac{3}{4}}$$
$$= \frac{4t^{\frac{3}{4}}}{3}$$
$$= \frac{4}{3}\sqrt[4]{t^3}$$

Back to Unit 1: Assessment

Media Attributions

Unit 2: Rules for integration

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Unit outcomes

By the end of this unit you will be able to:

- Apply the rules of integration to various functions.
- Calculate the definite integral.

What you should know

Before you start this unit, make sure you can:

- Define integration. To revise this, refer to unit 1 of this subject outcome.
- Apply the various techniques of differentiation by rule. To revise this, refer back to level 4 subject outcome 2.4 unit 1.
- Evaluate simple integrals by reversing the process of differentiation.

Try the following questions to make sure you are ready for this unit.

Find the indefinite integrals of:

1. $\int 5x \ dx$
2. $\int \sqrt{x} \ dx$
3. $\int \sqrt[3]{8x} \ dx$

Solutions

1. $$\begin{aligned}\int 5x \ dx &= \frac{5x^{1+1}}{1+1} + c \\ &= \frac{5x^2}{2} + c\end{aligned}$$
2.

$$\begin{aligned}\int \sqrt{x}\ dx &= \int x^{\frac{1}{2}}\ dx \\ &= \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}+c \\ &= \frac{x^{\frac{3}{2}}}{\frac{3}{2}}+c \\ &= \frac{2}{3}\sqrt{x^3}+c\end{aligned}$$

3.

$$\begin{aligned}\int \sqrt[3]{8x}\ dx &= \int 2x^{\frac{1}{3}}\ dx \\ &= 2\left(\frac{x^{\frac{4}{3}}}{\frac{4}{3}}\right) \\ &= \frac{3x^{\frac{4}{3}}}{2}+c\end{aligned}$$

Introduction

Integration and differentiation are inverse processes. The inverse relationship between differentiation and integration means that, for every statement about differentiation, there is a corresponding statement about integration. Finding the **antiderivative** is the same as finding an **indefinite integral**. The answer to an indefinite integral will always be some function.

Take note!

Given a derivative function $f(x)$

The antiderivative $= F(x)$

Finding the derivative of the antiderivative will take you back to the original function

$$F'(x) = f(x)$$

If $\frac{d}{dx}(F(x)) = f(x)$ then $\int f(x)dx = F(x)$.

In other words, if the derivative of $F(x)$ is $f(x)$, then an indefinite integral of $f(x)$ with respect to x is $F(x)$.

Note: we say ***an*** indefinite integral, not ***the*** indefinite integral. This is because an indefinite integral is not unique.

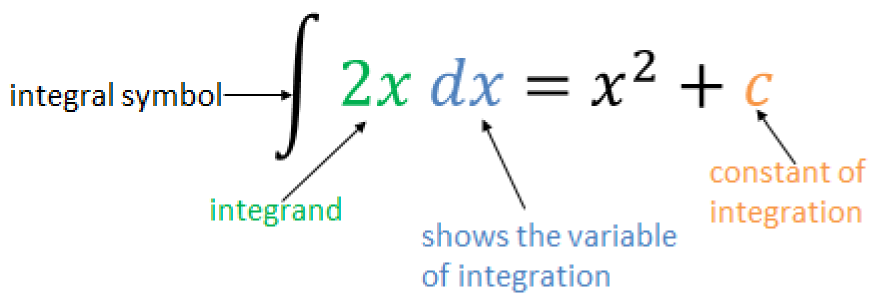
In unit 1 you learnt how to apply the power and constant coefficient rules for integration. These rules were fairly straightforward but there are many other integration rules, which get more complicated the more complex the functions become. In this unit we will go over the strategies that you will use when integrating different types of functions.

Integral rules

You have already seen, in the previous unit, that there are lots of functions with the same derivative. For example, $x^2+2,\ x^2+\frac{1}{4},\ x^2-5$ all have the same derivative $2x$. This is because when you differentiate the functions, the derivative of the constant terms will be zero. When we reverse the process of differentiation, we have no idea what the constant term may have been. So, we include in the answer a **constant of integration**. This is an unknown constant, which we call c.Therefore, we say the indefinite integral or antiderivative of $2x$ is x^2+c and this accounts for all the possible values of the constant term of the function.

We have defined integration as the inverse of differentiation but more precisely it is an indefinite integral that is the inverse of differentiation.

Take note!



The above integral is called an indefinite integral. Given a function $f(x)$, an indefinite integral of 'f' is its most general antiderivative. If $F(x)$ is an antiderivative of 'f' then $\int f(x)dx = F(x) + c$. The act of finding the antiderivatives of a function is referred to as integrating.

The rules for integration of functions are derived from the rules for differentiation.

Sum and difference rule:

$$\int (f(x) \pm g(x))dx = \int f(x)dx \pm \int g(x)dx$$

It sometimes helps to understand and remember rules like this if you state it in words. The above rule states that the integral of the sum (or difference) of two functions is the sum (or difference) of their integrals. We can easily extend this rule to include more than two terms.

For example, $\int (5x^3 - 7x^2 + 3x + 4)dx = \int 5x^3 dx - \int 7x^2 dx + \int 3x dx + \int 4dx$.

Take note!

You have already applied the following rules in the previous unit.

Power rule for integrals $n \neq -1$:

$$\int x^n \ dx = \frac{x^{n+1}}{n+1} + c$$

Constant coefficient rule:

$$\int kf(x) \ dx = k \int f(x) \ dx$$

Example 2.1

Find the following indefinite integrals:

1. $\int (5x^3 - 7x^2 + 3x + 4)dx$
2. $\int \frac{x^2 + 4\sqrt[3]{x}}{x} dx$

Solutions

1. Using properties of indefinite integrals, we can integrate each of the four terms in the integrand separately.
$\int (5x^3 - 7x^2 + 3x + 4)dx = \int 5x^3 dx - \int 7x^2 dx + \int 3x dx + \int 4dx$
From the constant coefficient rule each coefficient can be written in front of the integral sign.
$\int 5x^3 dx - \int 7x^2 dx + \int 3x dx + \int 4dx = 5\int x^3 dx - 7\int x^2 dx + 3\int x dx + 4\int 1dx$
Using the power rule for integrals, we get:
$5\int x^3 dx - 7\int x^2 dx + 3\int x dx + 4\int 1dx = \frac{5}{4}x^4 - \frac{7}{3}x^3 + \frac{3}{2}x^2 + 4x + c$
Note: We do not repeat the 'c' for each term, we use one 'c' for the entire integral.

2. Rewrite the integrand to simplify the exponents.
$$\int \frac{x^2+4\sqrt[3]{x}}{x}dx = \int \frac{x^2}{x} + \frac{4\sqrt[3]{x}}{x}dx$$
Then, to evaluate the integral, integrate each of these terms separately.
$$\begin{aligned}\int \frac{x^2}{x} + \frac{4\sqrt[3]{x}}{x}dx &= \int \frac{x^2}{x}dx + \int \frac{4x^{\frac{1}{3}}}{x}dx \\ &= \int xdx + \int 4x^{\frac{-2}{3}}dx \\ &= \frac{1}{2}x^2 + 4\left(\frac{x^{\frac{-2}{3}+1}}{\frac{-2}{3}+1}\right) + c \\ &= \frac{1}{2}x^2 + 12x^{\frac{1}{3}} + c\end{aligned}$$

Exercise 2.1

Find the following integrals using the rules for integration:

1. $\int (2x^3 + x^2 - 2x)dx$
2. $\int y(\frac{1}{y} + y^2)dy$
3. $\int (\sqrt{x}+\sqrt{3x})dx$

The full solutions are at the end of the unit.

Differentiation rules make integrating simple

The key to understanding integration is to understand differentiation. Integration becomes easier when you learn to recognise a given function as the derivative of another function. However, this does not mean you must memorise integration formulae from a table of integrals. There is no need to memorise the integration formulae if you know the differentiation ones.

A good way to become better at integrating is to become familiar with the list of derivatives and to practise recognising a function as the derivative of another function. All you need to do to integrate a function is to recall the way in which an integral is defined as an antiderivative.

Every time you integrate a function, pause and check your answer by differentiating it to see if you get back the original function. This is a very important habit to develop and will help you spot careless mistakes quickly.

For example, say we want to find $\int \cos x dx$, how can we do this using only what we know about derivatives?

We know that $\frac{d}{dx}(\sin x) = \cos x$, so $F(x) = \sin x$ is an antiderivative of $\cos x$. Therefore, every indefinite integral of $\cos x$ has the form $F(x) = \sin x + c$ and every function of the form $\sin x + c$ is an indefinite integral of $\cos x$.

Take note!

When integrating trigonometric functions, the angle is generally measured in radians.
You can convert between radians and degrees by recalling that:

$$\begin{aligned} 360^\circ &= 2\pi \\ \therefore 180^\circ &= \pi \end{aligned}$$

Let us look at one more example where knowing the derivative helps you find the integral.

Example 2.2

Find $\int \frac{\tan x}{\cos x} dx$.

Solution

First, rewrite the integrand.

$$\begin{aligned} \int \frac{\tan x}{\cos x} dx &= \int \tan x \cdot \frac{1}{\cos x} dx \\ &= \int \tan x \sec x dx \end{aligned}$$

Next, you need to think back to the derivative rules (subject outcome 2.4) or look at your formula sheet, to see if you recognise $\tan x \sec x$ as a derivative of another function.

From the derivative rules we see that $\frac{d}{dx} \sec x = \tan x \sec x$.

$$\therefore \int \tan x \sec x dx = \sec x + c$$

The natural logarithm

The $\int \frac{1}{x} dx$ needs special mention. How do we integrate the function $\frac{1}{x}$? You may be tempted to rewrite the function as x^{-1} and apply the power rule for integration. However, $\int x^n\ dx = \frac{x^{n+1}}{n+1} + c$ is not valid when $n = -1$. Why? Well, this is because the denominator would become zero and we cannot divide by 0.

In differential calculus (subject outcome 2.4) you learnt that the derivative of the natural logarithm function $\ln x$ is $\frac{1}{x}$. Since integration is the inverse of differentiation $\int \frac{1}{x} dx$ should be a function whose derivative is $\frac{1}{x}$

. So, this must mean $\int \frac{1}{x}dx = \ln x + c$. However, if x is negative then $\ln x$ is undefined so we must restrict x to positive values only. For $x > 0$, $\int \frac{1}{x}dx = \ln x + c$.

Take note!

Using derivatives, we can arrive at the following indefinite integrals when the functions are multiplied by some constant.

Differentiation rule		Indefinite integral
$\frac{d}{dx}k = 0$	→	$\int kdx = \int kx^0 dx = kx + c$
$\frac{d}{dx}ax^n = n \cdot ax^{n-1}$	→	$\int ax^n dx = \frac{ax^{n+1}}{n+1} + c \;\; for\ n \neq -1$
$\frac{d}{dx}\text{In}(ax) = \frac{a}{x};\ x > 0$	→	$\int \frac{a}{x} dx = a\text{ln}x + c;\ \ x > 0$
$\frac{d}{dx}(ae^{kx}) = ke^{kx}$	→	$\int ae^{kx} dx = \frac{ae^{kx}}{k} + c$
$\frac{d}{dx}a\sin kx = k \cdot a\cos kx$	→	$\int a\cos kx\, dx = \frac{a\sin kx}{k} + c$
$\frac{d}{dx}a\cos kx = -k \cdot a \sin kx$	→	$\int a \sin kx\, dx = \frac{-a\cos kx}{k} + c$
$\frac{d}{dx}a\tan kx = k \cdot a\sec^2 kx$	→	$\int a\sec^2 kx\, dx = \frac{a\tan kx}{k} + c$

Let us have a look at another example.

Example 2.3

With the use of derivative functions, integrate the following.

1. $\int e^x dx$
2. $\int \sin x dx$

3. $\int \sec^2 x dx$

Solutions

1. Since $\frac{d}{dx}(e^x) = e^x$ then $F(x) = e^x$ is an antiderivative of e^x. Therefore, every indefinite integral of e^x has the form $F(x) = e^x + c$ and every function of the form $e^x + c$ is an indefinite integral of e^x.
$\therefore \int e^x dx = e^x + c$
2. $\frac{d}{dx}(\cos x) = -\sin x$
$\therefore \int \sin x dx = -\cos x + c$
Note: $\int -\sin x dx = \cos x + c$
3. $\frac{d}{dx}(\tan x) = \sec^2 x$
$\therefore \int \sec^2 x dx = \tan x + c$

Now try this exercise.

Exercise 2.2

1. Each of the following statements is of the form $\int fx dx = F(x) + c$.Verify that each statement is correct by showing that $F'(x) = f(x)$:
   a. $\int (x + e^x)dx = \frac{1}{2}x^2 + e^x + c$
   b. $\int xe^x dx = xe^x - e^x + c$
   c. $\int x \cos x dx = x \sin x + \cos x + c$
2. Find the following indefinite integrals:
   a. $\int \tan x \cos x dx$
   b. $\int \frac{1}{x^2} + x dx$
   c. $f(x) = 5x^4 + 4x^5 + 2$
   d. $f(x) = \frac{1}{x^2}$
   e. $\int \left(2\sec^2 x + \frac{2}{x}\right) dx$

f. $\int (x - \frac{1}{x})^2 dx$ (Hint: Expand the integrand expression.)

The full solutions are at the end of the unit.

Definite integrals

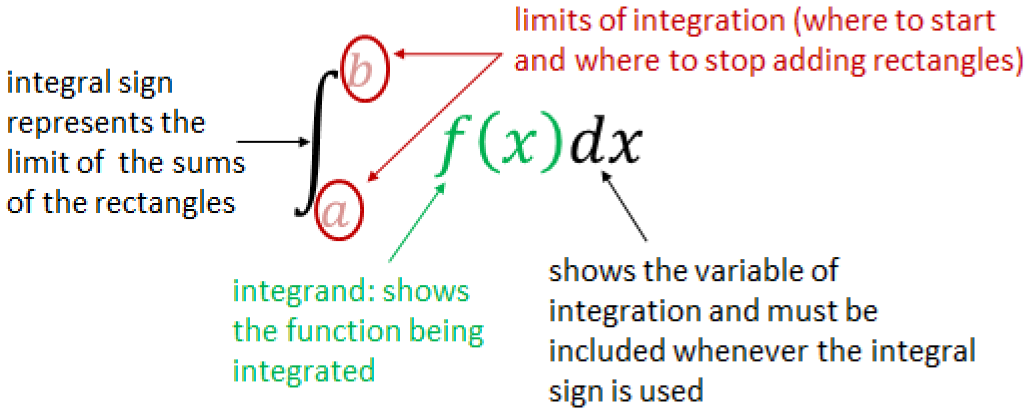
In Unit 1: Introduction to integration, we saw that we could represent the area under a curve using summation notation $\text{A} = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\Delta x$. A sum of this form is called a **Riemann sum**, named after the 19th-century mathematician Bernhard Riemann, who developed the idea.

Did you know?

Riemann sums are the formal application of the method of exhaustion. You have seen the area under a curve defined in terms of Riemann sums. You can click on this link for a graphical illustration of the definition of Riemann sums.

We saw in unit 1 that since differentiation and integration are inverses, we can easily find the area under a curve by finding definite integrals.

Take note!



This is called the **definite integral** and represents the area under a curve, over a given interval and bound by the x-axis.

We have shown that we represent the area under a curve between the x-axis and over a given interval by the definite integral $\int_a^b f(x)dx$. Will the answer to the definite integral be a value or a function?

The answer to the definite integral will be a numerical value because it is giving us the value of the area of a curve between the lower and upper limits of an interval.

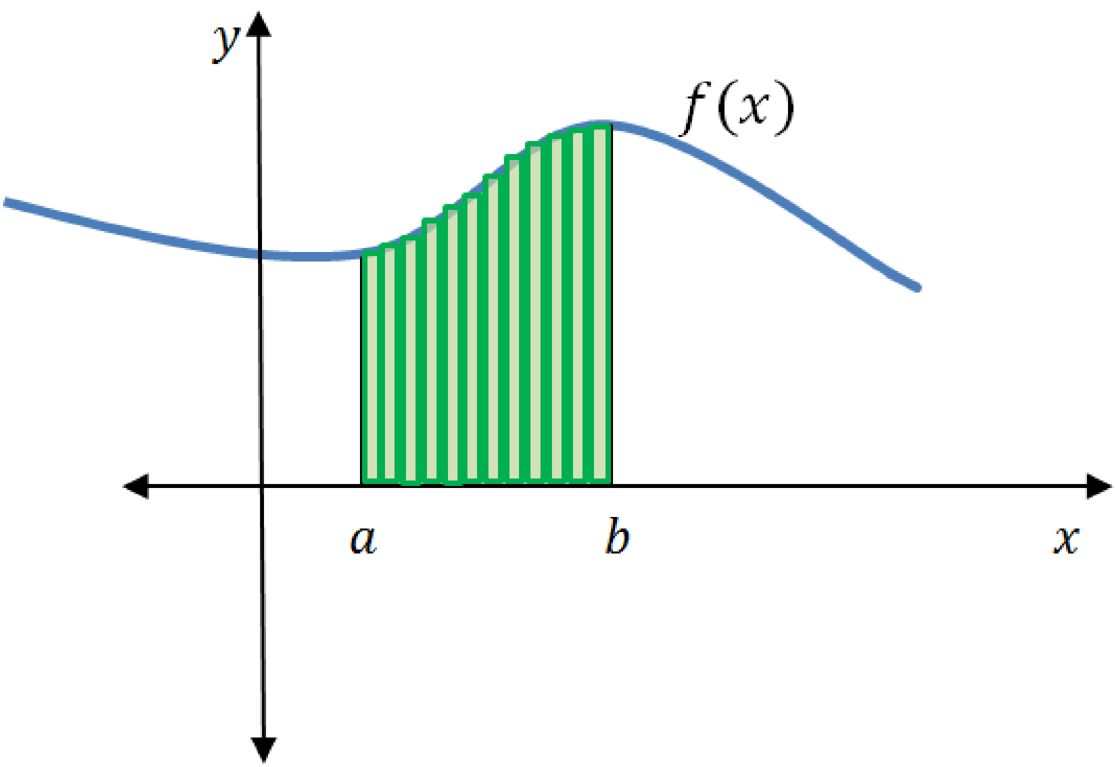


Figure 1: The area of a curve between the lower and upper limits of an interval

$$\int_a^b f(x)dx = F(b) - F(a)$$

$\int_a^b f(x)dx$ tells us that we need to integrate the function $f(x)$ with respect to x over the interval a to b.

$F(b) - F(a)$ means we need to find the value of the antiderivative at the upper limit b and subtract from that the value of the antiderivative at the lower limit a. The limits of integration are the x-values or the independent values.

We often use the notation $F(x)|_a^b$ to denote the expression $F(b) - F(a)$. We use the vertical bar and associated limits a and b to indicate that we should evaluate the function $F(x)$ at the upper limit and subtract the value of the function $F(x)$ at the lower limit.

Activity 2.1: Discover the definite integral for different limits of integration

Time required: 15 minutes

What you need:

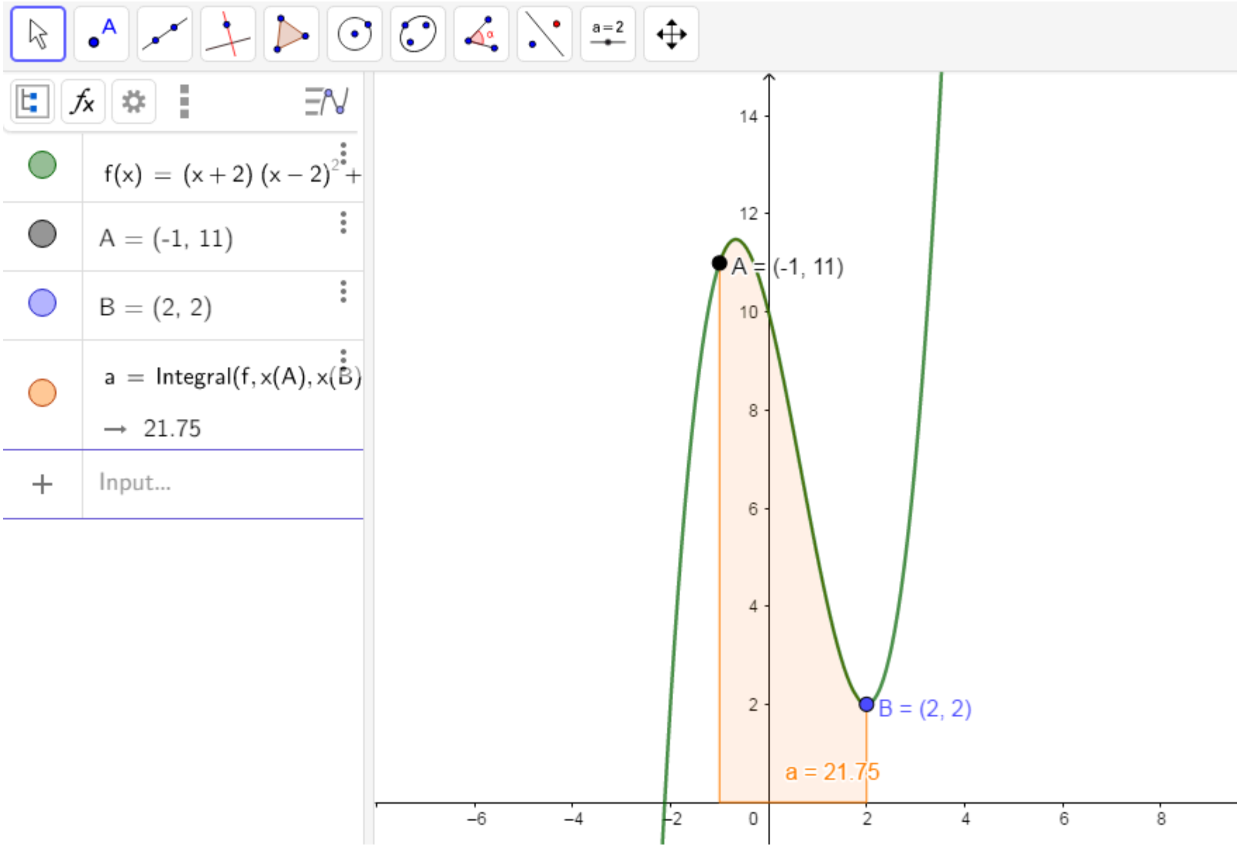
- a pen and paper
- an internet connection

What to do:

Find $\int_a^b (x^3 - 2x^2 - 4x + 10)dx$ for different values of a and b.

Step 1: Click on this link to go to the activity.

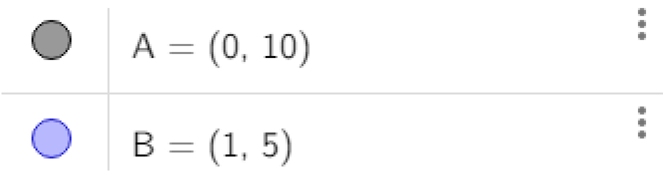
The following screen should appear.



Step 2: From the graph drawn in Geogebra, what is the value of $\int_a^b (x^3 - 2x^2 - 4x + 10)dx$ between $\mathrm{A}(-1; 11)$ and $\mathrm{B}(2; 2)$? (Hint: This is the value shown as 'Area a' on the graph.)

Step 3: Calculate $\int_{-1}^{2} (x^3 - 2x^2 - 4x + 10)dx$. Show all working. What have you calculated?

Step 4: Using the graph, drag and move point A to $(0; 10)$ and drag and move B to $(1; 5)$. You can achieve the same result by just entering the coordinates of A and B on the left of the Geogebra screen as shown here:



What are the limits of integration now?

Step 5: From the graph, read off the integral at the new limits of integration. What does this tell you?

Step 6: Calculate $\int_0^1 (x^3 - 2x^2 - 4x + 10)dx$. Show all working. Compare this to your answer from the graph. What do you notice?

What did you find?

- From the graph you can read off the value of $\int_a^b (x^3 - 2x^2 - 4x + 10)dx$, where $a = -1$ and $b = 2$. We see that the area between $\mathrm{A}(-1; 11)$ and $\mathrm{B}(2; 2)$ is 21.75.

- Using the rules for integration we can calculate $\int_{-1}^{2}(x^3-2x^2-4x+10)dx$ as follows:

$$\begin{aligned}\int_{-1}^{2}(x^3-2x^2-4x+10)dx &= \frac{x^4}{4}-\frac{2x^3}{3}-\frac{4x^2}{2}+10x\Big|_{-1}^{2} \\ &= \left[\frac{2^4}{4}-\frac{2(2)^3}{3}-2(2)^2+10(2)\right]-\left[\frac{(-1)^4}{4}-\frac{2(-1)^3}{3}-2(-1)^2+10(-1)\right] \\ &= \frac{32}{3}-(-\frac{133}{12}) \\ &= \frac{87}{4} \\ &= 21.75\end{aligned}$$

  You have calculated that the area under the curve between the x-axis and points $\text{A}(-1;11)$and $\text{B}(2;2)$ is 21.75. This is the same as the answer from the graph.
- The limits of integration are the x-values of the co-ordinates of A and B. Therefore, $a=0$ and $b=1$ are the limits of integration.
- From the graph we read that the area is 7.58. This means that the area under the curve and above the x-axis between $(0;10)$ and $(1;5)$ is 7.58.
- Use the rules of integration to integrate and then evaluate.

$$\begin{aligned}\int_{0}^{1}(x^3-2x^2-4x+10)dx &= \frac{x^4}{4}-\frac{2x^3}{3}-\frac{4x^2}{2}+10x\Big|_{0}^{1} \\ &= \left[\frac{1^4}{4}-\frac{2(1)^3}{3}-2(1)^2+10(1)\right]-[0] \\ &= \frac{91}{12} \\ &\approx 7.58\end{aligned}$$

The answer we calculated is the same as the one shown on the graph as we are also calculating the integral between $(0;10)$ and $(1;5)$. So we have confirmed that area under the curve and above the x-axis between $(0;10)$ and $(1;5)$ is7.58.

You may recall from unit 1 that we promised to show that $\int_{0}^{1}x^2dx=\frac{1}{3}$. This is the same as showing that the area under the curve $f(x)=x^2$ between the points $(0;0)$ and $(1,1)$ is equal to exactly $\frac{1}{3}$.

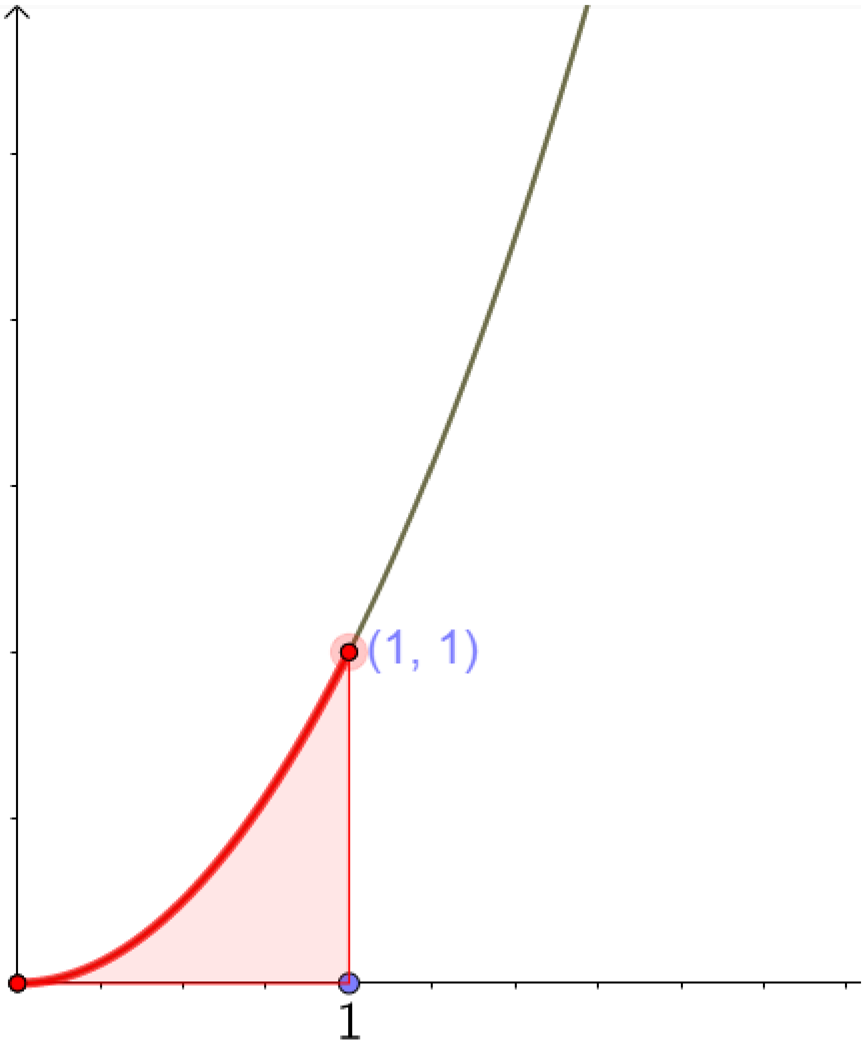


Figure 2: The area under the curve $f(x) = x^2$ *between the points* $(0;0)$ *and* $(1,1)$ *is equal to exactly* $\frac{1}{3}$

Using $\int_a^b f(x)dx = F(b) - F(a)$, we can evaluate the definite integral of $f(x) = x^2$ between $[0,1]$.

We rewrite this as $\int_0^1 x^2 dx$. Notice that the upper limit is one and the lower limit is zero and we have replaced $f(x)$ with the function x^2. $\int_0^1 x^2 dx = F(1) - F(0)$ but first we must integrate $f(x) = x^2$ using the power rule for integration. You will get that the antiderivative of x^2 is $\frac{x^3}{3}$.

$\int_0^1 x^2 dx = \frac{x^3}{3}\Big|_0^1$ Recall that the vertical line with the upper and lower limits means the evaluation of the antiderivative at one, $F(1)$, minus the antiderivative at zero $F(0)$.

By substituting: $F(1) = \frac{1^3}{3} = \frac{1}{3}$ and $F(0) = \frac{0^3}{3} = 0$

$$\therefore \left.\frac{x^3}{3}\right|_0^1 = \frac{1}{3} - 0 = \frac{1}{3}$$

In the previous unit we saw that this was the area under the curve $f(x) = x^2$ between $[0, 1]$ using the tiny rectangles, as the number of rectangles increased to infinity. So, we can see that evaluating the definite integral gives exactly the same answer as using the summation of tiny rectangles as $n \to \infty$. It was this idea that the founders of calculus used to show why differentiation and integration are inverse processes. The **fundamental theorem of calculus** brings together the two big concepts of integrals and derivatives. Its very name indicates how central this theorem is to the entire development of calculus.

The fundamental theorem of calculus and definite integrals:

If f is continuous over the interval $[a; b]$ and $F(x)$ is any antiderivative of $f(x)$, then $\int_a^b f(x)dx = F(x)|_a^b = F(b) - F(a)$

The theorem states that if we can find an antiderivative for the integrand, then we can evaluate the definite integral by evaluating the antiderivative at the endpoints of the interval and subtracting.

Did you know?

The fundamental theorem of calculus is made up of two parts. While we have covered both parts of the theorem, this video “Fundamental theorem of calculus (Part 2)” explains both versions of the fundamental theorem of calculus in more detail.

Fundamental theorem of calculus (Part 2) (Duration: 04.44)

Let us now look at an example.

Example 2.4

1. Evaluate $\int_1^4 \sqrt{t}(1 + t)dt$.
2. Find the definite integral of $f(x) = x^2 - 3x$ over the interval $[1; 5]$.

Solutions

1. Rewrite the function and simplify using exponent rules.

$$\int_1^4 \sqrt{t}(1+t)dt = \int_1^4 t^{\frac{1}{2}}(1+t)dt$$

$$\int_1^4 t^{\frac{1}{2}}(1+t)dt = \int_1^4 t^{\frac{1}{2}} + t^{\frac{3}{2}}\,dt$$ Apply the power rule of integration

$$\int_1^4 t^{\frac{1}{2}} + t^{\frac{3}{2}}\,dt = \left.\frac{t^{\frac{3}{2}}}{\frac{3}{2}} + \frac{t^{\frac{5}{2}}}{\frac{5}{2}}\right|_1^4$$ Use the vertical line with the limits of integration

$$= \left.\left(\frac{2}{3}t^{\frac{3}{2}} + \frac{2}{5}t^{\frac{5}{2}}\right)\right|_1^4$$

$$= \left[\frac{2}{3}(4)^{\frac{3}{2}} + \frac{2}{5}(4)^{\frac{5}{2}}\right] - \left[\frac{2}{3}(1)^{\frac{3}{2}} + \frac{2}{5}(1)^{\frac{5}{2}}\right]$$

$$= \frac{272}{15} - \frac{16}{15}$$

$$= \frac{256}{15}$$

2. Rewrite the question in integral notation.

$$\int_1^5 (x^2 - 3x)dx$$

$$\int_1^5 (x^2 - 3x)dx = \left.\frac{x^3}{3} - \frac{3x^2}{2}\right|_1^5$$

$$= \left[\frac{(5)^3}{3} - \frac{3(5)^2}{2}\right] - \left[\frac{(1)^3}{3} - \frac{3(1)^2}{2}\right]$$

$$= \frac{25}{6} - (-\frac{7}{6})$$

$$= \frac{16}{3}$$

Exercise 2.3

Evaluate the following definite integrals:

1. $\int_0^4 (3 - x)dx$
2. $\int_{-2}^2 (t^2 - 4)dt$
3. $\int_0^{2\pi} 3\cos 2\theta d\theta$

4. $\int_0^{\frac{\pi}{4}} 2\sec^2 x dx$
5. $\int_1^4 \left(\frac{1+x^2}{\sqrt{x}}\right) dx$

The full solutions are at the end of the unit.

Summary

In this unit you have learnt the following:

- How to use and apply the integral rules.
- How to find; $\int \frac{k}{x} dx, \int e^{kx} dx, \int a \sin kx dx, \int a \cos kx dx, \int a\sec^2 kx dx.$
- The fundamental theorem of calculus.
- How to evaluate the definite integral.

Unit 2: Assessment

Suggested time to complete: 35 minutes

1. Determine the following integrals (leave your answer with positive exponents and in surd form, where applicable):
    a. $\int (2x-3)(1-2x)dx$
    b. $\int \left(\frac{4-4x+x^2}{x-2}\right)dx$
    c. $\int \left(e^{3x} + \frac{1}{\cos^2 x} + \frac{\tan x}{\cos x}\right)dx$
    d. $\int \left(2\cos 3x + \frac{e^{4x}}{4} + \frac{2}{x}\right) dx$
    e. $\int \left(3\sec^2 2x + 2e^{3x} + \frac{1}{x}\right) dx$
2. Evaluate the following:
    a. $\int_1^3 \left(\frac{x+1}{x}\right)dx$
    b. $\int_0^{\pi} \cos 2x dx$
    c. $\int_1^2 \left(\frac{x^4+x^2}{x^3}\right) dx$

The full solutions are at the end of the unit.

Unit 2: Solutions

Exercise 2.1

1.

$$\begin{aligned}\int(2x^3+x^2-2x)dx &= 2\int x^3dx+\int x^2dx-2\int xdx\\ &= 2\cdot\frac{x^{3+1}}{4}+\frac{x^{2+1}}{3}-2\cdot\frac{x^{1+1}}{2}+c\\ &= \frac{x^4}{2}+\frac{x^3}{3}-x^2+c\end{aligned}$$

2.

$$\begin{aligned}\int y(\frac{1}{y}+y^2)dy &= \int(1+y^3)dy\\ &= \int 1dy+\int y^3dy\\ &= y^{0+1}+\frac{y^{3+1}}{4}+c\\ &= y+\frac{y^4}{4}+c\end{aligned}$$

3.

$$\begin{aligned}\int(\sqrt{x}+\sqrt{3x})dx &= \int\sqrt{x}dx+\int\sqrt{3x}\\ &= \int x^{\frac{1}{2}}dx+\sqrt{3}\int x^{\frac{1}{2}}dx\\ &= \frac{x^{\frac{1}{2}+1}}{\frac{3}{2}}+\sqrt{3}\cdot\frac{x^{\frac{1}{2}+1}}{\frac{3}{2}}+c\\ &= \frac{2}{3}x^{\frac{3}{2}}+\frac{2\sqrt{3}}{3}x^{\frac{3}{2}}+c\\ &= \frac{2+2\sqrt{3}}{3}x^{\frac{3}{2}}+c\end{aligned}$$

Back to Exercise 2.1

Exercise 2.2

1.

a. $\int(x+e^x)dx=\frac{1}{2}x^2+e^x+c$

$$\begin{aligned}F(x)&=\frac{1}{2}x^2+e^x+c\\ F'(x)&=x+e^x\\ f(x)&=x+e^x\\ \therefore F'(x)&=f(x)\end{aligned}$$

The statement $\int(x+e^x)dx=\frac{1}{2}x^2+e^x+c$ is correct.

b. $\int xe^xdx=xe^x-e^x+c$

$$F(x) = xe^x - e^x + c$$
$$F'(x) = 1 \cdot e^x + xe^x - e^x + 0 \text{ Use the product rule}$$
$$= xe^x$$
$$f(x) = xe^x$$
$$\therefore F'(x) = f(x)$$

The statement $\int xe^x dx = xe^x - e^x + c$ is correct.

c. $\int x \cos x dx = x \sin x + \cos x + c$

$$F(x) = x \sin x + \cos x + c$$
$$F'(x) = \sin x + x \cos x - \sin x$$
$$= x \cos x$$

Remember: $D_x[x \sin x] = \sin x + x \cos x$ by the product rule

$$f(x) = x \cos x$$
$$\therefore F'(x) = f(x)$$

The statement $\int x \cos x dx = x \sin x + \cos x + c$ is correct.

2.

a.

$$\int \tan x \cos x dx = \int \frac{\sin x}{\cos x} \cos x dx \text{ Rewrite } \tan x \text{ and simplify}$$
$$= \int \sin x dx$$
$$\int \sin x dx = \cos x + c$$

b.

$$\int \frac{1}{x^2} + x dx = \int \frac{1}{x^2} dx + \int x dx$$
$$= \int x^{-2} dx + \int x dx$$
$$= \frac{x^{-2+1}}{-2+1} + \frac{x^{1+1}}{1+1} + c \text{ Use the power rule for integration}$$
$$= -x^{-1} + \frac{1}{2}x^2 + c$$
$$= \frac{-1}{x} + \frac{1}{2}x^2 + c \text{ Write answer with positive exponent}$$

c. $f(x) = 5x^4 + 4x^5 + 2$ Given $f(x)$ so we can rewrite as $\int (5x^4 + 4x^5 + 2)dx$

$$\int (5x^4 + 4x^5 + 2)dx = \frac{5x^5}{5} + \frac{4x^6}{6} + \frac{2x^{0+1}}{1} + c$$
$$= x^5 + \frac{2}{3}x^6 + 2x + c$$

d.

$$f(x) = \frac{1}{x^2}$$
$$\therefore \int \frac{1}{x^2} dx = \int x^{-2} dx$$
$$= \frac{x^{-1}}{-1} + c$$
$$= -\frac{1}{x} + c$$

e.

$$\int \left(2\sec^2 x + \frac{2}{x}\right) dx = \int 2\sec^2 x dx + \int \frac{2}{x} dx$$
$$= 2 \tan x + 2\text{In} x + c$$

f.

$$\begin{aligned}\int (x-\frac{1}{x})^2dx &= \int x^2-2+\frac{1}{x^2}dx \text{ Multiply out integrand first}\\ &= \int x^2dx - \int 2dx + \int x^{-2}dx\\ &= \frac{x^3}{3} - 2x + \left(\frac{x^{-1}}{-1}\right) + c\\ &= \frac{1}{3}x^3 - 2x - \frac{1}{x} + c\end{aligned}$$

Back to Exercise 2.2

Exercise 2.3

1.

$$\begin{aligned}\int_0^4 (3-x)dx &= \int_0^4 3dx - \int_0^4 xdx\\ &= 3x - \frac{x^2}{2}\Big|_0^4\\ &= \left(3(4) - \frac{(4)^2}{2}\right) - \left(3(0) - \frac{(0)^2}{2}\right)\\ &= 12-8\\ &= 4\end{aligned}$$

2.

$$\begin{aligned}\int_{-2}^2 (t^2-4)dt &= \frac{t^3}{3} - 4t\Big|_{-2}^2\\ &= \left(\frac{(2)^3}{3} - 4(\frac{2}{3})\right) - \left(\frac{(-2)^3}{3} - 4(\frac{-2}{3})\right)\\ &= 0\end{aligned}$$

3.

$\int_0^{2\pi} 3\cos 2\theta d\theta$ Use $\frac{d}{dx}a\sin kx = ka\cos kx$ to get $\int a\cos kxdx = \frac{a\sin kx}{k} + c$.

$\int_0^{2\pi} 3\cos 2\theta d\theta = \frac{3\sin 2\theta}{2}\Big|_0^{2\pi}$ Where $a=3, k=2$

$$\begin{aligned}&= \frac{3\sin 2(2\pi)}{2} - \frac{3\sin 2(0)}{2}\\ &= \frac{3(0)}{2} - \frac{3(0)}{2}\\ &= 0\end{aligned}$$

4.

$$\begin{aligned}\int_0^{\frac{\pi}{4}} 2\sec^2 xdx &= 2\tan x|_0^{\frac{\pi}{4}}\\ &= 2\tan(\frac{\pi}{4}) - 2\tan(0) \text{ Using } \int a\sec^2 kxdx = \frac{a\tan kx}{k} + c\\ &= 2(1) - 2(0)\\ &= 2\end{aligned}$$

5.

$$\int_1^4 \left(\frac{1+x^2}{\sqrt{x}}\right)dx = \int_1^4 \left(\frac{1}{x^{\frac{1}{2}}} + \frac{x^2}{x^{\frac{1}{2}}}\right)dx \text{ Divide through by the denominator}$$

$$= \int_1^4 \left(x^{\frac{-1}{2}} + x^{\frac{3}{2}} \right) dx$$

$$= 2x^{\frac{1}{2}} + \frac{2x^{\frac{5}{2}}}{5} \Bigg|_1^4$$

$$= \left(2(4)^{\frac{1}{2}} + \frac{2(4)^{\frac{5}{2}}}{5} \right) - \left(2(1)^{\frac{1}{2}} + \frac{2(1)^{\frac{5}{2}}}{5} \right)$$

$$= \frac{84}{5} - \frac{12}{5}$$

$$= \frac{72}{5}$$

Back to Exercise 2.3

Unit 3: Assessment

1.

a.

Expand and simplify first.

$$\begin{aligned}\int (2x-3)(1-2x)dx &= \int \left(8x - 3 - 4x^2\right) dx \\ &= \int 8x dx - \int 3 dx - \int 4x^2 dx \\ &= \frac{8x^2}{2} - 3x - \frac{4x^3}{3} + c \\ &= 4x^2 - 3x - \frac{4}{3}x^3 + c\end{aligned}$$

b.

Factorise and cancel first.

$$\begin{aligned}\int \left(\frac{4 - 4x + x^2}{x - 2} \right) dx &= \int \left(\frac{(2-x)(2-x)}{-(2-x)} \right) dx \\ &= \int \left(-(2-x)\right) dx \\ &= \int (x-2) dx \\ &= \frac{x^2}{2} - 2x + c\end{aligned}$$

c.

$$\begin{aligned}\int \left(e^{3x} + \frac{1}{\cos^2 x} + \frac{\tan x}{\cos x} \right) dx &= \int \left(e^{3x} + \frac{1}{\cos^2 x} + \tan x \cdot \frac{1}{\cos x} \right) dx \\ &= \int \left(e^{3x} + \sec^2 x + \sec x \tan x \right) dx \\ &= \int e^{3x} dx + \int \sec^2 x dx + \int \sec x \tan x dx \\ &= \frac{e^{3x}}{3} + \tan x + \sec x + c \text{ Since } \frac{d}{dx} \sec x = \sec x \tan x\end{aligned}$$

d.

$$\int \left(2\cos 3x + \frac{e^{4x}}{4} + \frac{2}{x}\right) dx = \frac{2\sin 3x}{3} + \frac{e^{4x}}{4\cdot 4} + 2\ln x + c$$
$$= \frac{2}{3}\sin 3x + \frac{e^{4x}}{16} + 2\ln x + c$$

e.

$$\int \left(3\sec^2 2x + 2e^{3x} + \frac{1}{x}\right) dx = \frac{3\tan 2x}{2} + \frac{2e^{3x}}{3} + \ln x + c$$
$$= \frac{3}{2}\tan 2x + \frac{2}{3}e^{3x} + \ln x + c$$

2.

a.

$$\int_1^3 \left(\frac{x+1}{x}\right) dx = \int_1^3 \left(\frac{x}{x} + \frac{1}{x}\right) dx$$
$$= \int_1^3 \left(1 + \frac{1}{x}\right) dx$$
$$= x + \ln x|_1^3$$
$$= (3 + \ln 3) - (1 + \ln 1)$$
$$= 3 + \ln 3 - 1$$
$$= 2 + \ln 3$$

b.

$$\int_0^\pi \cos 2x dx = \frac{\sin 2x}{2}\Big|_0^\pi$$
$$= (\frac{\sin 2\pi}{2}) - (\frac{\sin 2(0)}{2})$$
$$= 0$$

c.

$$\int_1^2 \left(\frac{x^4 + x^2}{x^3}\right) dx = \int_1^2 \left(\frac{x^4}{x^3} + \frac{x^2}{x^3}\right) dx$$
$$= \int_1^2 \left(x + \frac{1}{x}\right) dx$$
$$= \frac{x^2}{2} + \ln x\Big|_1^2$$
$$= \left(\frac{2^2}{2} + \ln 2\right) - \left(\frac{1^2}{2} + \ln 1\right)$$
$$= 2 + \ln 2 - \frac{1}{2}$$
$$= \frac{3}{2} + \ln 2$$

Back to Unit 2: Assessment

Media Attributions

Unit 3: Determine the area under a curve

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Unit outcomes

By the end of this unit you will be able to:

- Find the area under a curve between two points and bound by the x-axis.

What you should know

Before you start this unit, make sure you can:

- Sketch cubic functions.
- Apply the rules of integration.
- Find the definite integral.
- Find the indefinite integral.

Try these questions to make sure you are ready for this unit:

1. Find: $\int \left(\frac{4-4x+x^2}{x-2}\right)dx$
2. Calculate: $\int_{-1}^{2} (x^3 - 2x^2 - 4x + 10)dx$
3. Evaluate: $\int_0^{\pi} \cos 2x dx$

Solutions

1.
$$\begin{aligned}\int \left(\frac{4-4x+x^2}{x-2}\right)dx &= \int \left(\frac{(2-x)(2-x)}{-(2-x)}\right)dx \\ &= \int (-(2-x))\, dx \\ &= \int (x-2)dx \\ &= \frac{x^2}{2} - 2x + c\end{aligned}$$

2.

$$\int_{-1}^{2} (x^3 - 2x^2 - 4x + 10)dx = \frac{x^4}{4} - \frac{2x^3}{3} - \frac{4x^2}{2} + 10x\Big|_{-1}^{2}$$
$$= \left[\frac{2^4}{4} - \frac{2(2)^3}{3} - 2(2)^2 + 10(2)\right] - \left[\frac{(-1)^4}{4} - \frac{2(-1)^3}{3} - 2(-1)^2 + 10(-1)\right]$$
$$= \frac{32}{3} - (-\frac{133}{12})$$
$$= \frac{87}{4}$$
$$= 21.75$$

3.
$$\int_{0}^{\pi} \cos 2x dx = \frac{\sin 2x}{2}\Big|_{0}^{\pi}$$
$$= (\frac{\sin 2\pi}{2}) - (\frac{\sin 2\,(0)}{2})$$
$$= 0$$

Introduction

We have seen in units 1 and 2 of this subject outcome that we can find the area between a curve, the x-axis, and specific **ordinates**, by using integration. In this unit we will look at how to apply this idea in more complicated situations.

The area between a curve and the x-axis

Let us do an activity to revise how to find an area of a segment under a curve.

Activity 3.1: Find the areas of the segments contained between the curve y=x(x-1)(x-2) and the x-axis

Time required: 25 minutes

What you need:

- a pen and paper

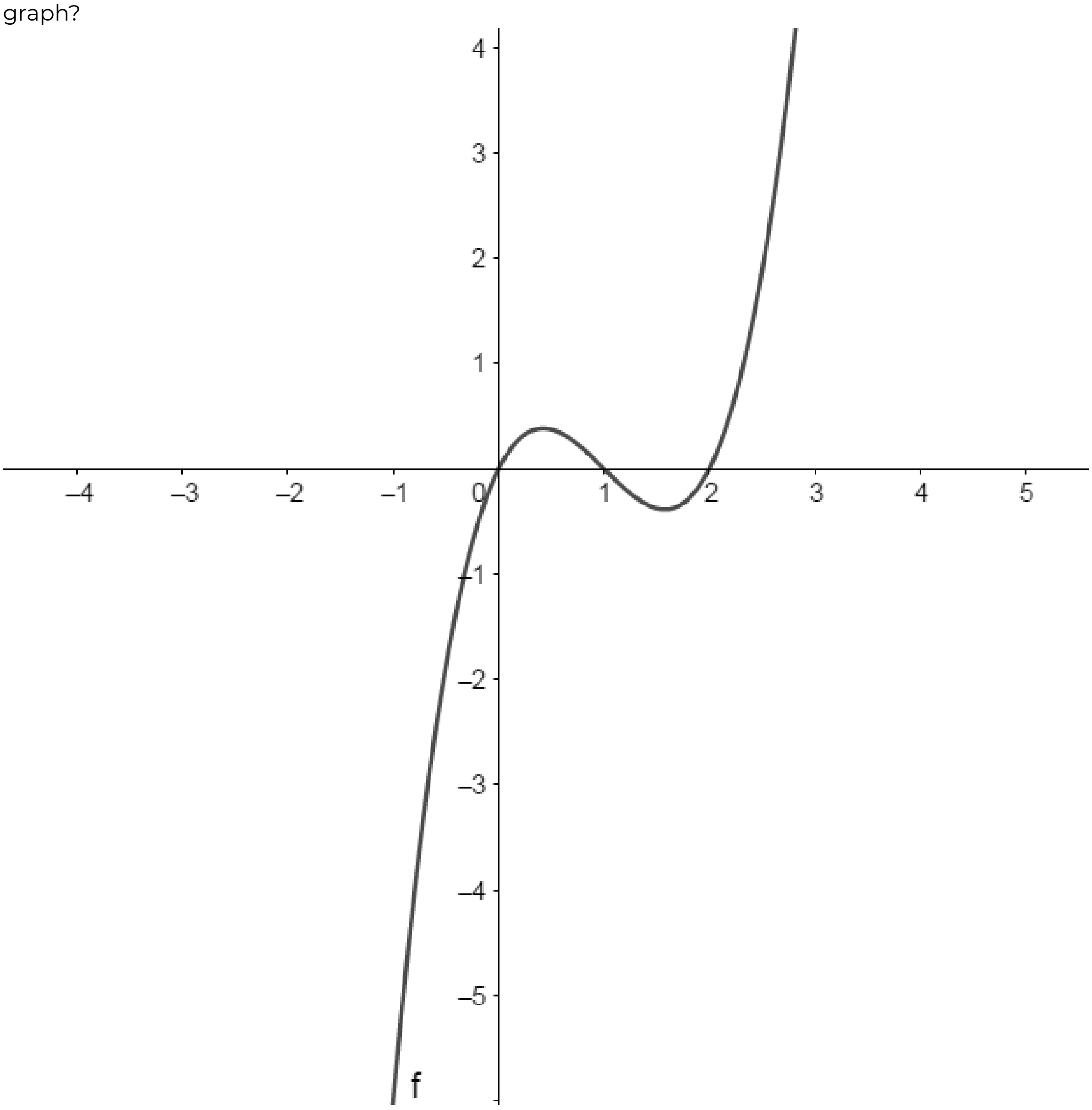
What to do:

1. Sketch the graph of $y = x(x-1)(x-2)$.
2. Write down the x-intercepts of the graph.
3. Mark the areas between the curve and the x-axis as A and B.
4. Write down the position of area A in relation to the x-axis.
5. Write down the position of area B in relation to the x-axis.
6. Decide if the positions of A and B make any difference to the area.
7. Calculate the definite integral of $x(x-1)(x-2)$ between the points $(0, 0)$and $(1, 0)$.
8. State what the value you have just calculated represents.

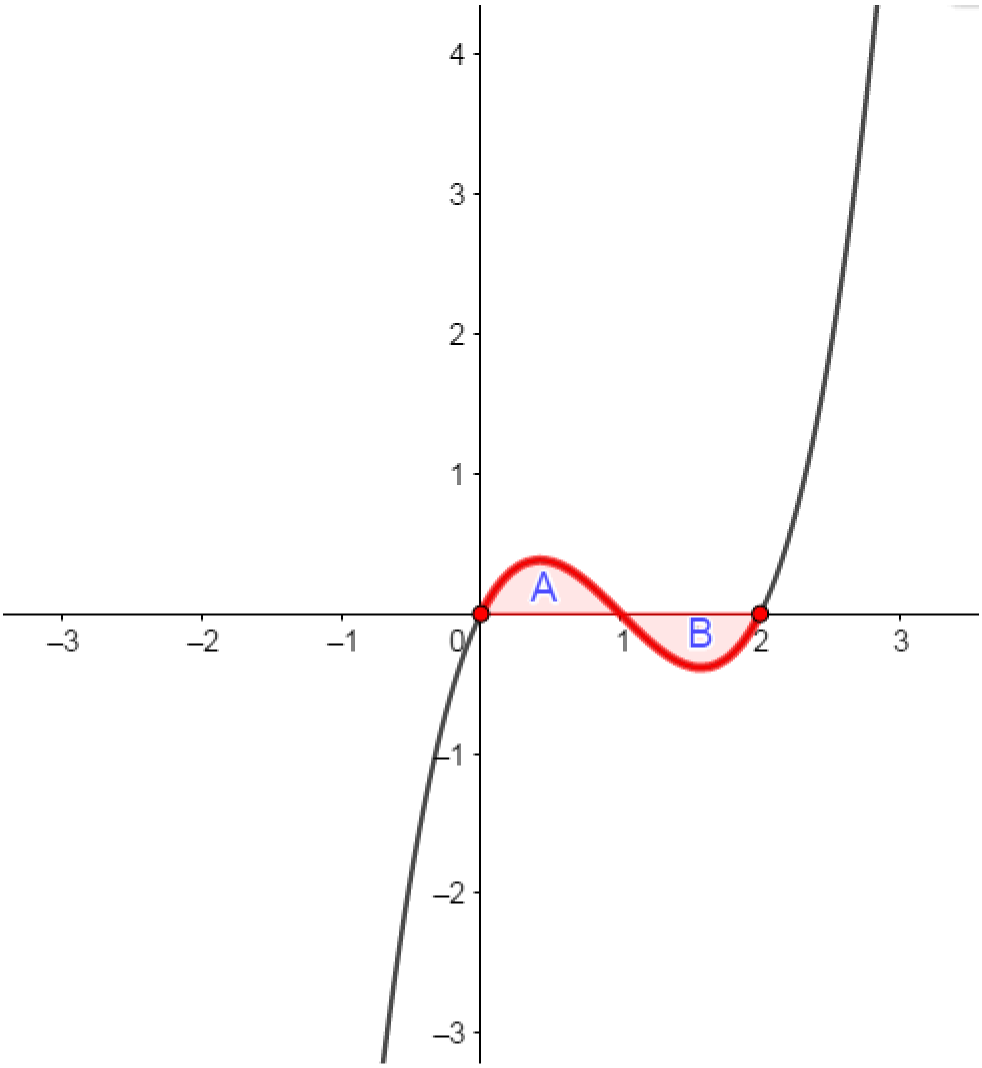
9. Calculate the definite integral of $x(x-1)(x-2)$ between the points $(1,0)$and $(2,0)$.
10. State what the value you have just calculated represents and why this value is negative.
11. Hence, find the total area contained between the curve and the x-axis.
12. Calculate the definite integral of $x(x-1)(x-2)$ between the points $(0,0)$ and $(2,0)$. Compare this answer to the answer you found for the areas of A and B previously.

What did you find?

- Compare the graph you drew to $y = x(x-1)(x-2)$ drawn here. Did you get the same shape of graph?



- The x-intercepts, which are calculated by letting $y = 0$, are $(0,0)$, $(1,0)$ and $(2,0)$.
- A and B shown on the graph represent the areas between the curve and the x-axis.



- We notice that area A lies above the x-axis and area B lies below the x-axis. In previous units we have only calculated areas using integration when the area of the curve is above the x-axis. Does the position of the curve make any difference to the area? Let us calculate the definite integrals over each interval separately so we can see what happens.
- We know that the area A is given by the integral from $x = 0$ to $x = 1$ of the curve $y = x(x-1)(x-2) = x^3 - 3x^2 + 2x$. Therefore:

$$\begin{aligned} A &= \int_0^1 (x^3 - 3x^2 + 2x)dx \\ &= \frac{x^4}{4} - \frac{3x^3}{3} + \frac{2x^2}{2}\Big|_0^1 \\ &= \frac{x^4}{4} - x^3 + x^2\Big|_0^1 \\ &= \left[\frac{1}{4} - 1 + 1\right] - \left[\frac{0}{4} - 0 + 0\right] \\ &= \frac{1}{4} \end{aligned}$$

- Area B is given by a similar integral, but now the limits of integration are from $x = 1$ to $x = 2$.

$$B = \int_1^2 (x^3 - 3x^2 + 2x)dx$$
$$= \frac{x^4}{4} - x^3 + x^2 \Big|_1^2$$
$$= \left[\frac{16}{4} - 8 + 4\right] - \left[\frac{1}{4} - 1 + 1\right]$$
$$= -\frac{1}{4}$$

- We see that the two integrals have the same magnitude but area A is above the x-axis and area B is below the x-axis, and the sign of the value B is negative. The actual or absolute value of the area B is $\frac{1}{4}$, so why does our calculation give a negative answer? In unit 1: Introduction to integration, when we summed all the tiny rectangles, we used $\sum f(x_i)\Delta x$. Recall that Δx was defined as a small positive increment in x but $f(x_i)$ was simply the y-value at the corresponding x-value. Clearly this y-value will be negative if the curve is below the x-axis, so in this case, the quantity $f(x_i)\Delta x$ gives a negative answer as this part of the curve is fully below the x-axis. The negative just tells us that the area lies below the curve, but the actual measurement of the area will be $\frac{1}{4}$.
- Area is always positive, but a definite integral can still produce a negative number (a net signed area, which is the area above the x-axis less the area below the x-axis). For example, if this were a profit function, a negative number indicates the company is operating at a loss over the given interval.
- To find the total area enclosed by the curve and the x-axis we add the area of A, $\frac{1}{4}$, to the absolute value of the area B, which is also $\frac{1}{4}$. Thus the total area enclosed by the curve and the x-axis is $\frac{1}{2}$.
- If we had decided to find the total area by calculating the definite integral of $x(x-1)(x-2)$ between the points $(0, 0)$ and $(2, 0)$, without drawing a sketch, what answer would we get?

We would find:

$$Area = \int_0^2 (x^3 - 3x^2 + 2x)dx$$
$$= \frac{x^4}{4} - x^3 + x^2 \Big|_0^2$$
$$= \left[\frac{16}{4} - 8 + 4\right] - \left[\frac{0}{4} - 0 + 0\right]$$
$$= 0$$

So according to this calculation, the area enclosed by the curve and the x-axis is zero. But, we know that this is not the case. We have a sketch to prove it. Clearly what has happened is that the values of the two areas with different signs have been added together in the process of integration, and they have cancelled each other out. A picture can often tell us more about a function than the results of computations.

Thus, the value of the integral evaluated between two x-values is not necessarily the value of the area between the curve, the x-axis and the two ordinates. So, we must be very careful when calculating areas to avoid this particular trap. The best way to find the area is to always draw a sketch of the curve over the required range of values of x .

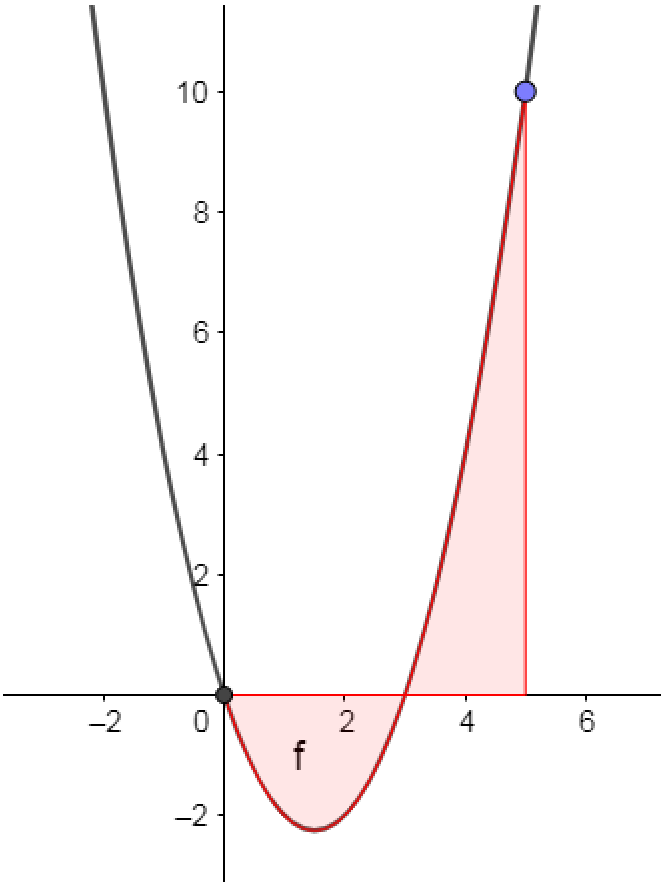
Take note!

When calculating the area between a curve and the x-axis, you should do separate calculations for the parts of the curve above the axis, and the parts of the curve below the axis. The integral for a part of the curve below the axis gives a negative area for that part. It is very helpful to draw a sketch of the curve for the required range of x-values, in order to see how many separate calculations will be needed.

There are times when you will be given a sketch and required to calculate the area based on the diagram. The next example goes through this type of question and shows two methods to find the area when one segment is below the curve and the other above.

Example 3.1

Find the area between the curve $y = x(x-3)$ from $x = 0$ to $x = 5$.



Solution

Method 1:

From the graph, we can see that we need to calculate the area between the curve, the x-axis and $x = 0$ and $x = 3$ first, and that we should expect this integral to give a negative answer because the area is completely below the x-axis.

$$\int_0^3 (x^2 - 3x)dx = \frac{x^3}{3} - \frac{3x^2}{2}\Big|_0^3$$
$$= \left[\frac{27}{3} - \frac{3(9)}{2}\right] - \left[\frac{0}{3} - \frac{3(0)}{3}\right]$$
$$= -4\tfrac{1}{2}$$

Next, we need to calculate the area between the curve, the x-axis, and $x = 3$ and $x = 5$.

$$\int_3^5 (x^2 - 3x)dx = \frac{x^3}{3} - \frac{3x^2}{2}\Big|_3^5$$
$$= \left[\frac{125}{3} - \frac{3(25)}{2}\right] - \left[\frac{27}{3} - \frac{3(9)}{3}\right]$$
$$= 8\tfrac{2}{3}$$

So, the total area is $4\tfrac{1}{2} + 8\tfrac{2}{3} = 13\tfrac{1}{6}$.

Method 2:

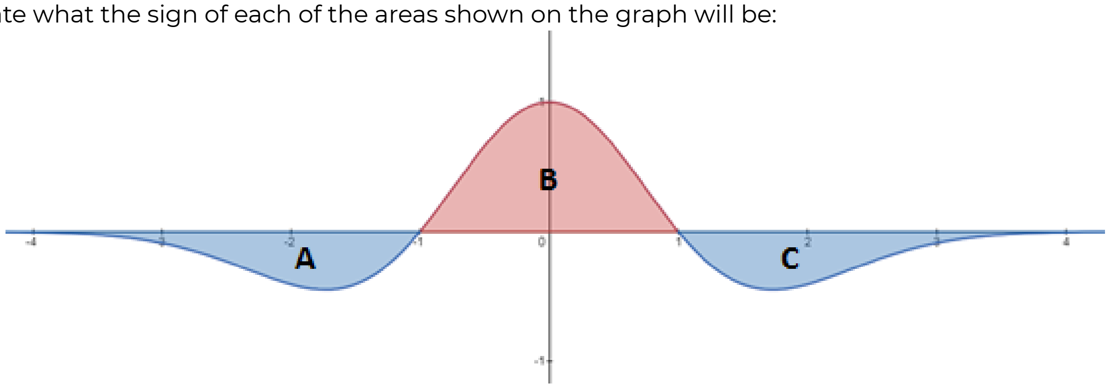
Another way we can arrive at the same answer is to represent the area below the curve using a negative sign in front of the integral.

$$-\int_0^3 (x^2 - 3x)dx + \int_3^5 (x^2 - 3x)dx$$

$$-\int_0^3 (x^2 - 3x)dx + \int_3^5 (x^2 - 3x)dx = -(\frac{x^3}{3} - \frac{3x^2}{2})\Big|_0^3 + (\frac{x^3}{3} - \frac{3x^2}{2})\Big|_3^5$$
$$= -\left[-4\tfrac{1}{2}\right] + \left[8\tfrac{2}{3}\right]$$
$$= 13\tfrac{1}{6}$$

Exercise 3.1

1. State what the sign of each of the areas shown on the graph will be:



2. Find the area bounded by the curve $y = x^2 + x + 4$, the x-axis and the ordinates $x = 1$ and $x = 3$.
3. Find the area contained by the curve $y = x(x - 1)(x + 1)$ and the x-axis.

4. Calculate the value of $\int_{-1}^{1} x(x - 1)(x + 1)dx$. Compare your answer with that obtained in question three and explain what has happened.

The full solutions are at the end of the unit.

Summary

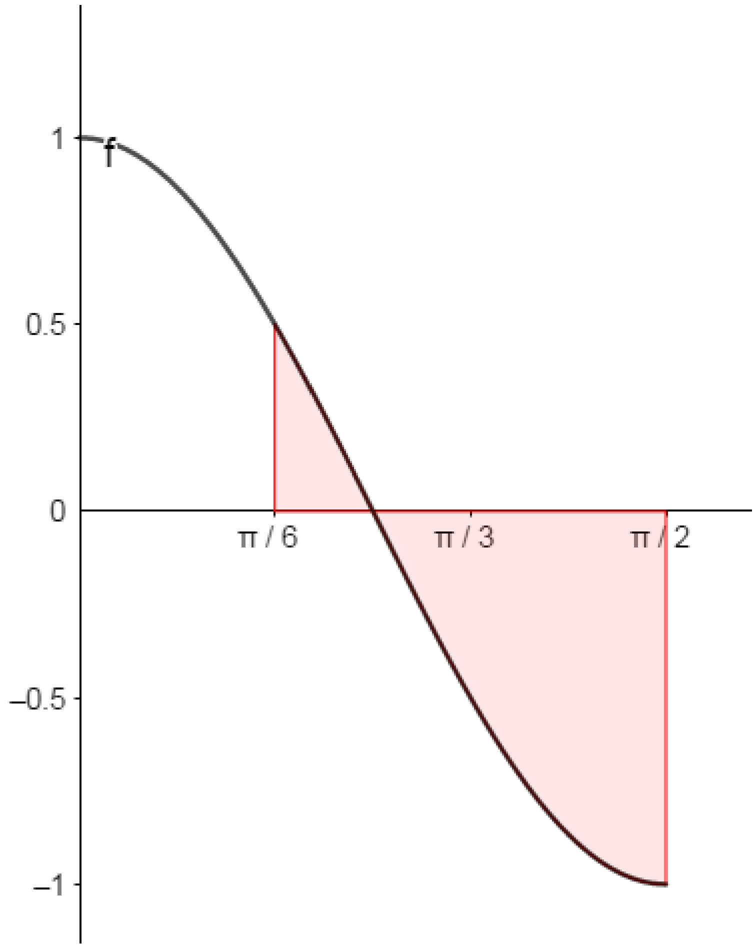
In this unit you have learnt the following:

- How to find the area of a curve above the x-axis over a given interval.
- How to find the area of a curve below the x-axis over a given interval.
- How to find the total area when a curve crosses the x-axis.
- To use a sketch to arrive at the correct calculation of the area under a curve.

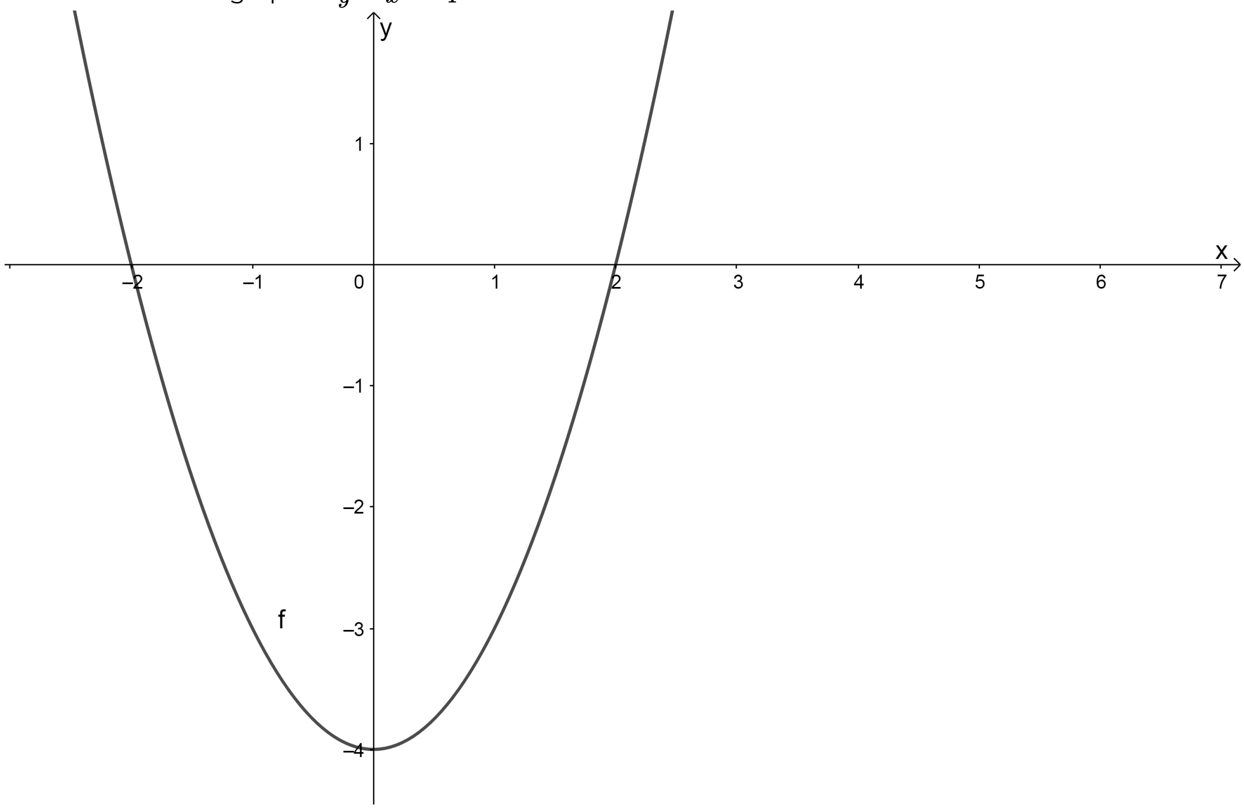
Unit 3: Assessment

Suggested time to complete: 15 minutes

1. Given $f(x) = x^2 - 1$:
    a. Sketch the curve $f(x)$ and clearly shade the region bounded by f and the lines, $x = 1$ and $x = 2$.
    b. Calculate the size of the area of the shaded region using integration.
2. Given below is the graph of $y = \cos 2x$ over the interval $\left[0, \frac{\pi}{2}\right]$. Calculate the area bounded by the curve and the x-axis from $x = \frac{\pi}{6}$ to $x = \frac{\pi}{2}$.



3. Given below is the graph of $y = x^2 - 4$.



Determine the area between the graph of $f(x)$, the x-axis and the x-intercepts.

The full solutions are at the end of the unit.

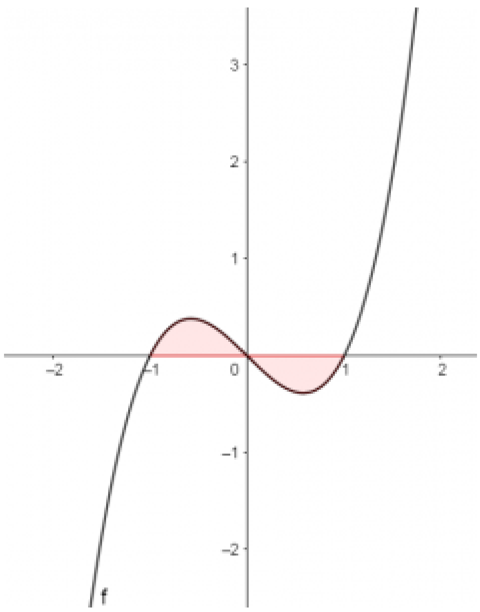
Unit 3: Solutions

Exercise 3.1

1. Area A is completely below the x-axis so the sign of the integral over that interval will be negative. Area B is fully above the x-axis so the sign of the integral over that interval will be positive. Area C is wholly below the x-axis so the sign of the integral over that interval will be negative.
2. $y = x^2 + x + 4$ has no real roots so the entire curve is above the x-axis. In other words, the graph does not cross the x-axis. Since, the required area is entirely above the x-axis we can simply evaluate the integral between the required limits.

$$\int_1^3 (x^2 + x + 4)dx = \frac{x^3}{3} + \frac{x^2}{2} + 4x\bigg|_1^3$$
$$= \left[\frac{27}{9} + \frac{9}{2} + 12\right] - \left[\frac{1}{3} + \frac{1}{2} + \frac{1}{4}\right]$$
$$= 20\tfrac{2}{3}$$

3.



The graph crosses the x-axis at $x = -1; x = 0; x = 1$ so the two areas will have different signs. Calculate each area separately.

Area above the x-axis:

$$\int_{-1}^0 (x^3 - x)dx = \frac{x^4}{4} - \frac{x^2}{2}\bigg|_{-1}^0$$
$$= \left[\frac{0}{4} - \frac{0}{2}\right] - \left[\frac{(-1)^4}{4} - \frac{(-1)^2}{2}\right]$$
$$= \frac{1}{4}$$

Area below the x-axis:

$$\begin{aligned}\int_0^1 (x^3 - x)dx &= \frac{x^4}{4} - \frac{x^2}{2}\Big|_0^1 \\ &= \left[\frac{(1)^4}{4} - \frac{(1)^2}{2}\right] - \left[\frac{0}{4} - \frac{0}{2}\right] \\ &= -\frac{1}{4}\end{aligned}$$

The total area: $\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$.

OR

$$\begin{aligned}\int_{-1}^0 (x^3 - x)dx - \int_0^1 (x^3 - x)dx &= \int_{-1}^0 (x^3 - x)dx - \int_0^1 (x^3 - x)dx \\ &= \frac{x^4}{4} - \frac{x^2}{2}\Big|_{-1}^0 - \frac{x^4}{4} - \frac{x^2}{2}\Big|_0^1 \\ &= \left[\frac{1}{4}\right] - \left[\frac{-1}{4}\right] \\ &= \frac{1}{2}\end{aligned}$$

4.

$$\begin{aligned}\int_{-1}^1 (x^3 - x)dx &= \frac{x^4}{4} - \frac{x^2}{2}\Big|_{-1}^1 \\ &= \left[\frac{(1)^4}{4} - \frac{(1)^2}{2}\right] - \left[\frac{(-1)^4}{4} - \frac{(-1)^2}{2}\right] \\ &= -\frac{1}{4} - (-\frac{1}{4}) \\ &= 0\end{aligned}$$

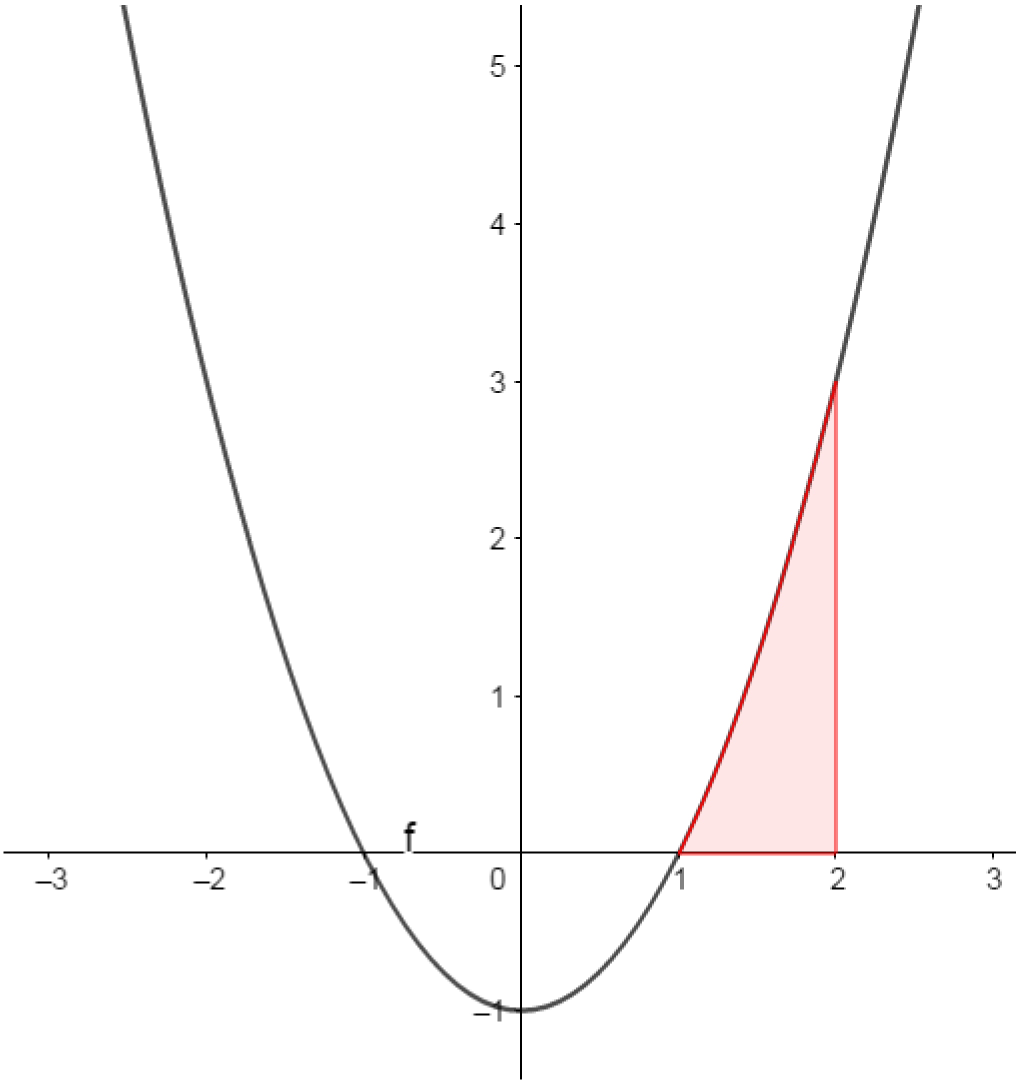
The areas between $x = -1$ and $x = 0$, and between $x = 0$ and $x = 1$, are equal in size. However, one area is above the axis and one is below the axis, so that the integrals have opposite signs and cancel out. Therefore, this does not give the actual area of this curve between those points.

Back to Exercise 3.1

Unit 3: Assessment

1.

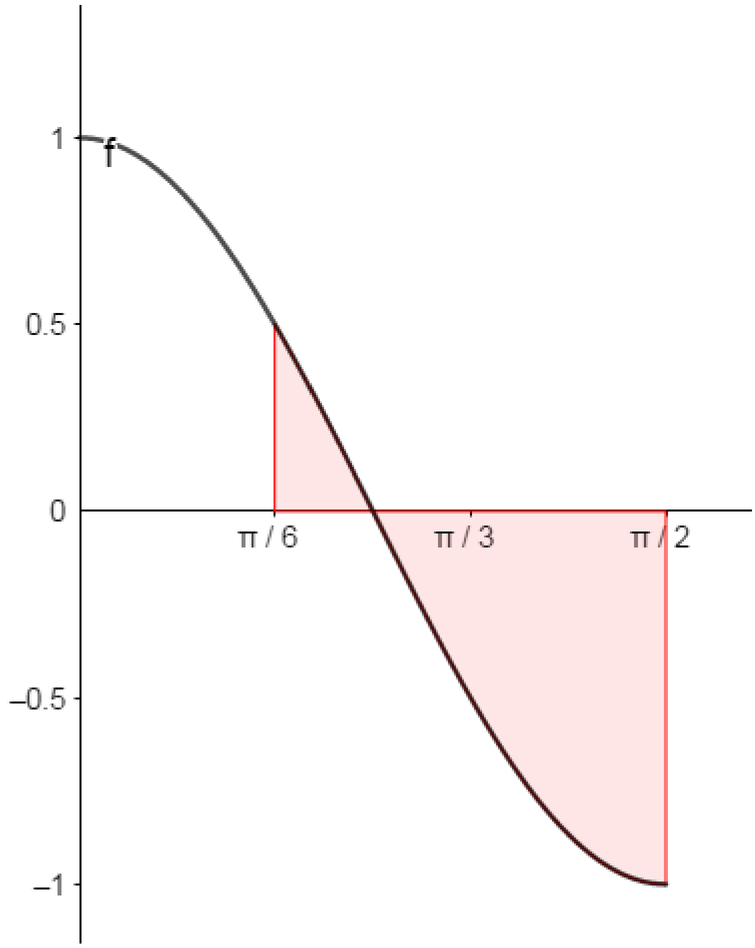
a.



b.

$$\int_1^2 (x^2-1)dx = \frac{x^3}{3} - x\Big|_1^2$$
$$= \left[\frac{2^3}{3} - 2\right] - \left[\frac{1^3}{3} - 1\right]$$
$$= \frac{4}{3}$$

2. First, find the x-intercept. When $y = 0$, then $x = \frac{\pi}{4}$.



Next, find the integral from $x = \frac{\pi}{6}$ to $x = \frac{\pi}{4}$ and add that to the integral from $x = \frac{\pi}{4}$ to $x = \frac{\pi}{2}$.

$$\begin{aligned}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cos 2x dx &= \frac{\sin 2x}{2}\bigg|_{\frac{\pi}{6}}^{\frac{\pi}{4}} \\ &= \frac{\sin 2(\frac{\pi}{4})}{2} - \frac{\sin 2(\frac{\pi}{6})}{2} \\ &= \frac{\sin(\frac{\pi}{2})}{2} - \frac{\sin(\frac{\pi}{3})}{2} \\ &= \frac{1}{2} - \frac{\sqrt{3}}{4} \\ &= \frac{2-\sqrt{3}}{4}\end{aligned}$$

AND

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos 2x dx = \frac{\sin 2x}{2}\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= \frac{\sin 2(\frac{\pi}{2})}{2} - \frac{\sin 2(\frac{\pi}{4})}{2}$$

$$= \frac{\sin \pi}{2} - \frac{\sin(\frac{\pi}{2})}{2}$$

$$= 0 - \frac{1}{2}$$

$$= -\frac{1}{2}$$

Total area: $\frac{2-\sqrt{3}}{4} + \frac{1}{2} = 0.57$.

3. Since the area we are interested in is completely below the x-axis, you can find the integral from $x = -2$ all the way to $x = 2$.

$$\int_{-2}^{2} (x^2 - 4)dx = \frac{x^3}{3} - 4x\Big|_{-2}^{2}$$

$$= \left[\frac{(2)^3}{3} - 4(2)\right] - \left[\frac{(-2)^3}{3} - 4(-2)\right]$$

$$= -\frac{16}{3} - \frac{16}{3}$$

$$= -\frac{32}{3}$$

$\therefore$ area is $\frac{32}{3}$.

Back to Unit 3: Assessment

Media Attributions

SUBJECT OUTCOME VIII

SPACE, SHAPE AND MEASUREMENT: USE THE CARTESIAN CO-ORDINATE SYSTEM TO DERIVE AND APPLY EQUATIONS

Subject outcome

Subject outcome 3.1: Use the Cartesian co-ordinate system to derive and apply equations

Learning outcomes

- Use the Cartesian coordinate system to derive and apply the equation of a circle (any centre).
- Use the Cartesian coordinate system to derive and apply the equation of a tangent to a circle given a point on the circle. Note that:
  - Straight lines to be written in the following forms only: $y = mx + c$, $y - y_1 = m(x - x_1)$ and/or $ax + by + c = 0$ (general form).
  - Learners are expected to know and be able to use as an axiom 'the tangent to a circle is perpendicular to the radius drawn to the point of contact.'

Unit 1 outcomes

By the end of this unit you will be able to:

- Find the equation of a circle centred at the origin.
- Find the equation of a circle with centre (a, b).
- Write the equation of the circle in standard form.

Unit 2 outcomes

By the end of this unit you will be able to:

- Find the gradient of a tangent to a circle using analytical geometry.
- Find the equation of a tangent to the circle using analytical geometry.

- Find the equation of a tangent to a circle at the point of contact with the radius.

Unit 1: Determine the equation of a circle with any centre

DYLAN BUSA

Unit outcomes

By the end of this unit you will be able to:

- Find the equation of a circle centred at the origin.
- Find the equation of a circle with centre (a, b).
- Write the equation of the circle in standard form.

What you should know

Before you start this unit, make sure you can:

- State the theorem of Pythagoras for right-angled triangles. Refer to level 2 subject outcome 3.3 unit 2 if you need help with this.
- Calculate the distance between two points on the Cartesian plane. Refer to level 2 subject outcome 3.3 unit 2 if you need help with this.
- Calculate the midpoint of a line segment between two points on the Cartesian plane. Refer to level 2 subject outcome 3.3 unit 2 if you need help with this.
- Use the Cartesian coordinate system to derive the equation of a straight line through two given points using the gradient-point form or the gradient-intercept form. Refer to level 3 subject outcome 3.2 unit 1 if you need help with this.
- Complete the square of a quadratic expression. Refer to level 3 subject outcome 2.2 unit 3 if you need help with this.
- Find the coordinates of a point (x, y) after it has been reflected about the axes and lines $y = x$ and $y = -x$. Refer to level 2 subject outcome 3.4 unit 1 if you need help with this.

Introduction

The circle is one of the most common and important geometric shapes in mathematics. It is, after all, what defines one of the most famous numbers in the world, π (pi). π, remember, is the ratio of a circle's circumference to its diameter (see figure 1) and it pops up in all sorts of interesting and useful places.

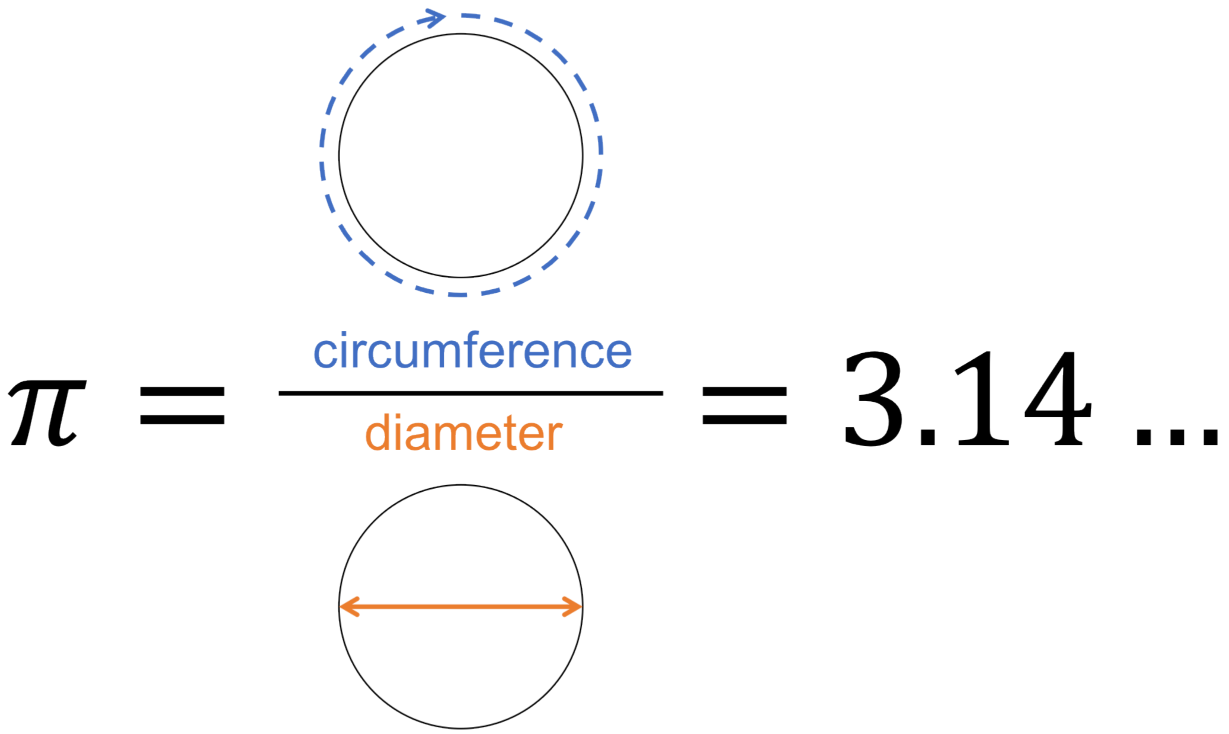


Figure 1: The definition of π

The circle, and in particular the unit circle (circle with a radius of 1 centred on the origin), is also foundational to trigonometry which is a key link between algebra and geometry.

So, I think you will agree, the circle is important.

However, we did not deal with it when examining functions, because it is not a function. The circle fails the basic test of a function – the vertical line test. Draw any circle on the Cartesian plane and it is easy to see why. Except for two points on each side of the circle, a vertical line cuts the circle twice (see figure 2). Therefore, almost all of the input values correspond to more than one output value.

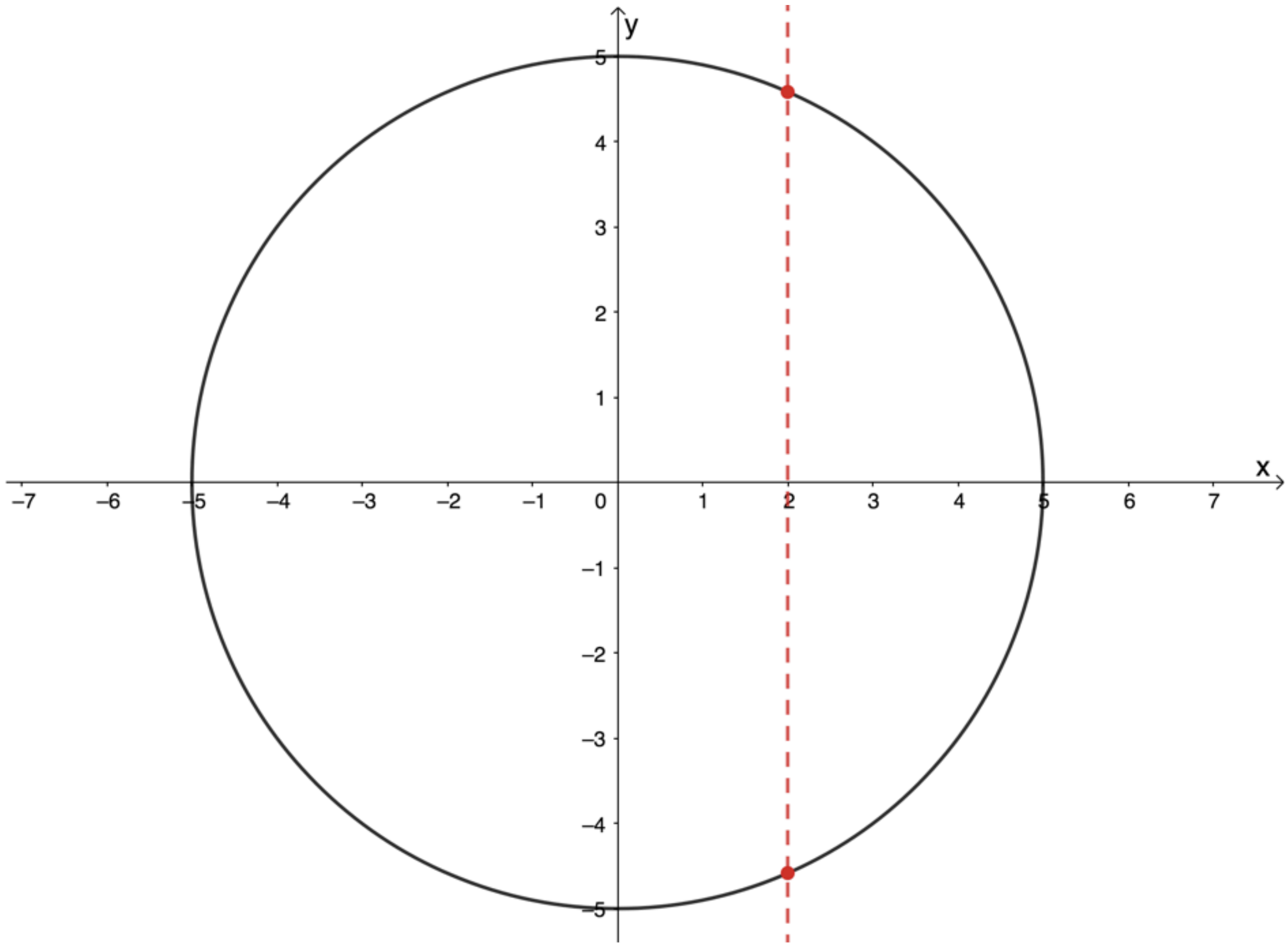


Figure 2: The circle fails the vertical line test for a function

But just because the circle is not a function does not mean that there is not a relation that relates all the input and output pairs that together make up the circle. In this unit we will discover what this equation is.

The equation of a circle with centre the origin

By definition, a circle is a flat shape consisting of all the points in a plane that are an equal distance away from its centre. In other words, every point on the circumference of a circle is the same distance (equidistant) from its centre.

You might already know what the equation is for a circle centred on the origin. If not, complete activity 1.1 to find out.

Activity 1.1: Find the equation of a circle

Time required: 15 minutes

What you need:

- a pen or pencil
- a blank piece of paper or graph paper

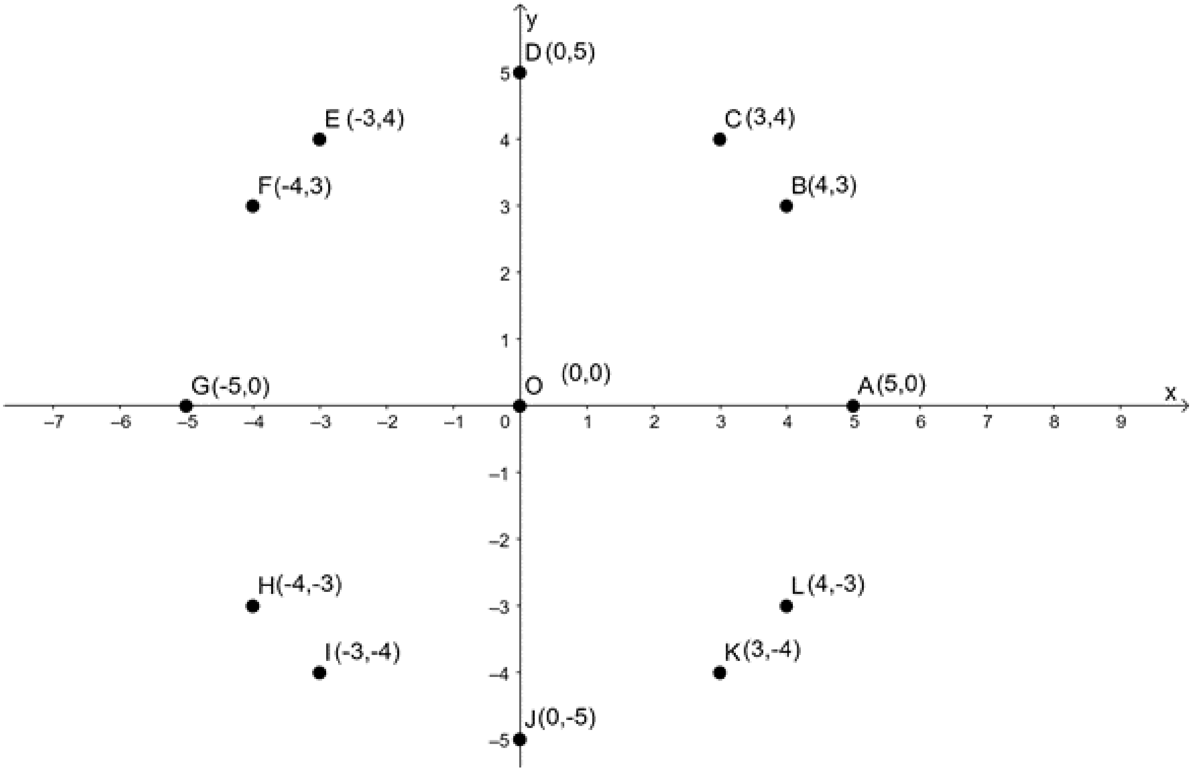
What to do:

For this activity, you will need to draw a system of axes. Use a scale of $1 \text{ cm} = 1 \text{ unit}$ and make sure your axes extend about 5 units in all directions.

1. Plot the following points on your Cartesian plane:
   $O(0,0)$, $A(5,0)$, $B(4,3)$, $C(3,4)$, $D(0,5)$, $E(-3,4)$, $F(-4,3)$, $G(-5,0)$, $H(-4,-3)$, $I(-3,-4)$ $J(0,-5)$, $K(3,-4)$, $L(4,-3)$
2. Determine the following distances using the distance formula $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$:
   OA, OB, OC, OD, OE, OF, OG, OH, OI, OJ, OK, OL
3. What do you notice about the length of each line segment? What general term could we give to these line segments?
4. What object do these points form if you join them up with a smooth curve?
5. If the point $P(x,y)$ lies on this shape, use the distance formula to determine an expression for the length of OP.
6. Can you deduce an equation for a circle centre the origin?

What did you find?

1. Plot the following points on your Cartesian plane:



2. $OA = \sqrt{(5-0)^2 + (0-0)^2} = \sqrt{25} = 5$
   $OB = \sqrt{(4-0)^2 + (3-0)^2} = \sqrt{16+9} = \sqrt{25} = 5$
   $OC = \sqrt{(3-0)^2 + (4-0)^2} = \sqrt{9+16} = \sqrt{25} = 5$

$OD = \sqrt{(0-0)^2 + (5-0)^2}\sqrt{25} = 5$

$OE = \sqrt{(-3-0)^2 + (4-0)^2} = \sqrt{9+16} = \sqrt{25} = 5$

$OF = \sqrt{(-4-0)^2 + (3-0)^2} = \sqrt{16+9} = \sqrt{25} = 5$

$OG = \sqrt{(-5-0)^2 + (0-0)^2} = \sqrt{25} = 5$

$OH = \sqrt{(-4-0)^2 + (-3-0)^2} = \sqrt{16+9} = \sqrt{25} = 5$

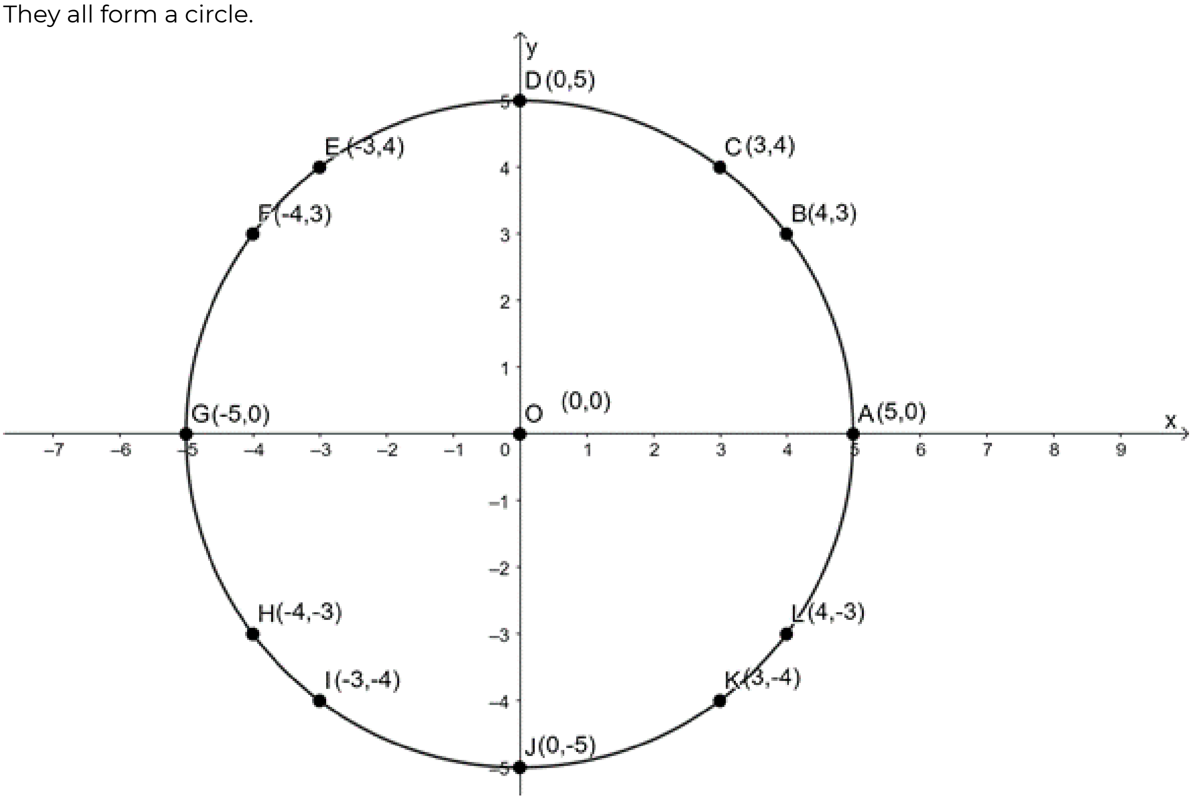
$OI = \sqrt{(-3-0)^2 + (-4-0)^2} = \sqrt{9+16} = \sqrt{25} = 5$

$OJ = \sqrt{(0-0)^2 + (-5-0)^2} = \sqrt{25} = 5$

$OK = \sqrt{(3-0)^2 + (-4-0)^2} = \sqrt{9+16} = \sqrt{25} = 5$

$OL = \sqrt{(4-0)^2 + (-3-0)^2} = \sqrt{16+9} = \sqrt{25} = 5$

3. Each of the line segments is the same length (5 units). We could call these line segments the radius of a circle.
4. They all form a circle.



5. $OP = \sqrt{(x-0)^2 + (y-0)^2} = \sqrt{x^2 + y^2}$.
6. Since OP is the radius (r) of the circle:
   $r = \sqrt{x^2 + y^2}$ or $r^2 = x^2 + y^2$
   We have the equation of a circle centre the origin.

In activity 1.1 we discovered what the equation of a circle centre the origin is. This equation is based on the theorem of Pythagoras (see figure 3).

In ΔOPQ:

$$OP^2 = OQ^2 + PQ^2 \qquad \text{(Pythagoras)}$$

But $OP = r$ and $OQ = x - 0$ and $PQ = y - 0$

$$\therefore r^2 = (x-0)^2 + (y-0)^2$$

$$\therefore r^2 = x^2 + y^2$$

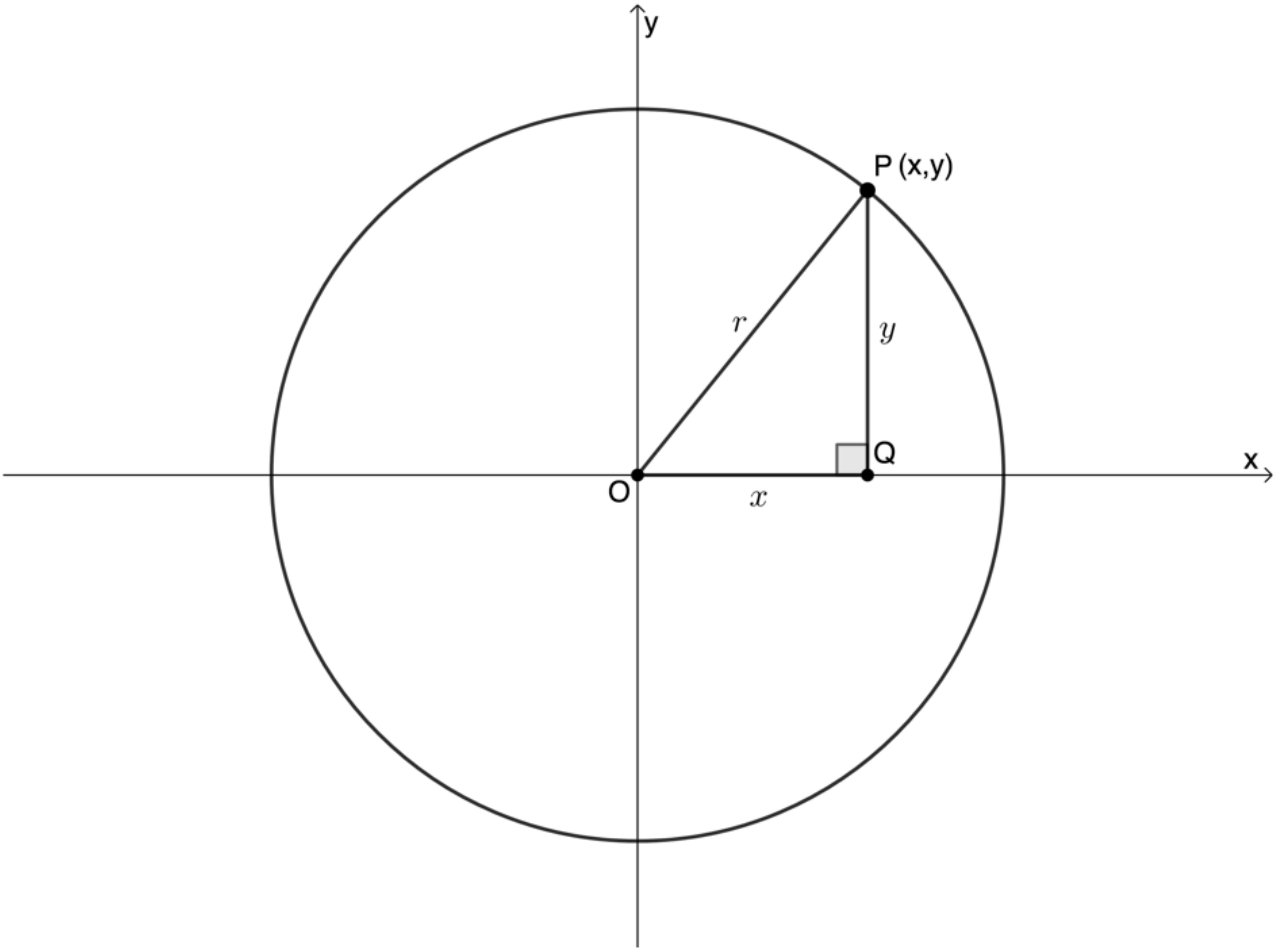


Figure 3: Derivation of the equation of a circle centre origin using Pythagoras

Equation of a circle centre the origin:

$x^2 + y^2 = r^2$

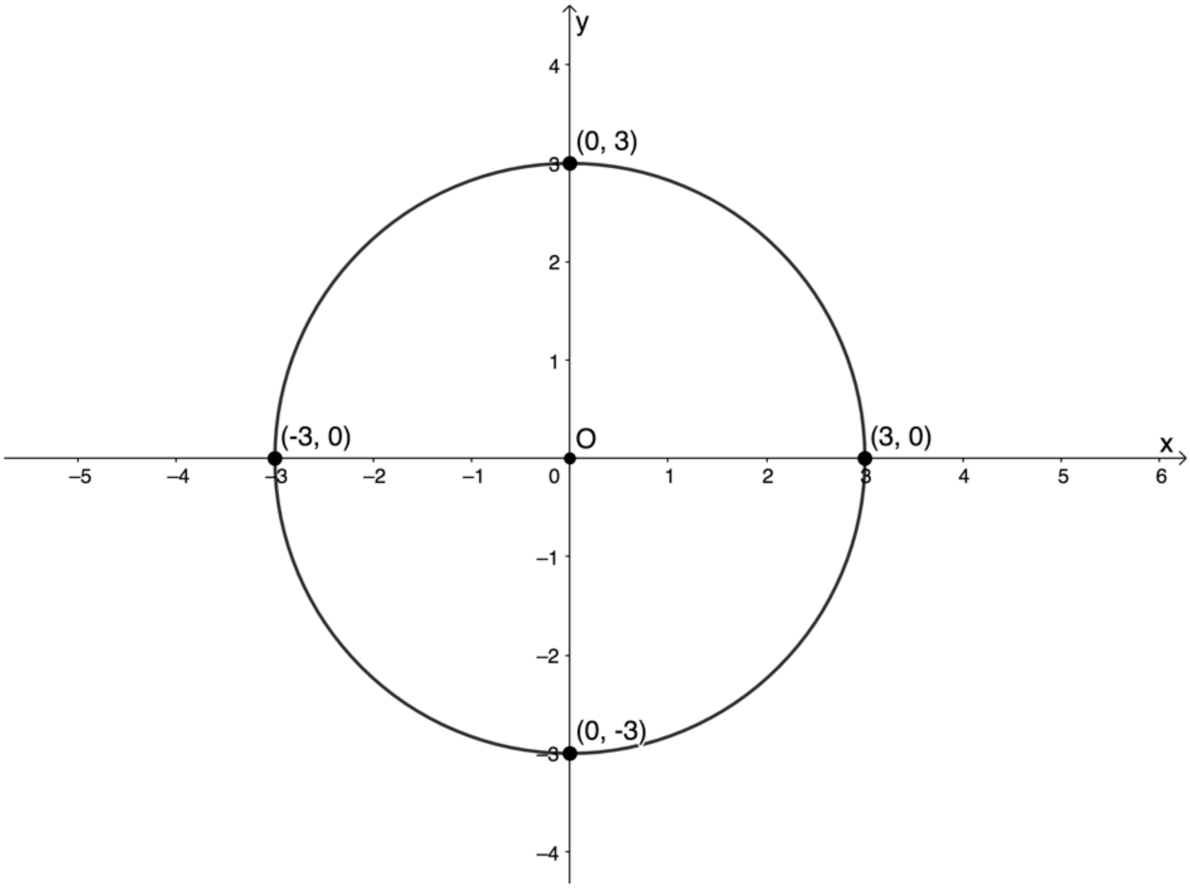
Example 1.1

A circle with centre the origin has a radius of 3 units.

1. Determine the equation of the circle.
2. Sketch the circle on the Cartesian plane.
3. Show that the point $(-\sqrt{4}, \sqrt{5})$ lies on the circle.

Solutions

1. The circle has its centre the origin. Therefore, its equation is $x^2 + y^2 = r^2$. We are told that the radius is 3 units. Therefore, the equation is $x^2 + y^2 = (3)^2$ or $x^2 + y^2 = 9$.
2. We know that the centre is the origin and the radius is 3 units. Therefore, the circle will pass through the points $(3, 0)$, $(0, 3)$, $(-3, 0)$ and $(0, -3)$.



3. The equation of the circle is $x^2 + y^2 = 9$. Does the point $(-\sqrt{4}, \sqrt{5})$ satisfy the equation?

$$\begin{aligned} \text{LHS} &= x^2 + y^2 \\ &= \left(-\sqrt{4}\right)^2 + \left(\sqrt{5}\right)^2 \\ &= 4 + 5 \\ &= 9 = \text{RHS} \end{aligned}$$

Therefore, the point does lie on the circle.

Example 1.2

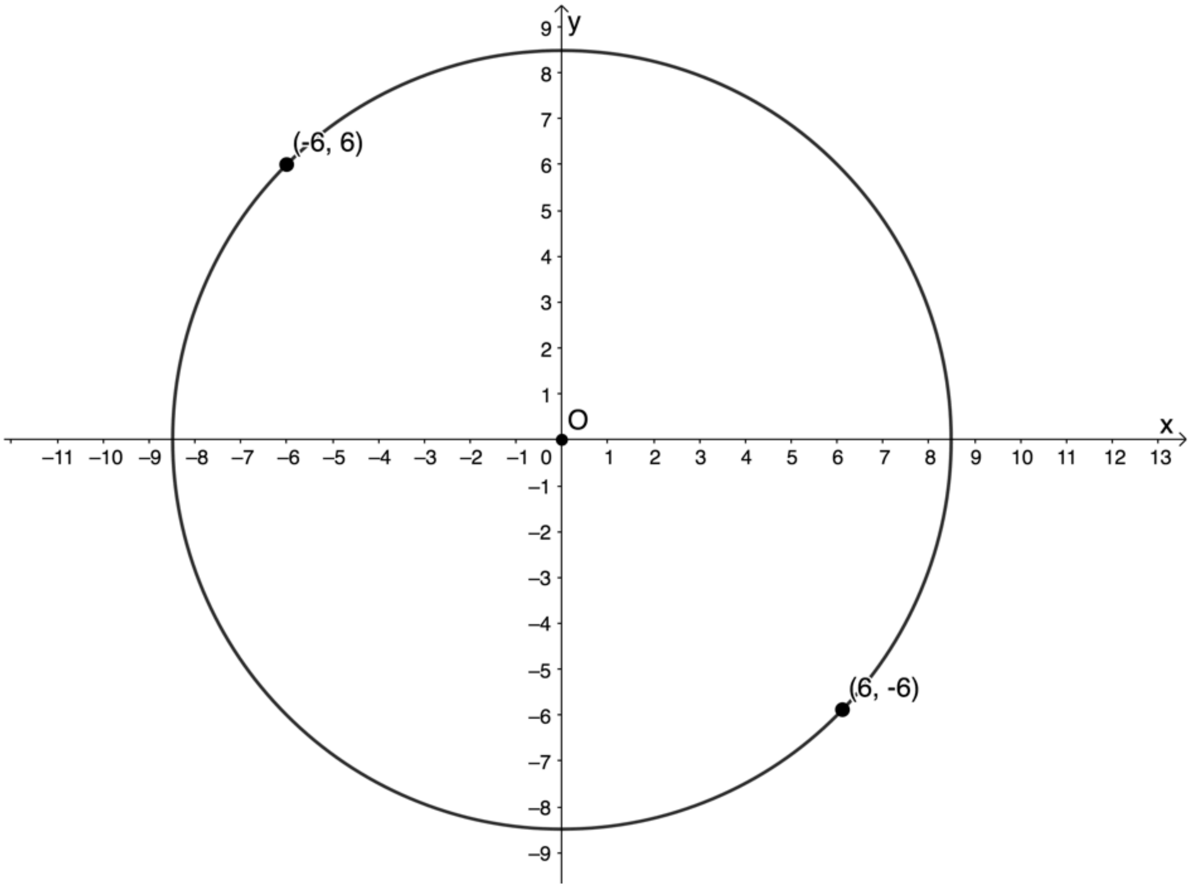
A circle centre the origin passes through the points $A(-6, 6)$ and $B(6, -6)$.

1. Plot the points and draw a rough sketch of the circle.
2. Determine the equation of the circle.
3. Calculate the length of AB.

4. Explain why AB is a diameter of the circle.

Solutions

1. Here is a sketch of the circle:



2. $x^2 + y^2 = r^2$. We have two possible points we can substitute into the equation. We will substitute $A(-6, 6)$.

$$x^2 + y^2 = r^2$$
$$\therefore (-6)^2 + 6^2 = r^2$$
$$\therefore 36 + 36 = r^2$$
$$\therefore r^2 = 72$$

The equation of the circle is $x^2 + y^2 = 72$.

3.

$$\begin{aligned} d_{AB} &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(-6-6)^2 + (6-(-6))^2} \\ &= \sqrt{(-12)^2 + 12^2} \\ &= \sqrt{144+144} \\ &= \sqrt{288} \\ &= 2\sqrt{72} \end{aligned}$$

4. The radius of the circle is $\sqrt{72}$. Therefore, AB is twice the radius of the circle. Also, a circle centre the origin is symmetrical about the axes as well as about the lines $y = x$ and $y = -x$. The points A and B are symmetrical about the line $y = -x$ meaning that AB passes through the centre. Therefore,

AB is a straight line passing through the centre and twice the radius i.e. AB is the diameter.

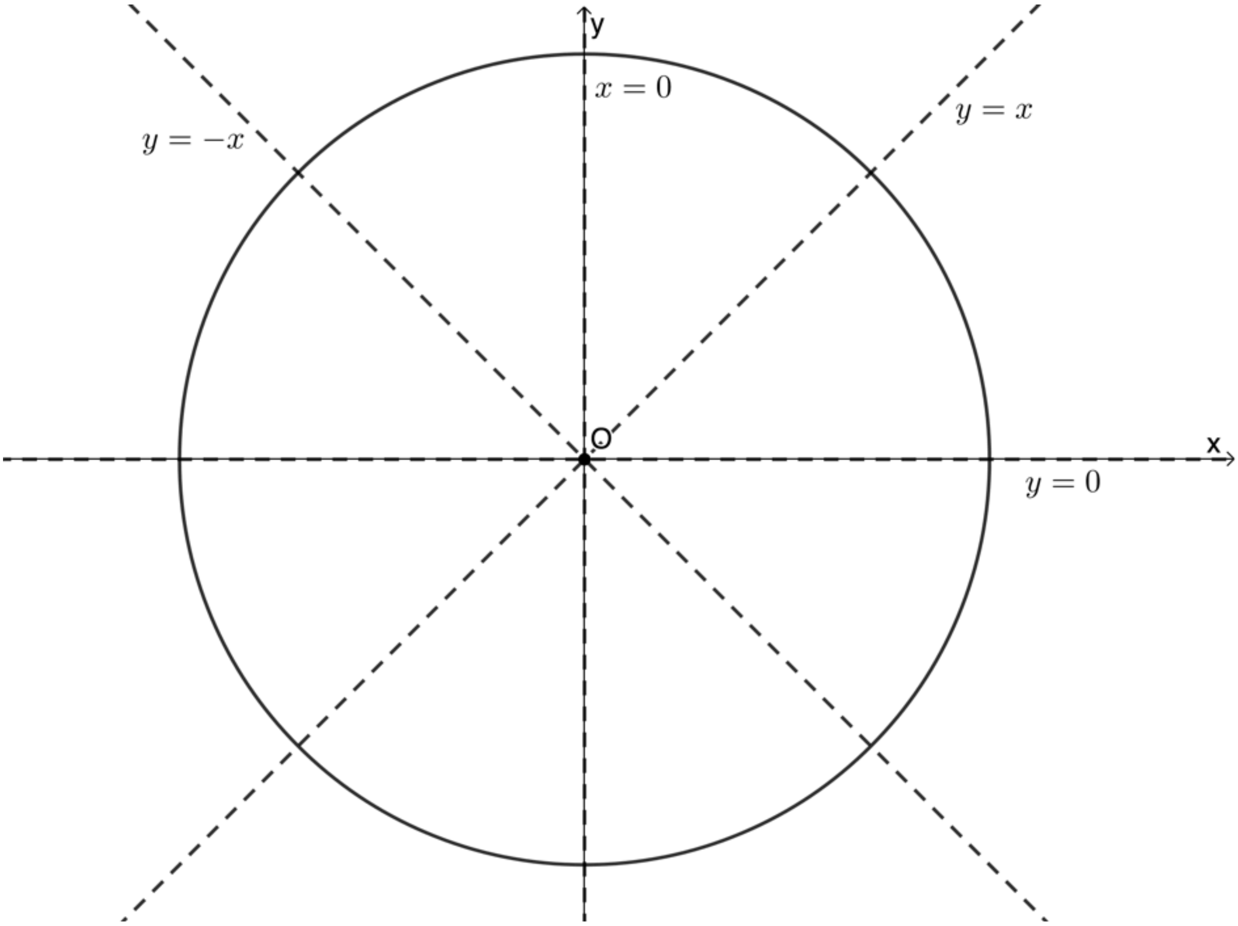
Note: You could also find the equation of AB and then show that the point $O(0,0)$ lies on this point to demonstrate that AB passes through the centre. In both cases, showing that a line is a diameter requires you to show that it is:

1. a straight line passing through the centre of the circle
2. twice the length of the radius.

An important characteristic of the circle centre the origin is its symmetry. The shape is very symmetrical. It is symmetrical about both axes as well as about the lines $y = x$ and $y = -x$.

Take note!

The circle centre the origin is symmetrical about both axes as well as about the lines $y = x$ and $y = -x$.



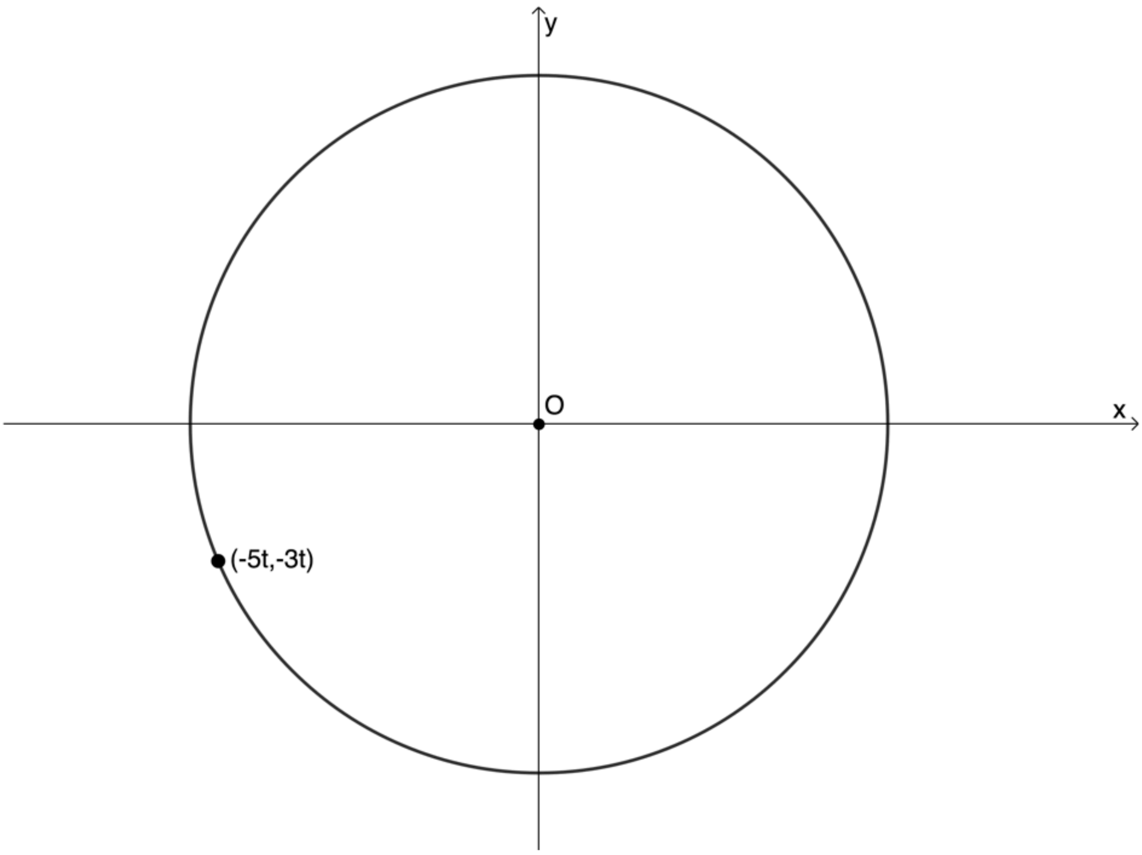
Take note!

To prove that a line is a diameter you must prove that it is:

1. a straight line passing through the centre of the circle
2. twice the length of the radius.

Exercise 1.1

1. Given $x^2 + y^2 = 100$:
    a. Determine the radius.
    b. Make a sketch of the circle.
2. Given $x^2 + y^2 = \frac{16}{9}$:
    a. Determine the radius.
    b. Make a sketch of the circle.
3. Given $4x^2 + 4y^2 = 80$:
    a. Determine the radius.
    b. Make a sketch of the circle.
4. Determine the equation of the circle in each case:
    a. With centre the origin and a radius of 6.
    b. With centre $(0, 0)$ and $r = \sqrt{13}$.
    c. With centre the origin passing through $(-3, -7)$.
    d. Passing through $(2s, 3s)$ and with centre $(0, 0)$.
    e.



5. Is $\sqrt{90}+x^2-y^2=0$ the equation of a circle with centre the origin?
6. Given a circle with centre the origin, with a radius of $r=\sqrt{50}$, determine the coordinates of the points on the circle whose x-values are three times their y-value.

The full solutions are at the end of the unit.

The equation of a circle centre (a,b)

There are two ways to think about the equation of a circle with centre not the origin but any point (a,b) on the Cartesian plane.

The first is to think about translating or shifting the circle centre the origin. Consider the parabola $y=2x^2$. We know that the graph of $y=2x^2+1$ is the same as $y=2x^2$ except that the graph has been shifted one unit up (see figure 4).

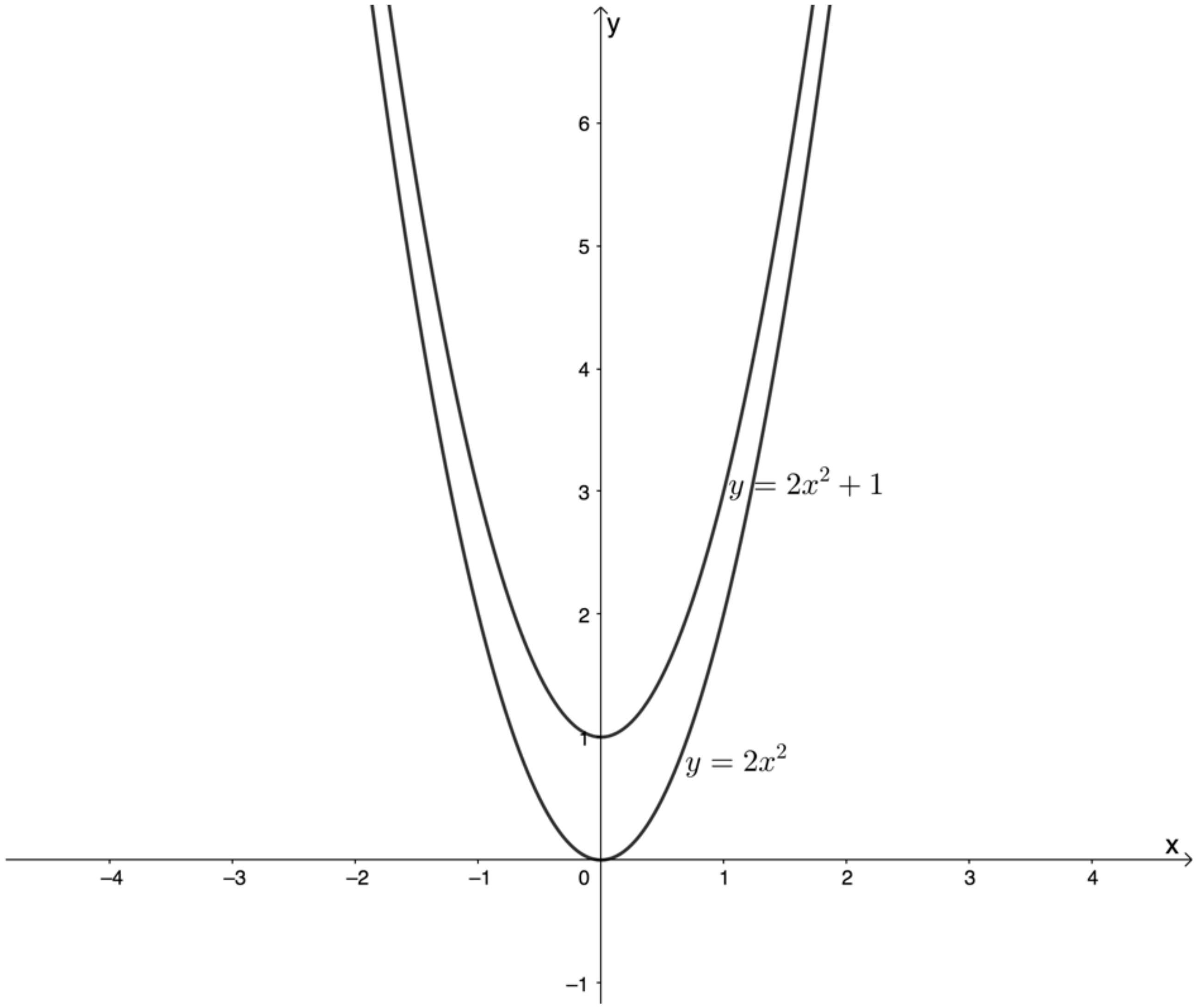


Figure 4: $y = 2x^2$ *shifted one unit up*

We can rewrite the equation $y = 2x^2 + 1$ as $y - 1 = 2x^2$ or $(y - 1) = 2x^2$.

Now, how do you think the circle defined by $x^2 + (y - 1)^2 = 25$ is different from that defined by $x^2 + y^2 = 25$? $x^2 + (y - 1)^2 = 25$ is the same except that the circle has been shifted one unit up and the centre is now the point $(0, 1)$ (see figure 5).

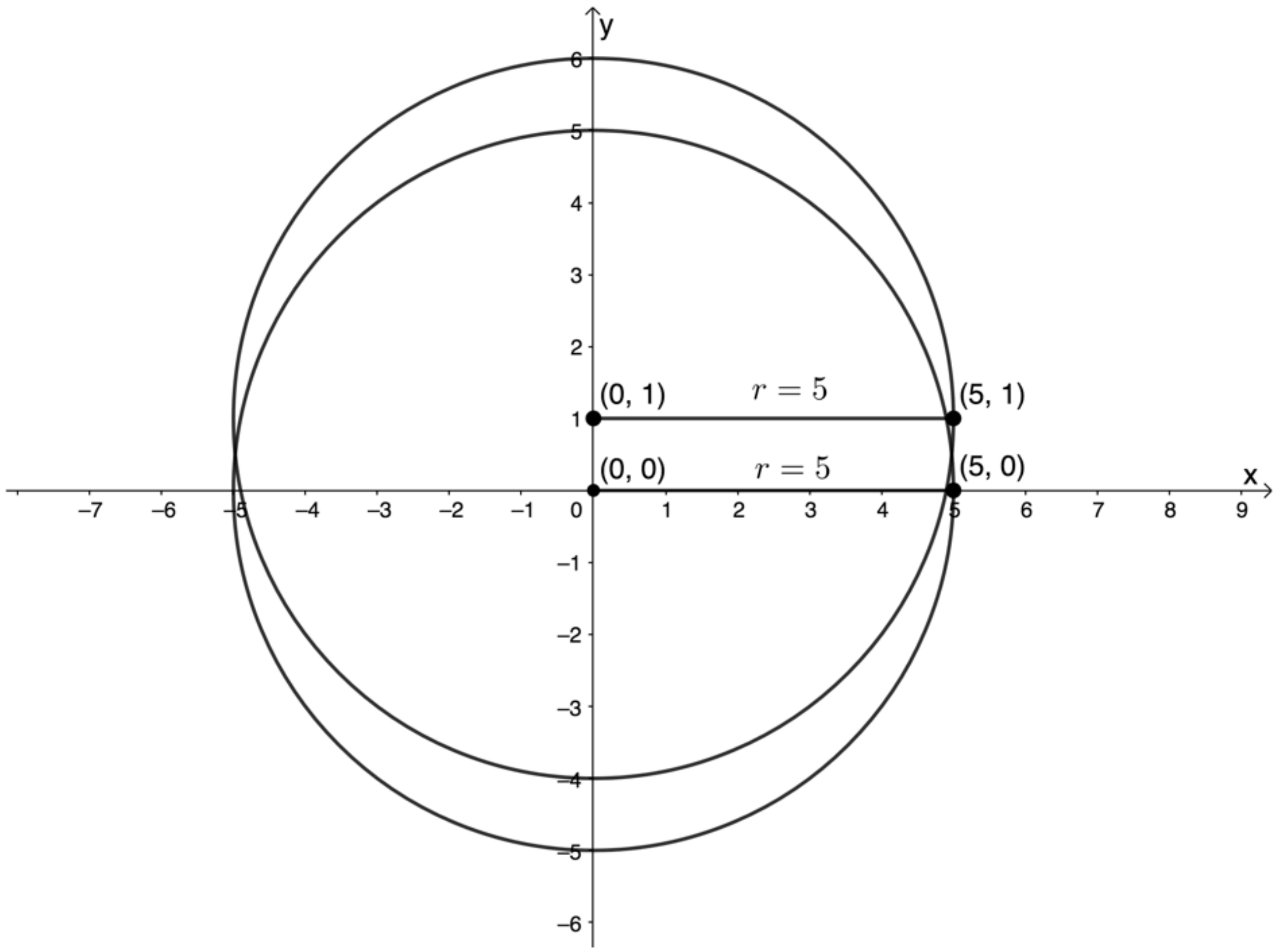


Figure 5: $x^2 + y^2 = 25$ *shifted one unit up*

Consider the parabola $y = 2x^2$ again. We know that the graph of $y = 2(x-1)^2$ is the same as $y = 2x^2$ except that it has been shifted one unit to the right (see figure 6).

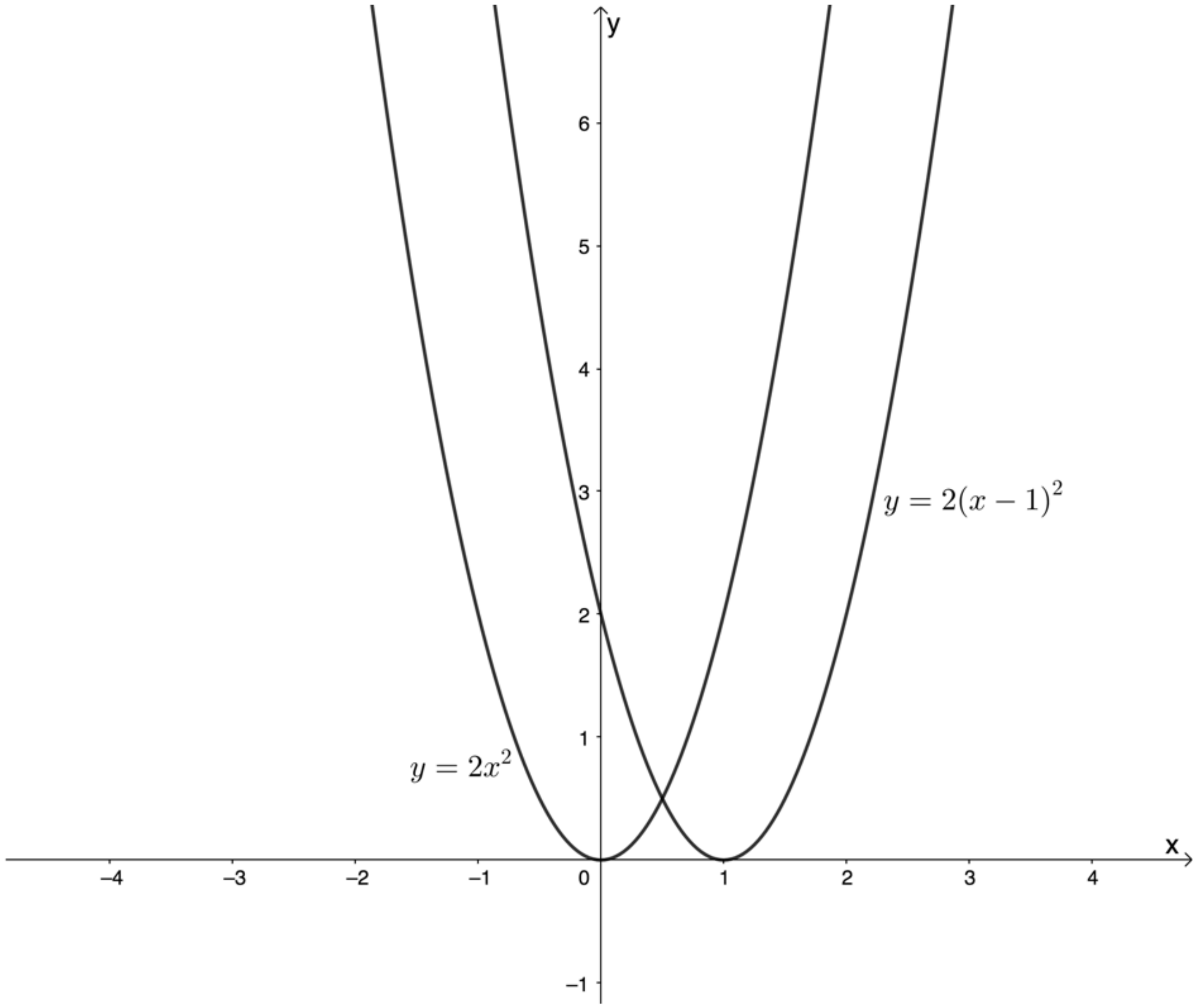


Figure 6: $y = 2x^2$ shifted one unit to the right

In the same way, the circle $(x-1)^2 + y^2 = 25$ is the same as $x^2 + y^2 = 25$ just shifted one unit to the right. The centre is now the point $(1, 0)$ (see figure 7).

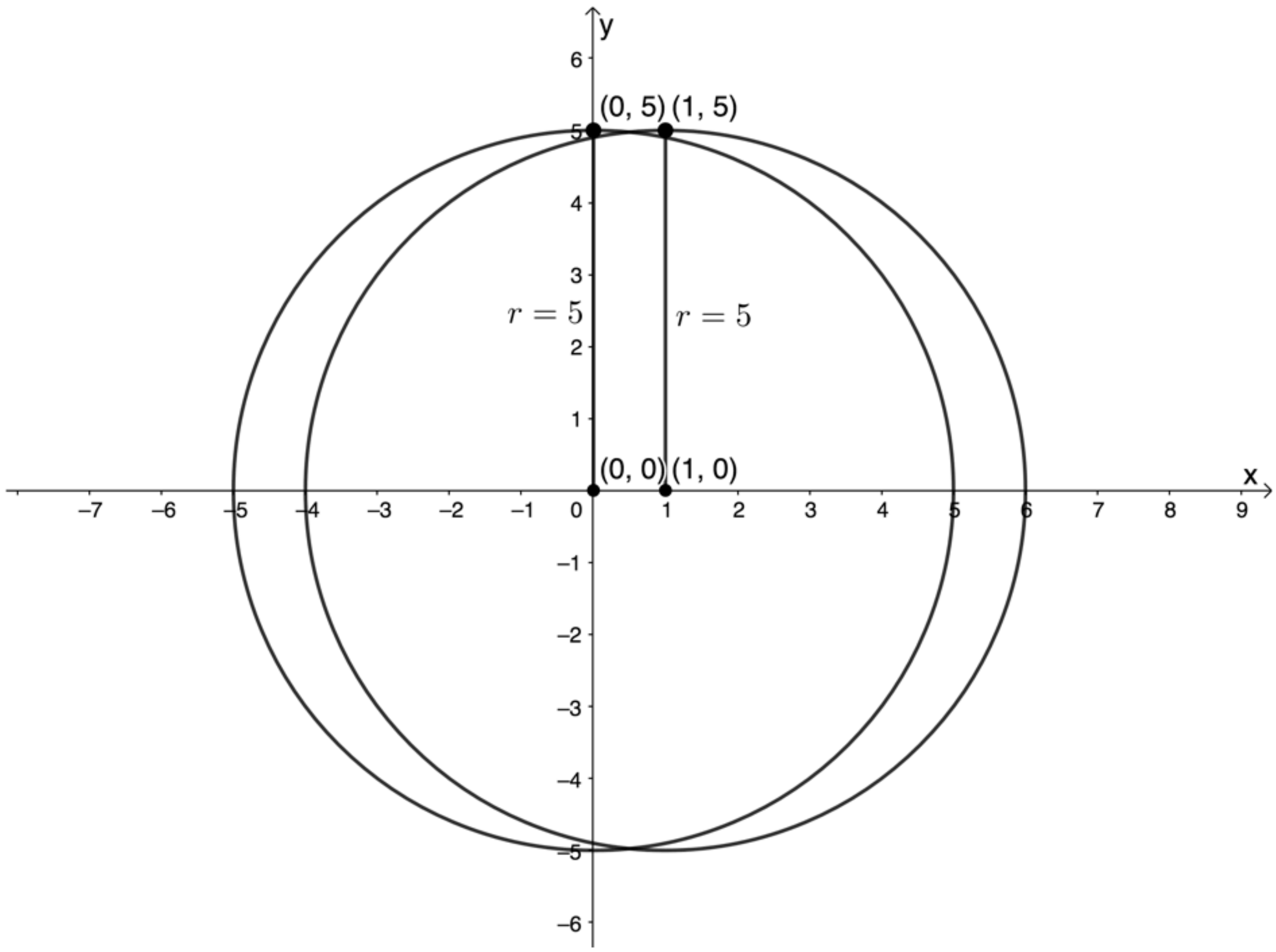


Figure 7: $x^2 + y^2 = 25$ *shifted one unit to the right*

Therefore, we can generalise and say that the equation $(x-a)^2 + (y-b)^2 = r^2$ is the equation of a circle radius r and with centre (a, b).

The other way to consider this is geometrically. Figure 8 shows a circle centre (a, b) and passing through $p(x, y)$ with a radius r.

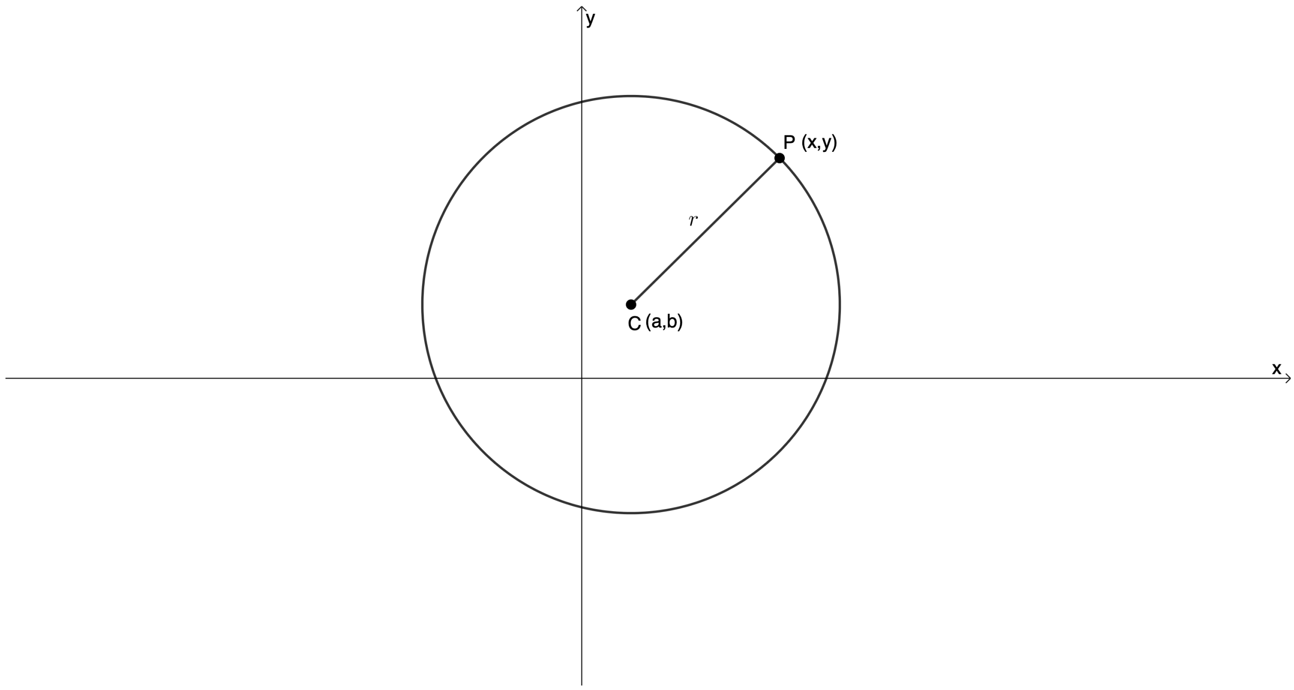


Figure 8: Circle centre (a, b) passing through $P(x, y)$ with radius r

We can determine the distance PC using the distance formula.

$$\begin{aligned} d_{PC} &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(x - a)^2 + (y - b)^2} \end{aligned}$$

But we know that the length of PC is the radius. Therefore $d_{PC} = r$. Therefore:

$$\begin{aligned} r &= \sqrt{(x - a)^2 + (y - b)^2} \\ \therefore r^2 &= (x - a)^2 + (y - b)^2 \\ \therefore (x - a)^2 + (y - b)^2 &= r^2 \end{aligned}$$

With this equation, we can think of the circle with centre the origin as a special case.

$$\begin{aligned} (x - a)^2 + (y - b)^2 &= r^2 \text{ but } (a, b) = (0, 0) \\ \therefore (x - 0)^2 + (y - 0)^2 &= r^2 \\ \therefore x^2 + y^2 &= r^2 \end{aligned}$$

Equation of a circle with centre at (a, b):
$(x - a)^2 + (y - b)^2 = r^2$

Example 1.3

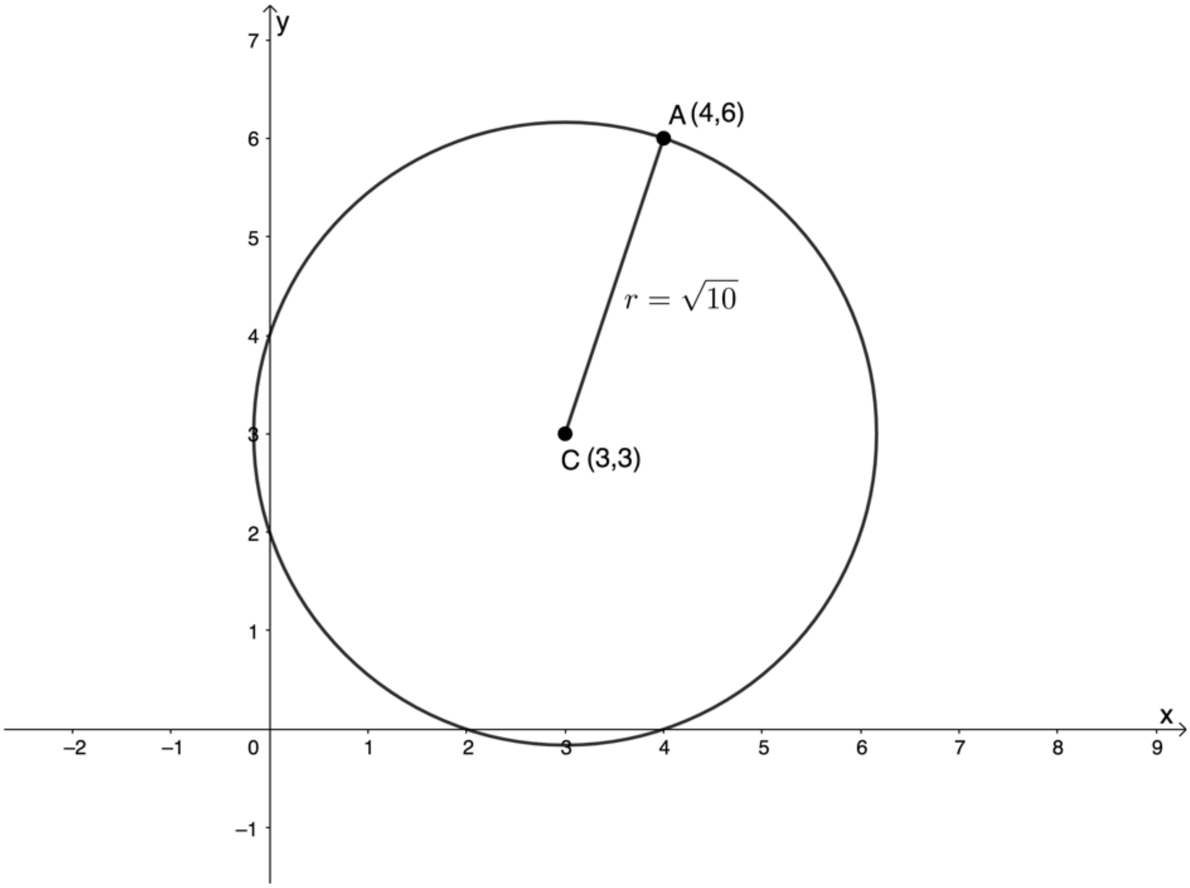
$A(4, 6)$ is a point on a circle centre $(3, 3)$:

1. Determine the equation of the circle.

2. Draw a sketch of the circle.
3. Does the point $B(\frac{3}{2}, 2)$ lie on the circle?
4. Does the circle cut the y-axis?

Solutions

1. Because the circle has centre $(3, 3)$, we know that the equation will be of the form $(x-3)^2 + (y-3)^2 = r^2$. We need to find r which we can do by finding the distance between the point $A(4, 6)$ and the centre $(3, 3)$.
$$\begin{aligned} r &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(4-3)^2 + (6-3)^2} \\ &= \sqrt{1+9} \\ &= \sqrt{10} \end{aligned}$$
Therefore, $r^2 = 10$ and the equation of the circle is $(x-3)^2 + (y-3)^2 = 10$.
2. The easiest way to draw a sketch of a circle centre (a, b) and passing through a point (x, y) is to use a pair of compasses. Set the point on the centre and your pencil on the point to get the radius, and draw the resulting circle.



3. If the point $B(\frac{3}{2}, 2)$ lies on the circle, it will satisfy the equation of the circle.

$$
\begin{aligned}
\text{LHS} &= (x-3)^2+(y-3)^2\\
&= \left(\frac{3}{2}-3\right)^2+(2-3)^2\\
&= \left(-\frac{3}{2}\right)^2+(-1)^2\\
&= \frac{9}{4}+1\\
&= \frac{9+4}{4}\\
&= \frac{13}{4} \neq \text{RHS}
\end{aligned}
$$

Therefore, the point $B(\frac{3}{2},2)$ does not lie on the circle.

4. For the circle to cut the y-axis, there must be a point on the circle with an x-coordinate of zero. We need to substitute $x=0$ into the equation to see if there is at least one real solution for y.

$$
\begin{aligned}
(0-3)^2+(y-3)^2 &= 10\\
\therefore (-3)^2+(y-3)^2 &= 10\\
\therefore 9+y^2-6y+9 &= 10\\
\therefore y^2-6y+8 &= 0\\
\therefore (y-4)(y-2) &= 0\\
\therefore y=4 \text{ or } y &= 2
\end{aligned}
$$

There are two points $(0,2)$ and $(0,4)$ where the circle cuts the y-axis.

Note: You can also find any x- or y-intercepts to help you make an accurate sketch of a circle centre (a,b).

Example 1.4

Determine the coordinates of the centre of the circle and its radius if the equation of the circle is $3x^2+12x+3y^2-24y+21=0$.

Solution

This question is basically asking us to rewrite the equation of the circle centre (a,b) in the standard form $(x-a)^2+(y-b)^2=r^2$ rather than the current expanded form. Because the standard form of the equation of a circle contains perfect squares, we know that we will need to complete the squares for x and for y.

Step 1: Get the coefficients of the x^2 and y^2 terms equal to 1

$$
\begin{aligned}
3x^2+12x+3y^2-24y+21 &= 0\\
\therefore x^2+4x+y^2-8y+7 &= 0
\end{aligned}
$$

Step 2: Complete the squares

Remember, to complete the squares, we need to add half the coefficients of the x and y terms squared. Don't forget to keep the equation balanced by subtracting these values as well.

$$
\begin{aligned}
x^2+4x+y^2-8y+7 &= 0\\
\therefore x^2+4x+4+y^2-8y+16+7-4-16 &= 0\\
\therefore (x+2)^2+(y-4)^2 &= 13
\end{aligned}
$$

Step 3: Answer the question
The centre of the circle is the point $(-2, 4)$ and the radius is $r = \sqrt{13}$.

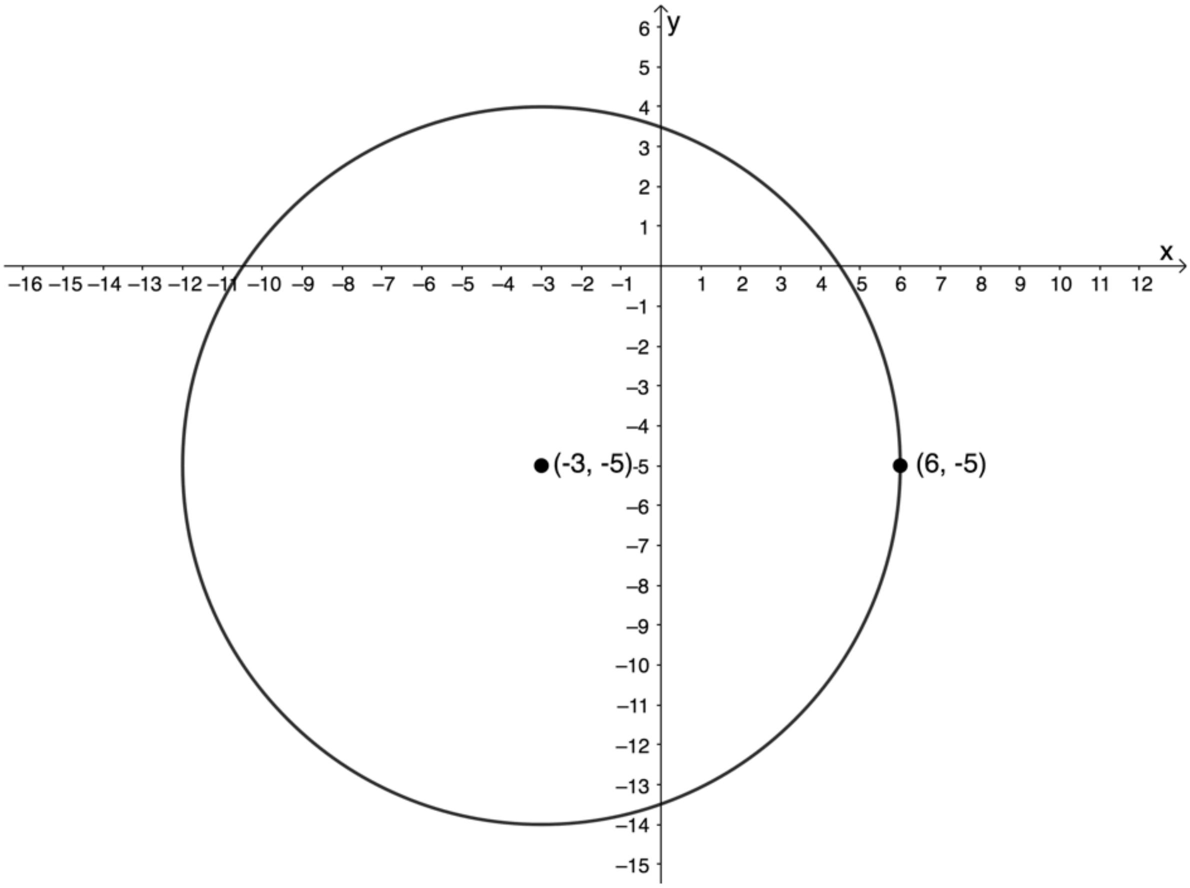
Example 1.5

Given a circle centre the origin with $r = 9$:

1. Write down the equation of the circle if it is shifted 3 units to the left and 5 units down.
2. Draw a sketch of the circle.
3. Write down the equation of the circle that is symmetrical to the circle in question 1 about the line $y = -x$.
4. Sketch the circle in question 3 on the same set of axes as the graph in question 2.

Solutions

1. The equation of the circle we were given is $x^2 + y^2 = 81$. The equation of the shifted circle will be $(x+3)^2 + (y+5)^2 = 81$.
2. We know the centre of the circle but we need at least one point that the circle passes through. An easy way to find this point is to move either horizontally or vertically from the centre a distance of 9 units (the radius) as indicated below.



3. To find the coordinates of a point symmetrical about the line $y = -x$, each x value must be

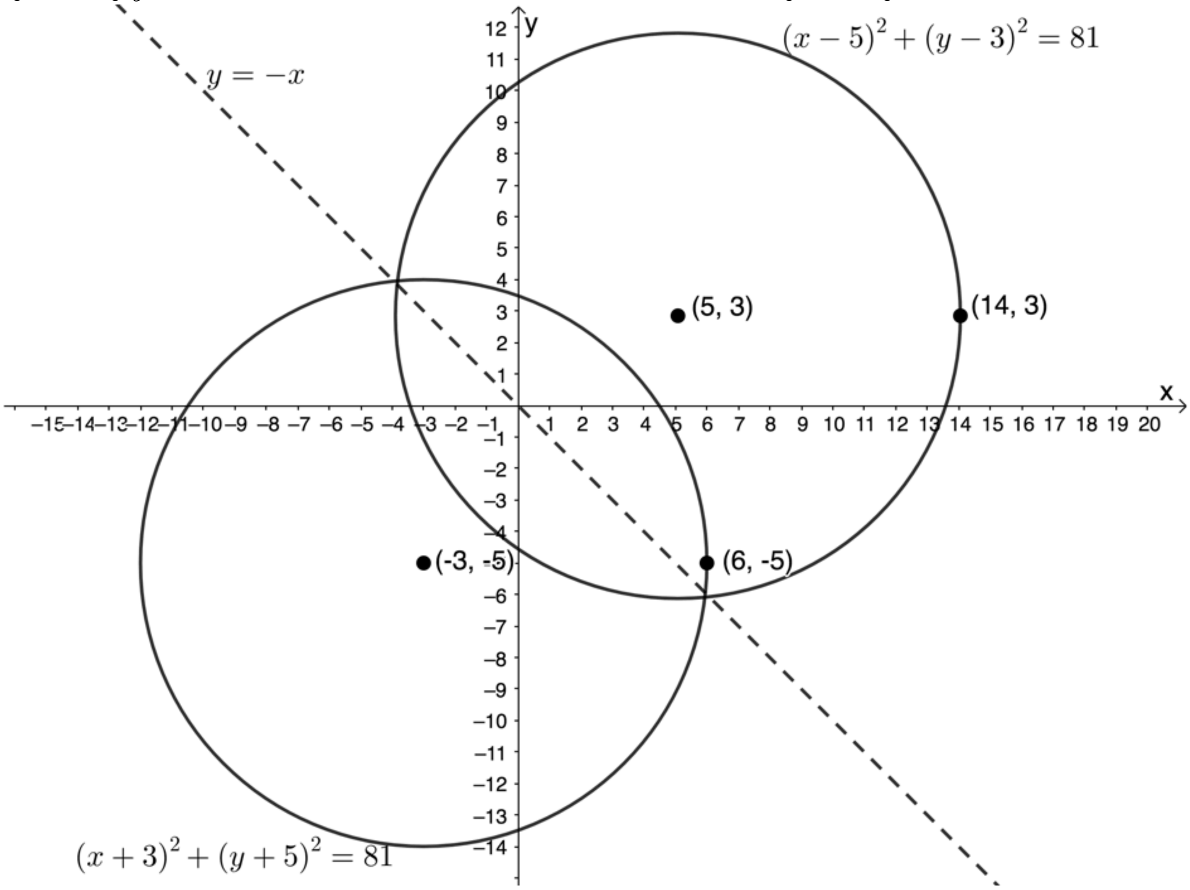
replaced with $-y$ and each y value replaced with $-x$. Therefore, the equation of the symmetrical circle will be:

$$((-y)+3)^2+((-x)+5)^2=81$$
$$\therefore (5-x)^2+(3-y)^2=81$$

We can expand the equation and then simplify into standard form.

$$(5-x)^2+(3-y)^2=81$$
$$\therefore 25-10x+x^2+9-6y+y^2=81$$
$$\therefore x^2-10x+25+y^2-6y+9=81$$
$$\therefore (x-5)^2+(y-3)^2=81$$

4. Below are the two circles sketched on the same set of axes. Once again, to find another point on the symmetrical circle, we moved $r=9$ units horizontally to the right from the centre. The line of symmetry $y=-x$ has also been sketched to better illustrate the symmetry.



Exercise 1.2

1. Write down the equation of the following circles:
   1. Centre $(-3,0)$ and radius 4 units.
   2. Centre $C(-5,-1)$ and passing through the point $P(6,-2)$.
   3. $r=\sqrt{11}$ and centre $(2,-3)$.

2. Determine the centre and radius of the following circles:
   1. $5 - x^2 - 6x - 8y - y^2 = 0$
   2. $x^2 = 21 + 4y - y^2$
   3. $12x - 12y - 2x^2 - 2y^2 = 24$
3. A circle cuts the x-axis at $A(-3,0)$ and $B(3,0)$. If $r = \sqrt{21}$, determine the possible equation(s) of the circle.
4. $P(5,6)$ and $Q(-1,-2)$ are points on a circle. If PQ is a diameter of the circle, determine the equation of the circle.

The full solutions are at the end of the unit.

Summary

In this unit you have learnt the following:

- The equation of a circle centre (a,b) is given by $(x-a)^2 + (y-b)^2 = r^2$.
- The equation of a circle centre the origin is a special case and is given by $x^2 + y^2 = r^2$.
- The circle centre the origin is symmetrical about both axes and the lines $y = x$ and $y = -x$.

Unit 1: Assessment

Suggested time to complete: 60 minutes

1. Determine the equation of the circle:
   a. With centre the origin and a radius of $\frac{7}{3}$.
   b. With centre $(2,-5)$ and $r = 2\sqrt{11}$.
   c. With centre $C(-3,0)$ passing through $R(8,-4)$.
   d. Passing through $P(5q,6q)$ and centre $C(0,1)$.
2. Determine the value(s) of h if $(\sqrt{2},h)$ is a point on the circle $x^2 + y^2 = 17$.
3. $P(r,s)$ is a point on the circle with centre at the origin and a diameter of 70 cm. Determine the possible coordinates of P if the value of r is four times the value of s.

Question 4 adapted from Everything Maths Grade 12 Exercise 7-3 question 6

4. $P(-2,3)$ lies on a circle with centre at $(0,0)$.
   a. Determine the equation of the circle.
   b. Sketch the circle and label point P.
   c. If PQ is a diameter of the circle, determine the coordinates of Q.
   d. Calculate the length of PQ.
   e. Determine the equation of the line PQ.

f. Determine the equation of the line perpendicular to PQ and passing through the point P.

Question 5 adapted from Everything Maths Grade 12 Exercise 7-4 question 6

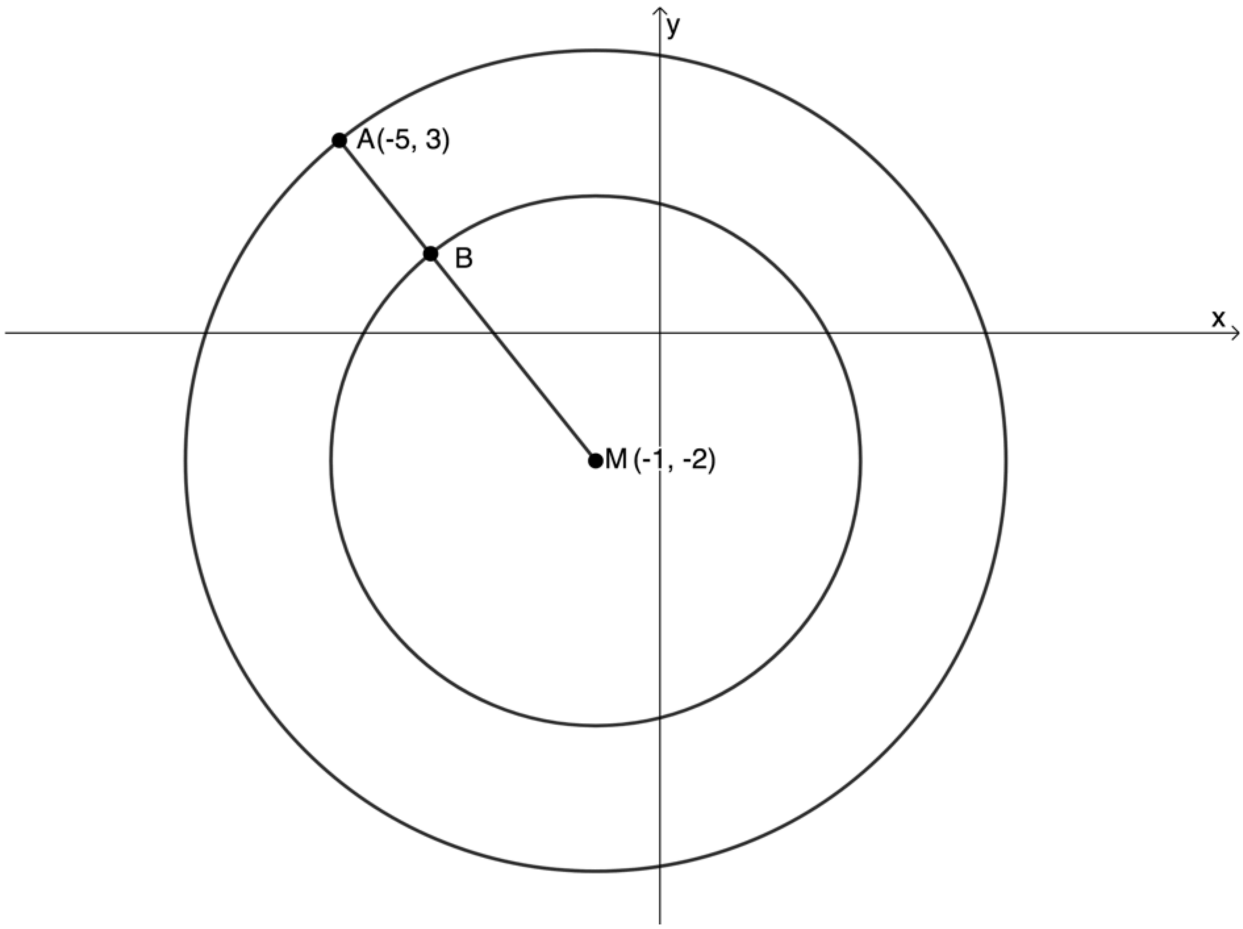
5. A circle with centre $N(4,4)$ passes through the points $K(1,6)$ and $L(6,7)$.
   a. Draw a sketch of the circle.
   b. Determine the equation of the circle.
   c. Determine the coordinates of M, the midpoint of KL.
   d. Show that $MN \perp KL$.
   e. Determine the equation of the line LN.
   f. If P is a point on the circle such that LP is a diameter, determine the coordinates of P.
6. A circle passes through the points $A(1,2)$ and $B(-3,2)$. If its centre lies on the line $6x+3y=-3$, determine the equation of the circle.

Question 7 adapted from Everything Maths Grade 12 Exercise 7-4 question 8

7. A circle with centre $(0,0)$ passes through the point $T(4,3)$.
   a. Determine the equation of the circle.
   b. If the circle is shifted 2 units to the right and 3 units down, determine the new equation of the circle.
   c. Draw a sketch of the original circle and the shifted circle on the same system of axes.
   d. If the circle in 7 b. is reflected about the x-axis, what will the centre of this new circle be.

Question 8 adapted from NC(V) Mathematics Level 4 Paper 2 November 2016 question 1.5

8. In the figure below two circles with a common centre $M(-1,-2)$ are shown. Radius MA of the larger circle cuts the smaller circle at B. Point A has coordinates $(-5,3)$.



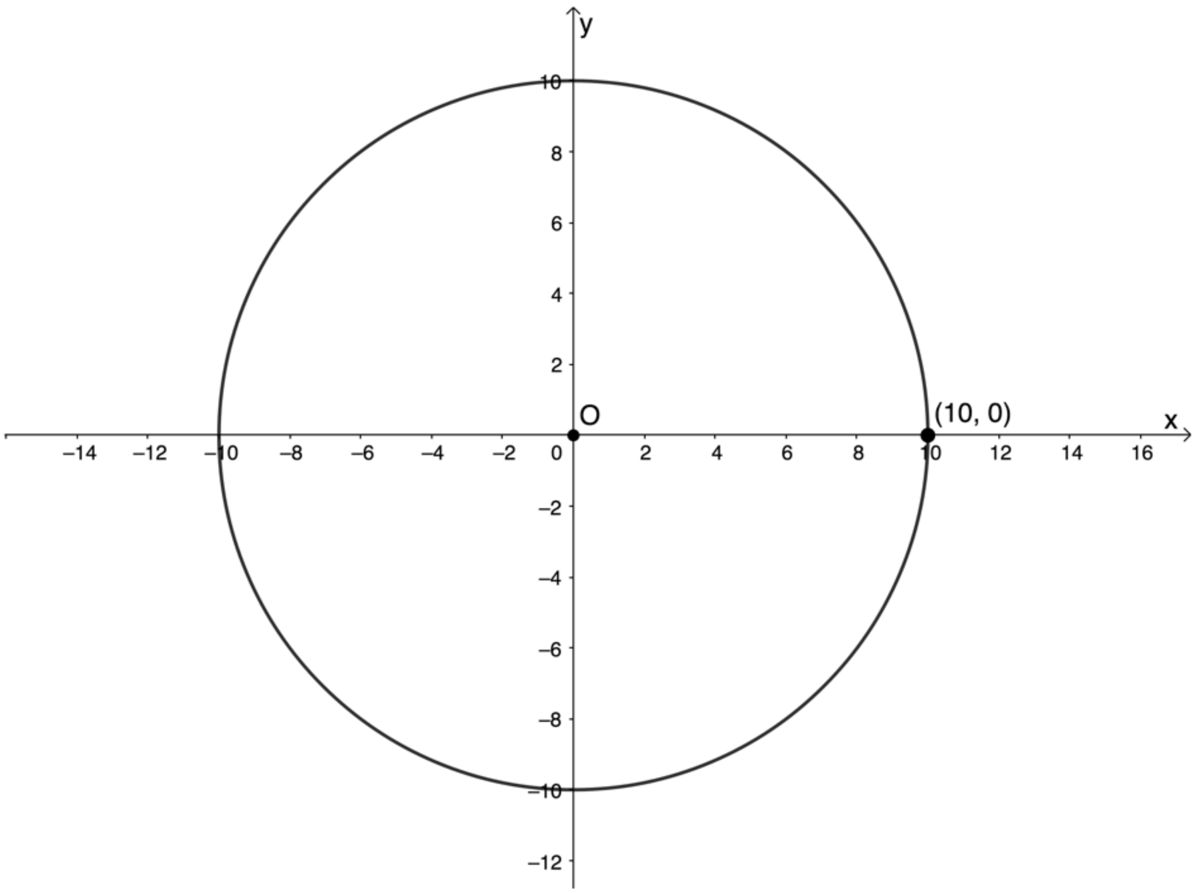
a. Determine the equation of the larger circle.
b. If the equation of the smaller circle is $x^2 + y^2 + 2x + 4y - 4 = 0$, determine the length AB.

The full solutions are at the end of the unit.

Unit 1: Solutions

Exercise 1.1

1.
   a.
      $r^2 = 100$
      $\therefore r = 10$
   b.



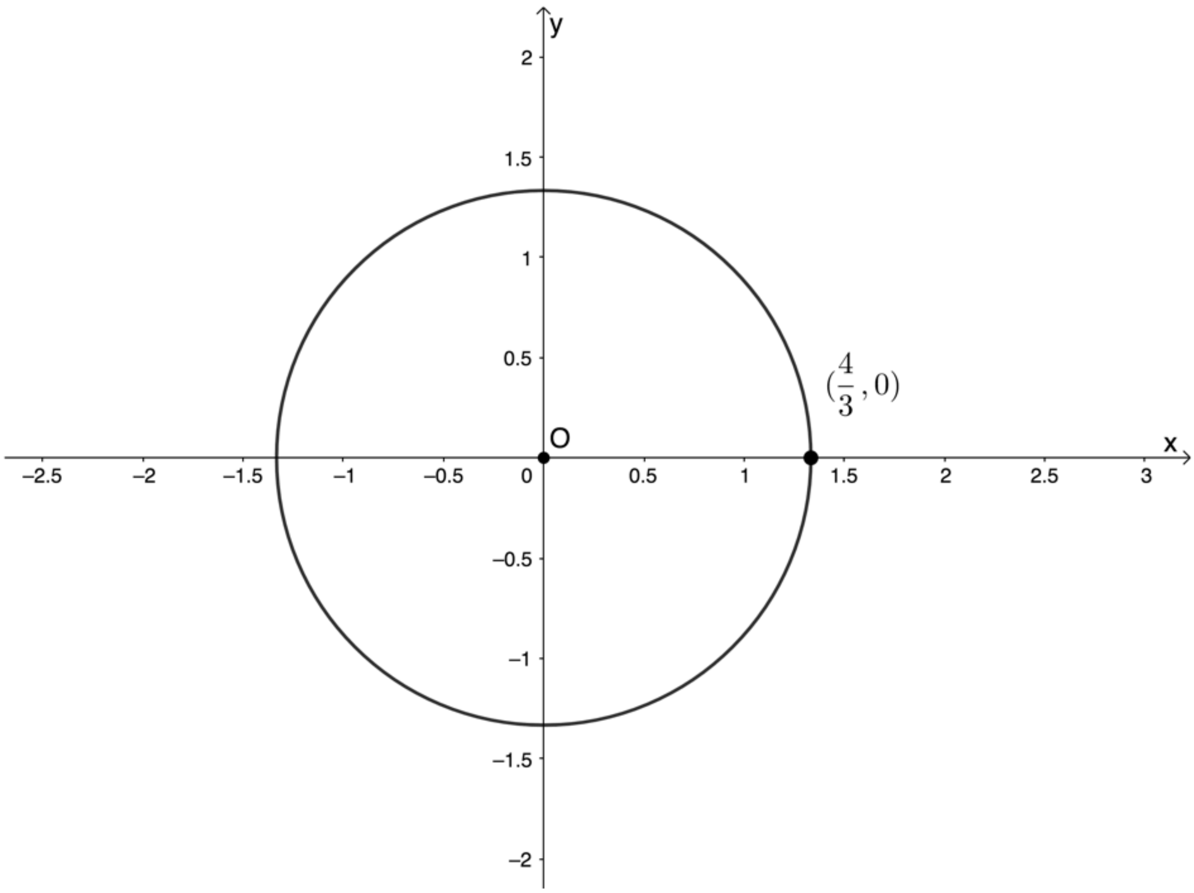
2.

a.

$$r^2 = \frac{16}{9}$$

$$\therefore r = \frac{4}{3}$$

b.



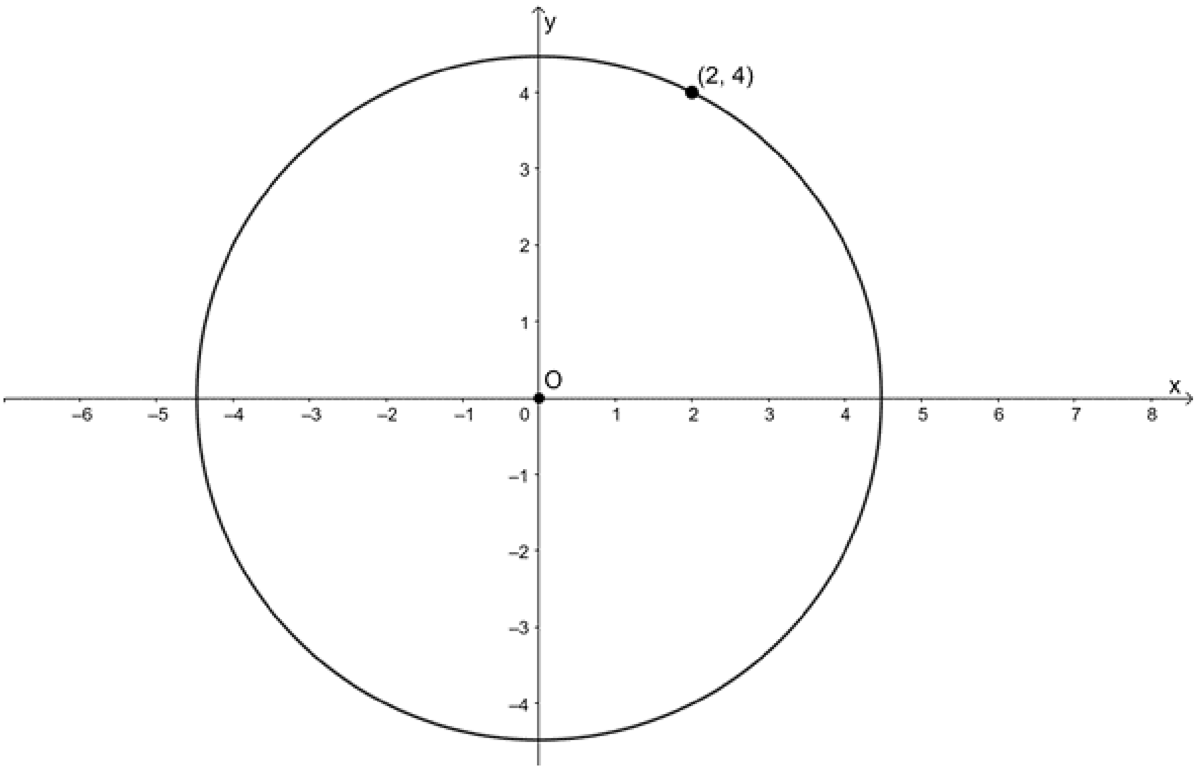
3.

a.

$$4x^2 + 4y^2 = 80$$
$$\therefore x^2 + y^2 = 20$$
$$r^2 = 20$$
$$\therefore r = \sqrt{20} = 2\sqrt{5}$$

b. We can find a point, with coordinates that are integers, that lies on the circle to help us sketch the circle more accurately.

$$20 = 16 + 4$$
$$= 4^2 + 2^2$$



4.

a. $x^2 + y^2 = 36$

b. $x^2 + y^2 = 13$

c.

$$(-3)^2 + (-7)^2 = 9 + 49$$
$$= 58$$
$$\therefore x^2 + y^2 = 58$$

d.

$$(2s)^2 + (3s)^2 = 4s^2 + 9s^2$$
$$= 13s^2$$
$$\therefore x^2 + y^2 = 13s^2$$

e.

$$(-5t)^2 + (-3t)^2 = 25t^2 + 9t^2$$
$$= 34t^2$$
$$\therefore x^2 + y^2 = 34t^2$$

5.

$$\sqrt{90} + x^2 - y^2 = 0$$
$$\therefore x^2 - y^2 = -\sqrt{90}$$
$$\therefore y^2 - x^2 = \sqrt{90}$$

This is not the equation of a circle with centre the origin.

6. $x^2 + y^2 = 50$ and $x = 3y$. Therefore:

$$(3y)^2 + y^2 = 50$$
$$\therefore 9y^2 + y^2 = 50$$
$$\therefore 10y^2 = 50$$
$$\therefore y^2 = 5$$
$$\therefore y = \pm\sqrt{5}$$

Substitute $y = +\sqrt{5}$:

$$x^2+\left(\sqrt{5}\right)^2=50$$
$$\therefore x^2+5=50$$
$$\therefore x^2=45$$
$$\therefore x=\pm\sqrt{45}$$
$$=\pm 3\sqrt{5}$$

Because the circle is symmetrical about the x-axis, we will get the same result substituting $y=-\sqrt{5}$.
Therefore, there are four possible points.
$(3\sqrt{5},\sqrt{5})$, $(-3\sqrt{5},-\sqrt{5})$, $(-3\sqrt{5},\sqrt{5})$ and $(3\sqrt{5},-\sqrt{5})$

Back to Exercise 1.1

Exercise 1.2

1.
   a. $(x+3)^2+y^2=16$
   b. Equation is of the form $(x+5)^2+(y+1)^2=r^2$. Length of CP is the radius.
   $$d_{CP}=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$
   $$=\sqrt{(-5-6)^2+(-1-(-2))^2}$$
   $$=\sqrt{(-11)^2+1^2}$$
   $$=\sqrt{121+1}$$
   $$=\sqrt{122}$$
   The equation is $(x+5)^2+(y+1)^2=122$.
   c. $(x-2)^2+(y+3)^2=11$
2. Determine the centre and radius of the following circles:
   a.
   $$5-x^2-6x-8y-y^2=0$$
   $$x^2+6x+y^2+8y=5 \qquad \text{Complete the squares}$$
   $$\therefore x^2+6x+9+y^2+8y+16=5+9+16$$
   $$\therefore (x+3)^2+(y+4)^2=30$$
   The centre is $(-3,-4)$, the radius is $\sqrt{30}$.
   b.
   $$x^2=21+4y-y^2$$
   $$x^2+y^2-4y=21 \qquad \text{Complete the square}$$
   $$\therefore x^2+y^2-4y-+4=21+4$$
   $$\therefore x^2+(y-2)^2=25$$
   The centre is $(0,2)$, the radius is 5.
   c.
   $$12x-12y-2x^2-2y^2=24$$
   $$\therefore 2x^2-12x+2y^2+12y=24$$
   $$\therefore x^2-6x+y^2+6y=12 \qquad \text{Complete the squares}$$
   $$\therefore x^2-6x+9+y^2+6y+9=12+9+9$$
   $$\therefore (x-3)^2+(y+3)^2=30$$
   The centre is $(3,-3)$, the radius is $\sqrt{30}$.
3. The equation of the circle is $(x-a)^2+(y-b)^2=21$. $A(-3,0)$ and $B(3,0)$ lie on the circle.
   Substitute $A(-3,0)$:

$(-3-a)^2+(0-b)^2=21$

$\therefore 9+6a+a^2+b^2=21$

$\therefore a^2+6a+9=21-b^2 \qquad (1)$

Substitute $B(3,0)$:

$(3-a)^2+(0-b)^2=21$

$\therefore 9-6a+a^2+b^2=21$

$\therefore a^2-6a+9=21-b^2 \qquad (2)$

Set $(1)=(2)$:

$a^2+6a+9=a^2-6a+9$

$\therefore 12a=0$

$\therefore a=0$

Substitute $a=0$ into (1):

$9=21-b^2$

$\therefore b^2=12$

$\therefore b=\pm\sqrt{12}$

$\therefore b=\pm 2\sqrt{3}$

The possible equations of the circle are $x^2+\left(y-2\sqrt{3}\right)^2=21$ or $x^2+\left(y+2\sqrt{3}\right)^2=21$.

4. If PQ is a diameter of the circle, then the midpoint of PQ is the centre.

$$\begin{aligned}\text{midpoint}_{PQ}&=\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)\\&=\left(\frac{5-1}{2},\frac{6-2}{2}\right)\\&=(2,2)\end{aligned}$$

The radius of the circle is the distance from the centre, C, to either P or Q.

$$\begin{aligned}d_{CP}&=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}\\&=\sqrt{(5-2)^2+(6-2)^2}\\&=\sqrt{3^2+4^2}\\&=\sqrt{9+16}\\&=\sqrt{25}\\&=5\end{aligned}$$

Therefore, the equation of the circle is $(x-2)^2+(y-2)^2=25$.

Back to Exercise 1.2

Unit 1: Assessment

1.

a. $x^2+y^2=\frac{49}{9}$

b. $(x-2)^2+(y+5)^2=44$

c.

$$\begin{aligned}r=d_{CR}&=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}\\&=\sqrt{(-3-8)^2+(0-(-4))^2}\\&=\sqrt{(-11)^2+4^2}\\&=\sqrt{121+16}\\&=\sqrt{137}\end{aligned}$$

The equation is $(x+3)^2+y^2=137$.

d.

$$\begin{aligned} r = d_{CP} &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\ &= \sqrt{(5q - 0)^2 + (6q - 1)^2} \\ &= \sqrt{25q^2 + 36q^2 - 12q + 1} \\ &= \sqrt{61q^2 - 12q + 1} \end{aligned}$$

The equation is $x^2 + (y-1)^2 = 61q^2 - 12q + 1$.

2. Substitute in $(\sqrt{2}, h)$:

$$\begin{aligned} (\sqrt{2})^2 + h^2 &= 17 \\ \therefore 2 + h^2 &= 17 \\ \therefore h^2 &= 15 \\ \therefore h &= \pm\sqrt{15} \end{aligned}$$

3. Diameter is 70 cm. Therefore, radius is 35 cm.
$r = 4s$. Substitute $x = r = 4s$ and $y = s$ into $x^2 + y^2 = 1\ 225$.

$$\begin{aligned} (4s)^2 + (s)^2 &= 1\ 225 \\ \therefore 16s^2 + s^2 &= 1\ 225 \\ \therefore 17s^2 &= 1\ 225 \\ \therefore s^2 &= \frac{1\ 225}{17} \\ \therefore s &= \pm\frac{35}{\sqrt{17}} \end{aligned}$$

Therefore:

$$\begin{aligned} r &= 4 \times \left(\pm\frac{35}{\sqrt{17}}\right) \\ &= \pm\frac{140}{\sqrt{17}} \end{aligned}$$

The possible coordinates of $P(r, s)$ are $\left(\pm\frac{140}{\sqrt{17}}, \pm\frac{35}{\sqrt{17}}\right)$.

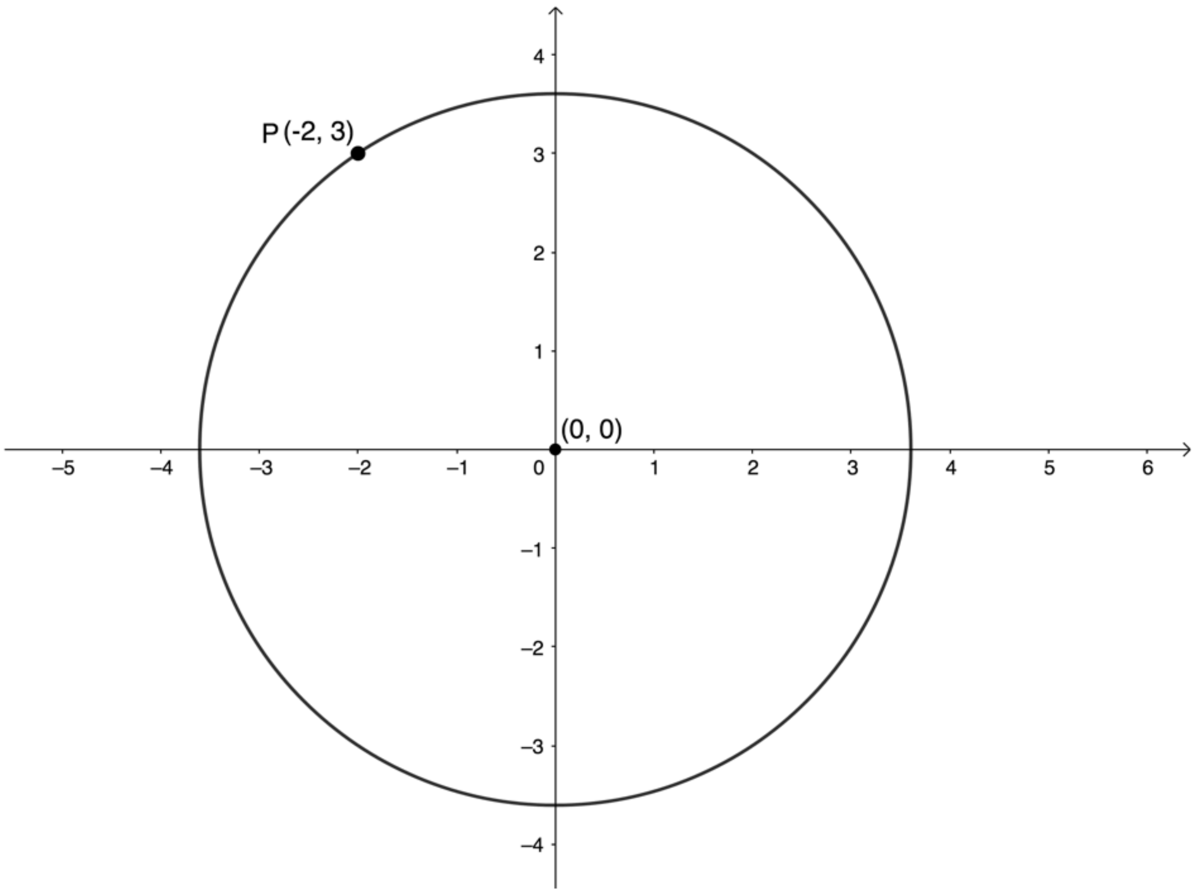
4.

a.

$$\begin{aligned} r^2 &= (-2)^2 + 3^2 \\ &= 4 + 9 \\ &= 13 \end{aligned}$$

The equation of the circle is $x^2 + y^2 = 13$.

b.



c. If PQ is a diameter then P and Q are symmetrical about the line $y = x$. Therefore $Q(3, -2)$.

d.

$$\begin{aligned} d_{PQ} &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\ &= \sqrt{(-2-3)^2 + (3-(-2))^2} \\ &= \sqrt{(-5)^2 + 5^2} \\ &= \sqrt{50} \\ &= 5\sqrt{2} \end{aligned}$$

e.

$$\begin{aligned} m_{PQ} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{3-(-2)}{-2-3} \\ &= -\frac{5}{5} \\ &= -1 \end{aligned}$$

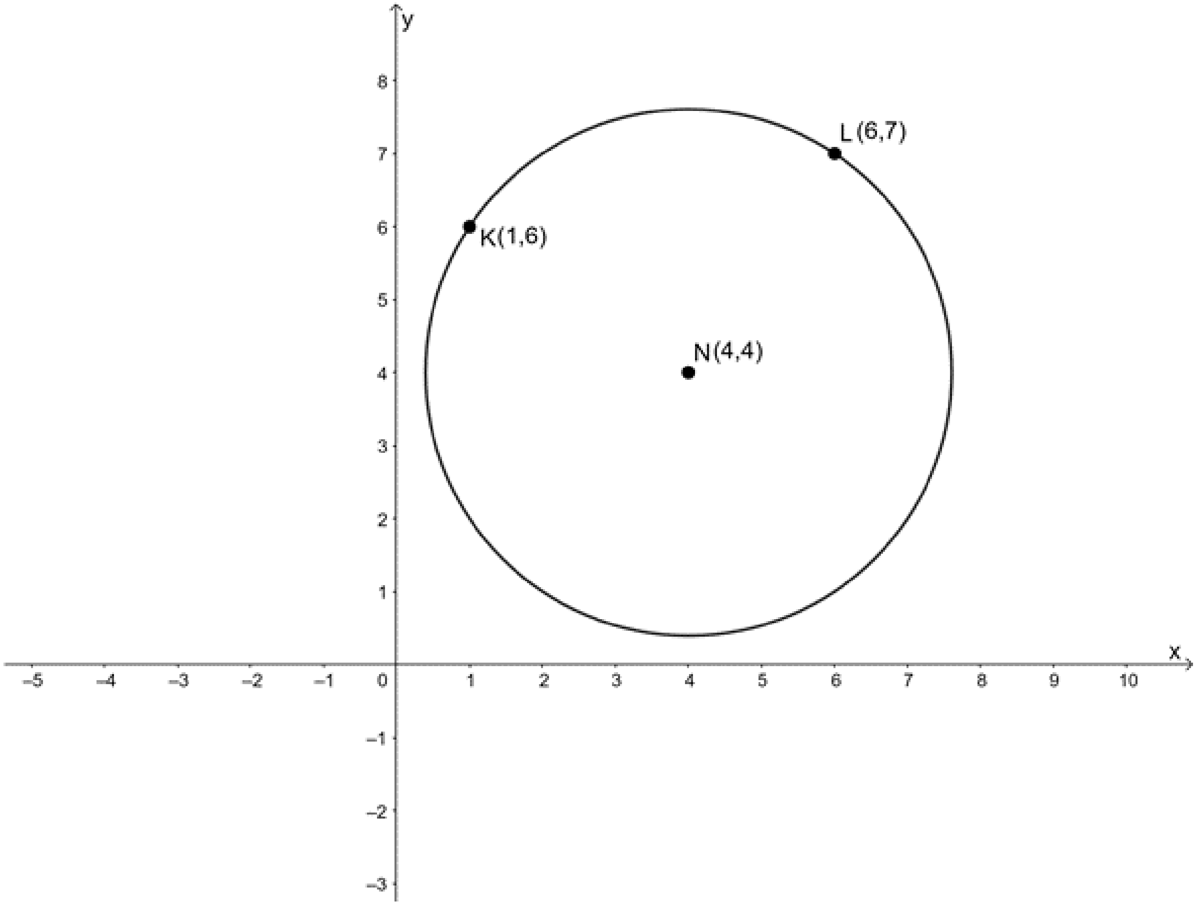
$$\begin{aligned} y - y_1 &= m(x - x_1) \\ \therefore y - 3 &= -1\,(x - (-2)) \\ \therefore y - 3 &= -(x+2) \\ \therefore y - 3 &= -x - 2 \\ \therefore y &= -x + 1 \end{aligned}$$

f. The gradient of the line perpendicular to PQ is $m = 1$.

$$y - y_1 = m(x - x_1)$$
$$\therefore y - 3 = 1\,(x - (-2))$$
$$\therefore y - 3 = (x + 2)$$
$$\therefore y - 3 = x + 2$$
$$\therefore y = x + 5$$

5.

a.



b.

$$r = d_{NK} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$
$$= \sqrt{(4-1)^2 + (4-6)^2}$$
$$= \sqrt{(-3)^2 + (-2)^2}$$
$$= \sqrt{9+4}$$
$$= \sqrt{13}$$

The equation is $(x-4)^2 + (y-4)^2 = 13$.

c.

$$\text{M=midpoint}_{KL} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$
$$= \left(\frac{1+6}{2}, \frac{6+7}{2}\right)$$
$$= (\frac{7}{2}, \frac{13}{2})$$

d.

$$m_{MN} = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{\frac{13}{2} - 4}{\frac{7}{2} - 4}$$
$$= \frac{\frac{5}{2}}{-\frac{1}{2}}$$
$$= \frac{5}{2} \times \left(-\frac{2}{1}\right) = -5$$
$$m_{KL} = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{7-6}{6-1}$$
$$= \frac{1}{5}$$
$m_{MN} \times m_{KL} = -1$. Therefore, $MN \perp KL$.

e. Equation of LN:
$$m_{LN} = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{7-4}{6-4}$$
$$= \frac{3}{2}$$
$$y - y_1 = m(x - x_1)$$
$$\therefore y - 4 = \frac{3}{2}(x-4)$$
$$\therefore y - 4 = \frac{3}{2}x - 6$$
$$\therefore y = \frac{3}{2}x - 2$$

f. Since N is the centre of the circle passing through point L, LN is a radius. Given that LP is a diameter, L, N and P lie on the same straight line. Equation of $LP$is thus:
$$y = \frac{3}{2}x - 2 \qquad (1)$$
$$(x-4)^2 + (y-4)^2 = 13 \qquad (2)$$
Substitute (1) into (2):
$$(x-4)^2 + \left(\left(\frac{3}{2}x - 2\right) - 4\right)^2 = 13$$
$$\therefore x^2 - 8x + 16 + \left(\frac{3}{2}x - 6\right)^2 = 13$$
$$\therefore x^2 - 8x + 16 + \frac{9}{4}x^2 - 18x + 36 = 13$$
$$\therefore \frac{13}{4}x^2 - 26x + 39 = 0$$
$$\therefore 13x^2 - 104x + 156 = 0$$
$$\therefore x^2 - 8x + 12 = 0$$
$$\therefore (x-2)(x-6) = 0$$
$$\therefore x = 2 \text{ or } x = 6$$
$x = 6$ is the x-coordinate of L. Therefore $x = 2$ is the x-coordinate of P.
Substitute $x = 2$ into (1):

$$y = \frac{3}{2}(2) - 2$$
$$= 1$$
$$P(2, 1)$$

6.

$$6x + 3y = -3$$
$$\therefore 3y = -6x - 3$$
$$\therefore y = -2x - 1$$

Every point on this line has coordinates $(x, -2x - 1)$. If the centre is C, $d_{AC} = d_{BC}$.

$$d_{AC} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$
$$= \sqrt{(1 - x)^2 + (2 - (-2x - 1))^2}$$
$$= \sqrt{1 - 2x + x^2 + (2x + 3)^2}$$
$$= \sqrt{1 - 2x + x^2 + 4x^2 + 12x + 9}$$
$$= \sqrt{5x^2 + 10x + 10}$$
$$d_{BC} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$
$$= \sqrt{(-3 - x)^2 + (2 - (-2x - 1))^2}$$
$$= \sqrt{9 + 6x + x^2 + (2x + 3)^2}$$
$$= \sqrt{9 + 6x + x^2 + 4x^2 + 12x + 9}$$
$$= \sqrt{5x^2 + 18x + 18}$$

But $d_{AC} = d_{BC}$. Therefore:

$$\sqrt{5x^2 + 10x + 10} = \sqrt{5x^2 + 18x + 18}$$
$$\therefore 5x^2 + 10x + 10 = 5x^2 + 18x + 18$$
$$\therefore 10x + 10 = 18x + 18$$
$$\therefore 8x = -8$$
$$\therefore x = -1$$
$$y = -2x - 1$$
$$= -2(-1) - 1$$
$$= 1$$

The centre of the circle is the point $(-1, 1)$.

$$r = d_{AC} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$
$$= \sqrt{(1 - (-1))^2 + (2 - 1)^2}$$
$$= \sqrt{4 + 1}$$
$$= \sqrt{5}$$

The equation of the circle is $(x + 1)^2 + (y - 1)^2 = 5$

7.

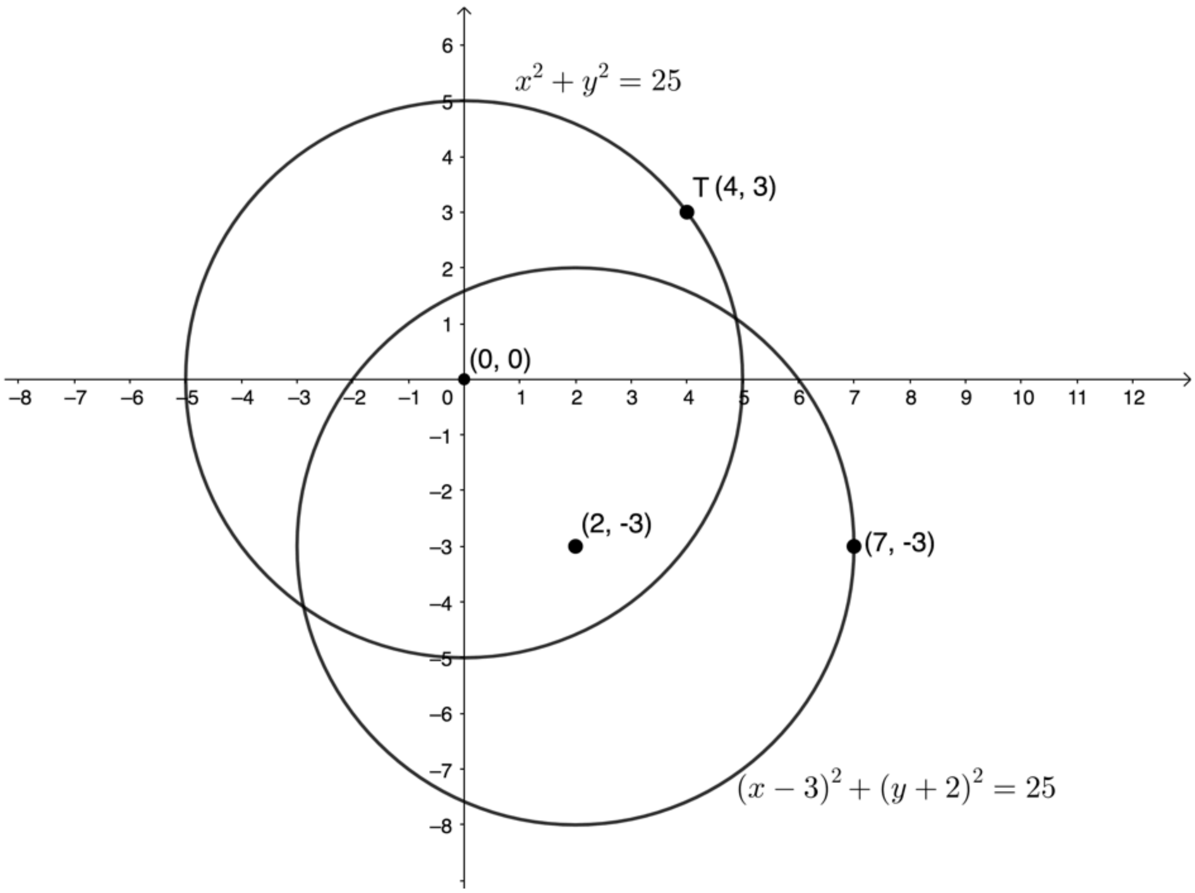
a.

$$r^2 = x^2 + y^2$$
$$= 4^2 + 3^2$$
$$= 16 + 9$$
$$= 25$$

The equation of the circle is $x^2 + y^2 = 25$.

b. $(x - 2)^2 + (y + 3)^2 = 25$

c.



d. The centre of new circle will be $(2, 3)$.

8.

a. The equation has the form $(x+1)^2 + (y+2)^2 = r^2$.

$$r = d_{MA} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$
$$= \sqrt{(-1-(-5))^2 + (-2-3)^2}$$
$$= \sqrt{4^2 + (-5)^2}$$
$$= \sqrt{16 + 25}$$
$$= \sqrt{41}$$

The equation is $(x+1)^2 + (y+2)^2 = 41$.

b.

$$x^2 + y^2 + 2x + 4y - 4 = 0$$
$$\therefore x^2 + 2x + y^2 + 4y = 4$$
$$\therefore x^2 + 2x + 1 + y^2 + 4y + 4 = 4 + 1 + 4$$
$$\therefore (x+1)^2 + (y+2)^2 = 9$$

Therefore, radius of the smaller circle is 3.

$$AB = AM - BM$$
$$= \sqrt{41} - 3$$

Back to Unit 1: Assessment

Media Attributions

Unit 2: Find the equation of a tangent to a circle

DYLAN BUSA

Unit outcomes

By the end of this unit you will be able to:

- Find the gradient of a tangent to a circle using analytical geometry.
- Find the equation of a tangent to the circle using analytical geometry.
- Find the equation of a tangent to a circle at the point of contact with the radius.

What you should know

Before you start this unit, make sure you can:

- Find the equations of straight lines using the two-point form, the gradient-point form or the gradient-intercept form, depending on the information available. Refer to level 3 subject outcome 3.2 unit 1 if you need help with this.
- Find the equations of parallel and perpendicular lines. Refer to level 3 subject outcome 3.2 unit 2 if you need help with this.

Introduction

From your work on limits, instantaneous rates of change and calculus, you are aware of what a **tangent** is (not to be confused with the trigonometric function tangent). The word 'tangent' comes from the Latin verb 'tangere' which means 'to touch' and this exactly describes what a tangent is. It is a straight line that another curve just touches at one point. It does not cross or intersect. It only touches at one point.

Consider the quadratic function $f(x) = (x-2)^2$. The parabola touches the x-axis at the point $(2, 0)$ (see figure 1). It does not cross the x-axis or touch it at more than one point. It touches it only at this point. We say that the x-axis is a tangent to the function $f(x) = (x-2)^2$ at the point $(2, 0)$.

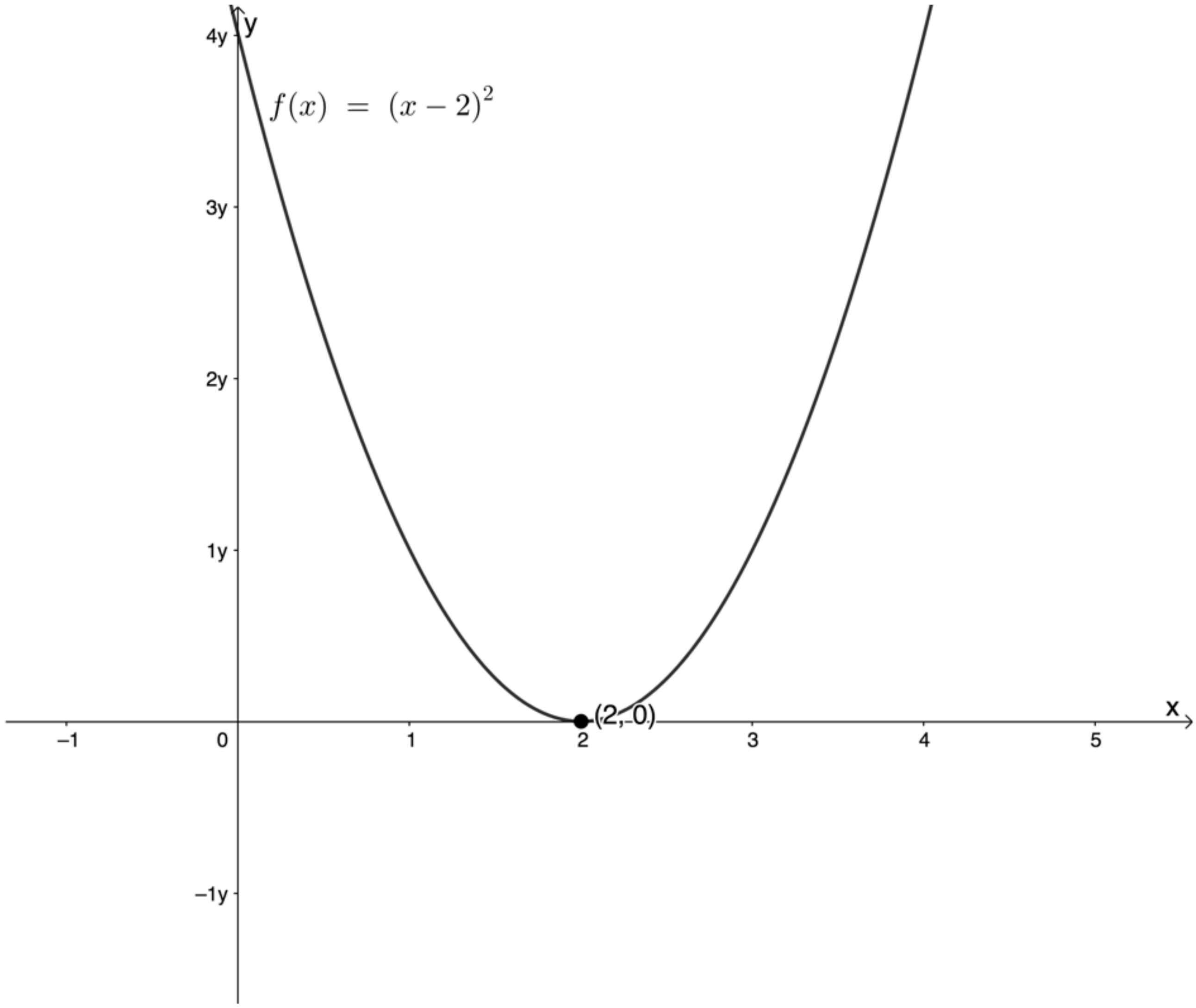


Figure 1: The x-axis is a tangent to $f(x) = (x-2)^2$ *at* $(2, 0)$

Of course, the function $f(x) = (x-2)^2$ does not only have this single tangent. It has infinitely many different tangents, each of which touches the parabola at one point. When we calculate the limit of a function such as $f(x) = (x-2)^2$ at a certain point we are calculating the **gradient of the tangent** to the curve at that point, which we take as the gradient of the curve itself at that point.

Now it is possible to use the techniques and methods of calculus to find the gradient of a tangent to a circle at any point but, because the circle is not a function, this is quite complicated. However, we can use analytical geometry to do this in a much simpler way.

The gradient of the tangent to a circle

Finding the gradient of a tangent to a circle at any point on the circle using analytical geometry is quite straightforward. But it relies on understanding the relationship of the tangent to the radius of the circle at the point of tangency. Let's investigate.

Activity 2.1: The tangent and the radius

Time required: 5 minutes

What you need:

- a pen or pencil

What to do:

In figure 2 is a sketch of the circle $x^2 + y^2 = 25$. On the circle is the point $T(3, 4)$. Touching the circle at point T is a tangent to the circle. $P(7, 1)$ is a point on the tangent line.

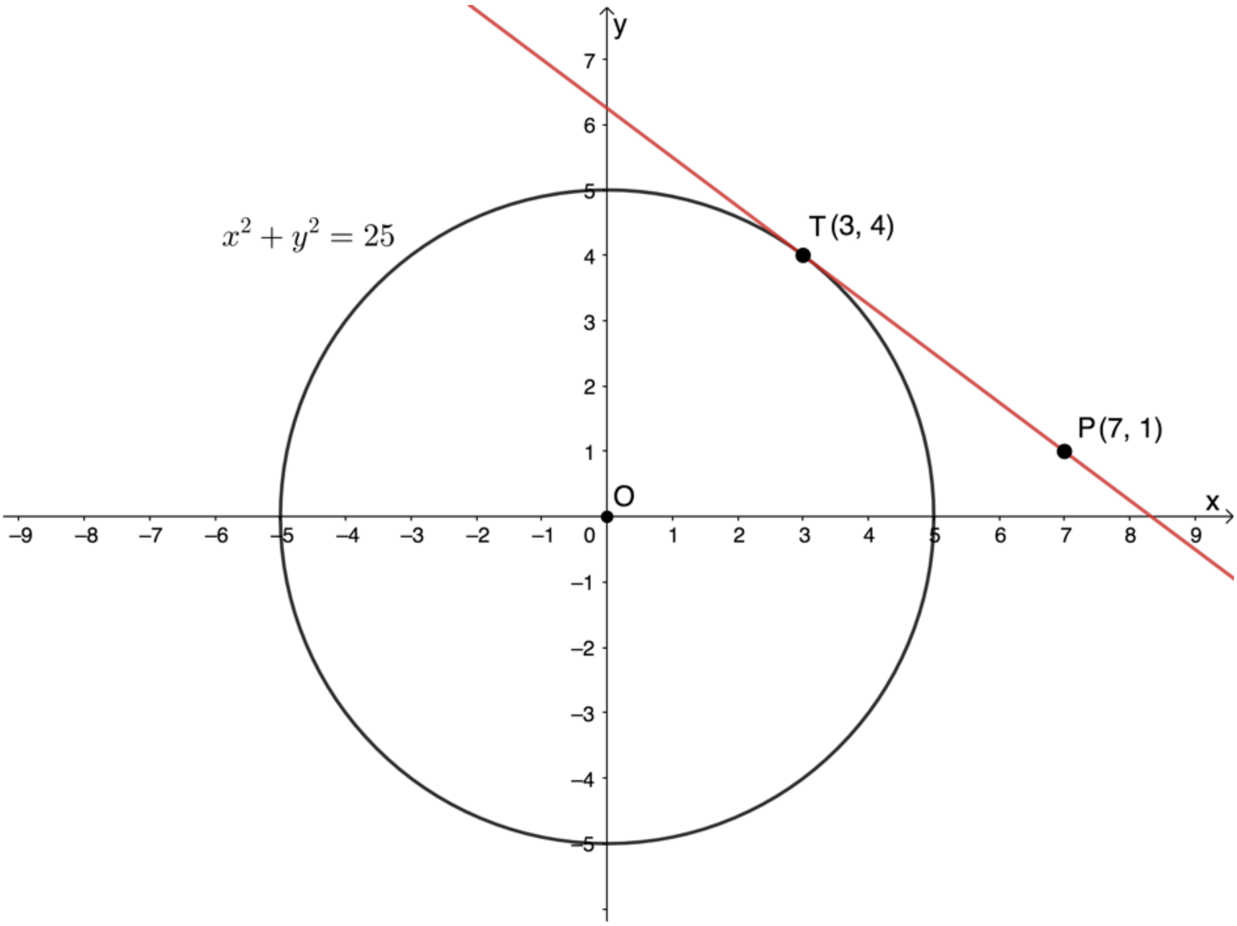


Figure 2: Circle $x^2 + y^2 = 25$ *with point* $T(3, 4)$

1. Determine the gradient of the tangent line PT.
2. Determine the gradient of the radius OT.
3. What can you say about the lines PT and OT?
4. What can you say generally about a tangent and a radius of the circle at the point of tangency?

What did you find?

1.

$$m_{PT} = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{4 - 1}{3 - 7}$$
$$= -\frac{3}{4}$$

2.

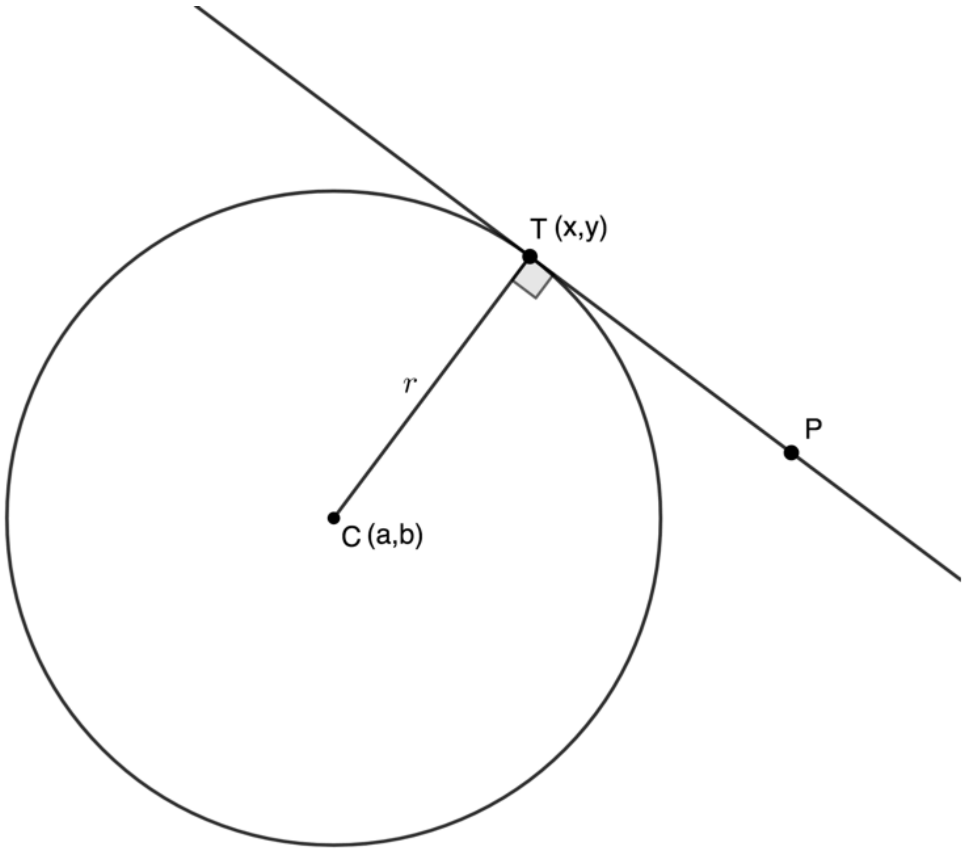
$$m_{OT} = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{4 - 0}{3 - 0}$$
$$= \frac{4}{3}$$

3. Because $m_{PT} \times m_{OT} = -1$ we can say that the lines PT and OT are perpendicular.
4. In general, the tangent to a circle is perpendicular to the radius of the circle at that point.

From activity 2.1 we see that the radius of a tangent is perpendicular to the tangent at the point of contact (or tangency).

Take note!

The radius r of a circle centre $C(a, b)$ is perpendicular to the tangent to the circle PT at the point of tangency $T(x, y)$.



Note

Spend some time exploring an "interactive circle tangent/radius gradient simulator".

Here you can drag the point T around the circle to see the change in the gradient of the tangent and the radius of the circle at the point of tangency and see that in all cases (except for the special cases of horizontal or vertical lines) the product of the two gradients is always -1.

Example 2.1

Find the gradient of the tangent to the circle $2x^2 + 16x + 2y^2 - 4y = 34$ at the point $T(-1, 2)$.

Solution

In order to find the gradient of the tangent to the circle at the point $T(-1,2)$, we need to find the gradient of the radius of the circle to this point. To start, we must get our circle equation into standard form so that we can determine the centre of the circle.

$$\begin{aligned} 2x^2+16x+2y^2-4y &= 34 && \text{Get the coefficients of } x^2 \text{ and } y^2 \text{ terms equal to 1} \\ \therefore x^2+8x+y^2-2y &= 17 && \text{Complete the squares} \\ \therefore x^2+8x+16+y^2-2y+1 &= 17+16+1 \\ \therefore (x+4)^2+(y-1)^2 &= 34 \end{aligned}$$

The centre of the circle is $C(-4,1)$. Now we can find the gradient of CT.

$$\begin{aligned} m_{CT} &= \frac{y_2-y_1}{x_2-x_2} \\ &= \frac{2-1}{-1-(-4)} \\ &= \frac{1}{3} \end{aligned}$$

We know that the tangent at T is perpendicular to the radius at T. Therefore, the gradient of the tangent at T is -3.

Exercise 2.1

Find the gradient of the tangent to the circle $3x^2+6x-36=-3y^2-12y$ at the following points:

1. $A(0,2)$
2. $B(-3,-1)$
3. $C(2,-6)$
4. $D(5,-3)$

The full solutions are at the end of the unit.

Find the equation of a tangent to a circle

Now that we know how to find the gradient of a tangent to a circle, we can use this information to help us find the equation of the tangent line.

To find the equations of tangents to circles, we use the same techniques as we learnt in level 3 subject outcome 3.2 unit 1, namely using the following:

- the two-point form
- the gradient–point form
- the gradient–intercept form.

Refer back to this unit now if you need to.

Example 2.2

Determine the equation of the tangent to the circle $x^2 + y^2 - 2y + 6x - 7 = 0$ at the point $T(-7, 2)$.

Solution

Step 1: Find the centre of the circle

If you don't know the centre, you need to find it. To do so, we must get the equation into standard form.

$$\begin{aligned} x^2 + y^2 - 2y + 6x - 7 &= 0 \\ \therefore x^2 + 6x + y^2 - 2y &= 7 \quad \text{Complete the squares} \\ \therefore x^2 + 6x + 9 + y^2 - 2y + 1 &= 7 + 9 + 1 \\ \therefore (x+3)^2 + (y-1)^2 &= 17 \end{aligned}$$

The centre of the circle is $C(-3, 1)$.

Step 2: Determine the gradient of the radius of the circle to the point of tangency

$$\begin{aligned} m_{CT} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{2-1}{-7-(-3)} \\ &= -\frac{1}{4} \end{aligned}$$

Step 3: Determine the gradient of the tangent

The tangent at $T(-7, 2)$ is perpendicular to the radius to $T(-7, 2)$. Therefore, the gradient of the tangent is 4.

Step 4: Find the equation of the tangent

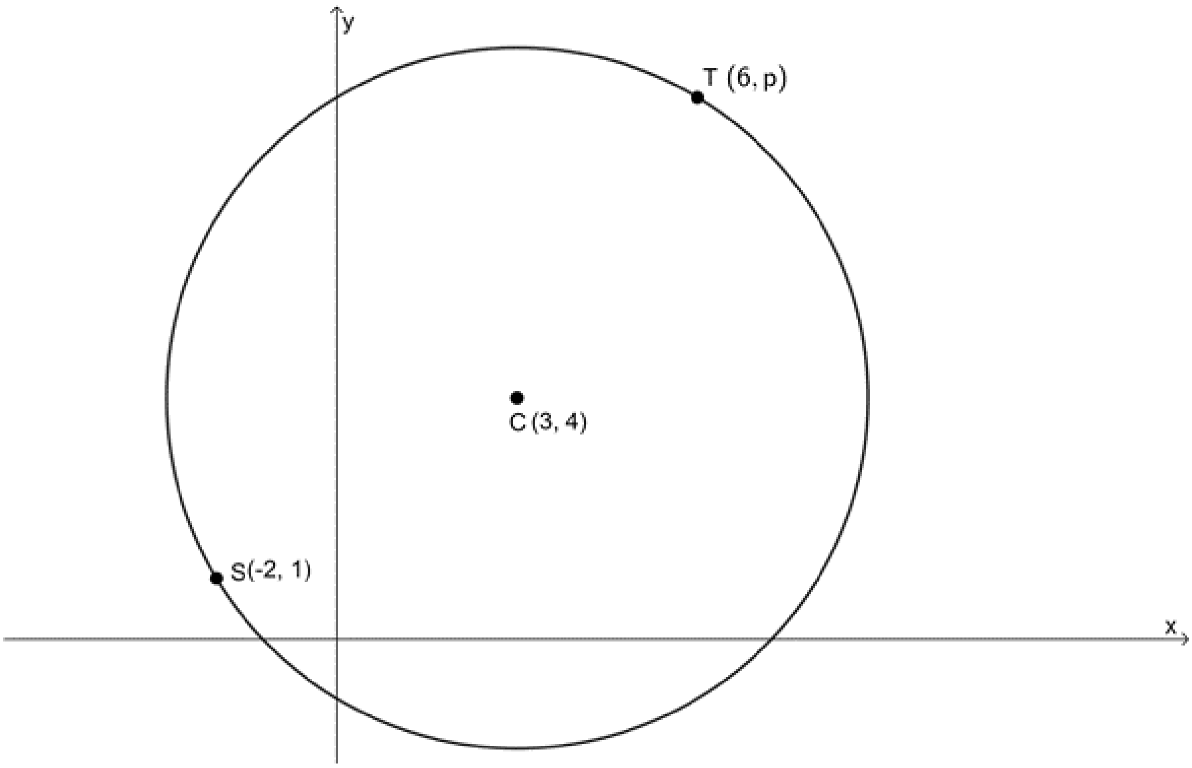
We know the gradient of the tangent and a point $T(-7, 2)$ that it passes through. Therefore, we can use the gradient-point form.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ \therefore y - 2 &= 4\left(x - (-7)\right) \\ \therefore y - 2 &= 4x + 28 \\ \therefore y &= 4x + 30 \end{aligned}$$

The equation of the tangent at $T(-7, 2)$ is $y = 4x + 30$.

Exercise 2.2

1. A circle with centre $C(2, -2)$ has a tangent at $A(-2, -1)$. Find the equation of the tangent.
2. $C(3, 4)$ is the centre of the circle passing through $S(-2, 1)$ and $T(6, p)$.



a. Determine the equation of the circle.
b. Determine the value of p.
c. Determine the equation of the tangent at T.

Question 3 adapted from Everything Maths Grade 12 Worked example 14

3. Determine the equations of the tangents to the circle $x^2 + (y-1)^2 = 80$ if they are both parallel to the line $y = \frac{1}{2}x + 1$.

The full solutions are at the end of the unit.

Summary

In this unit you have learnt the following:

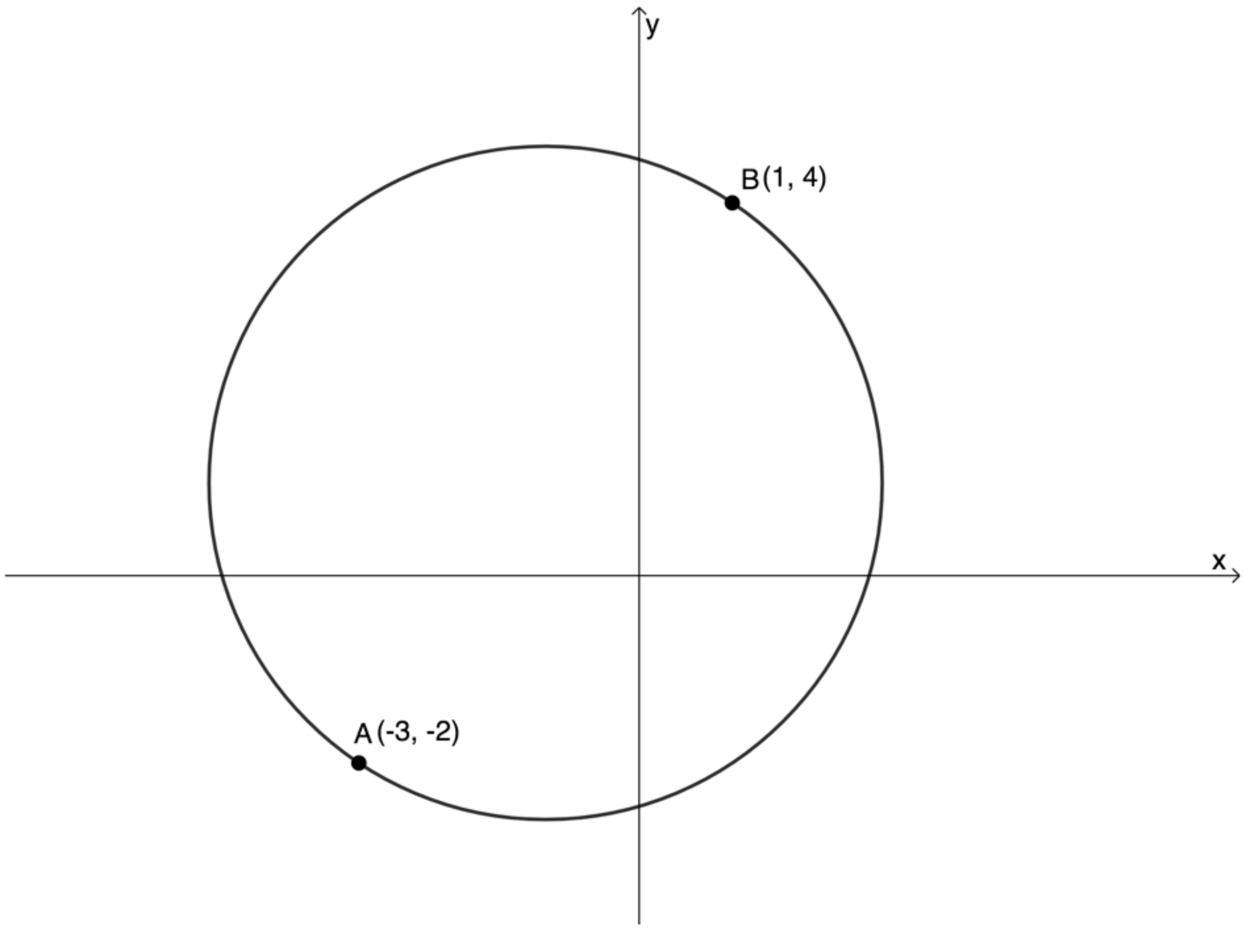
- That the tangent to a circle is perpendicular to the radius of the circle at the point of tangency.

Unit 2: Assessment

Suggested time to complete: 50 minutes

Question 1 adapted from NC(V) Mathematics Level 4 Paper 2 November 2013 question 1.5

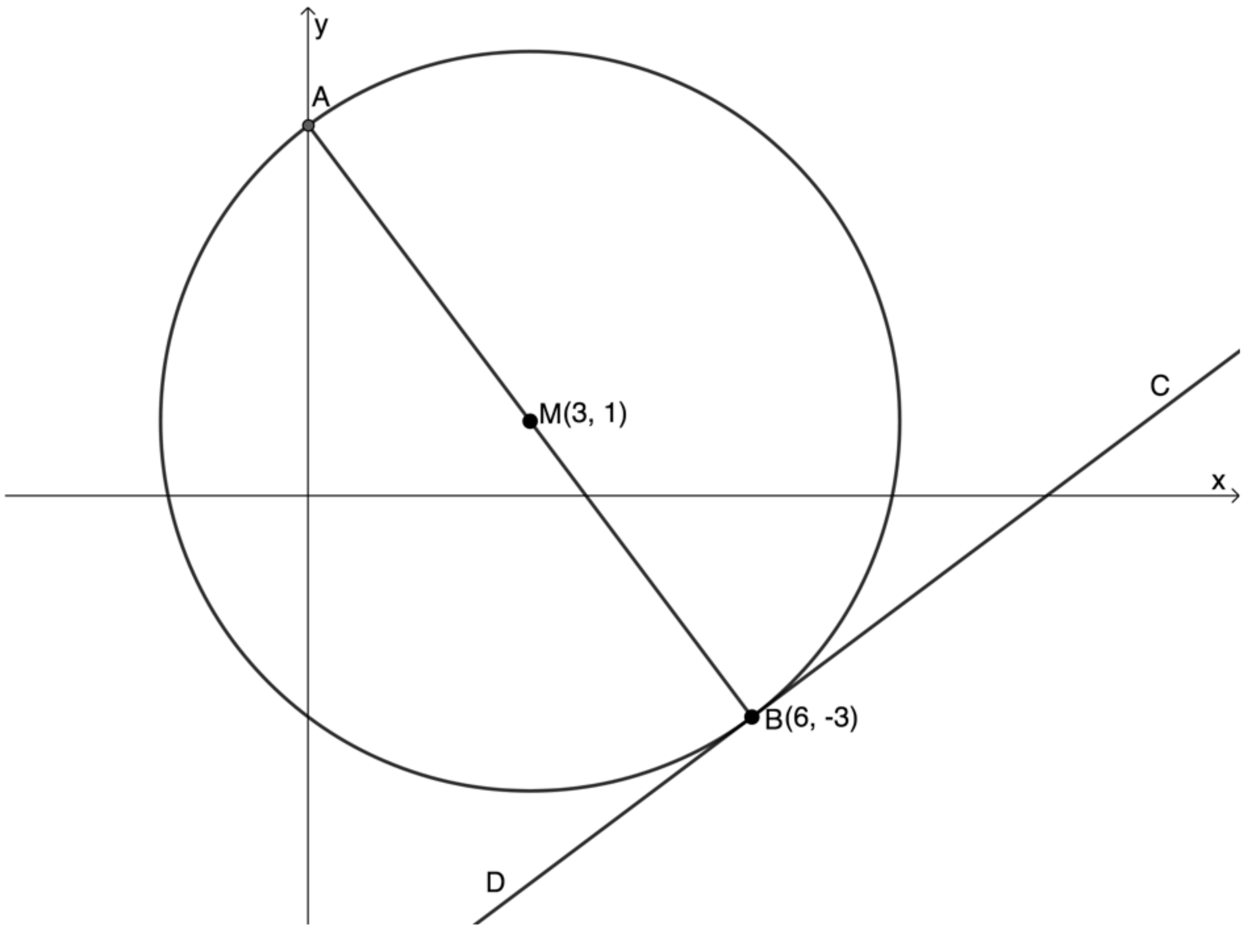
1. Points $A(-3, -2)$ and $B(1, 4)$ lie on a circle such that AB is a diameter of the circle.



a. Determine the equation of the circle.
b. Determine the tangent to the circle at A.

Question 2 adapted from NC(V) Mathematics Level 4 Paper 2 November 2012 question 1.4

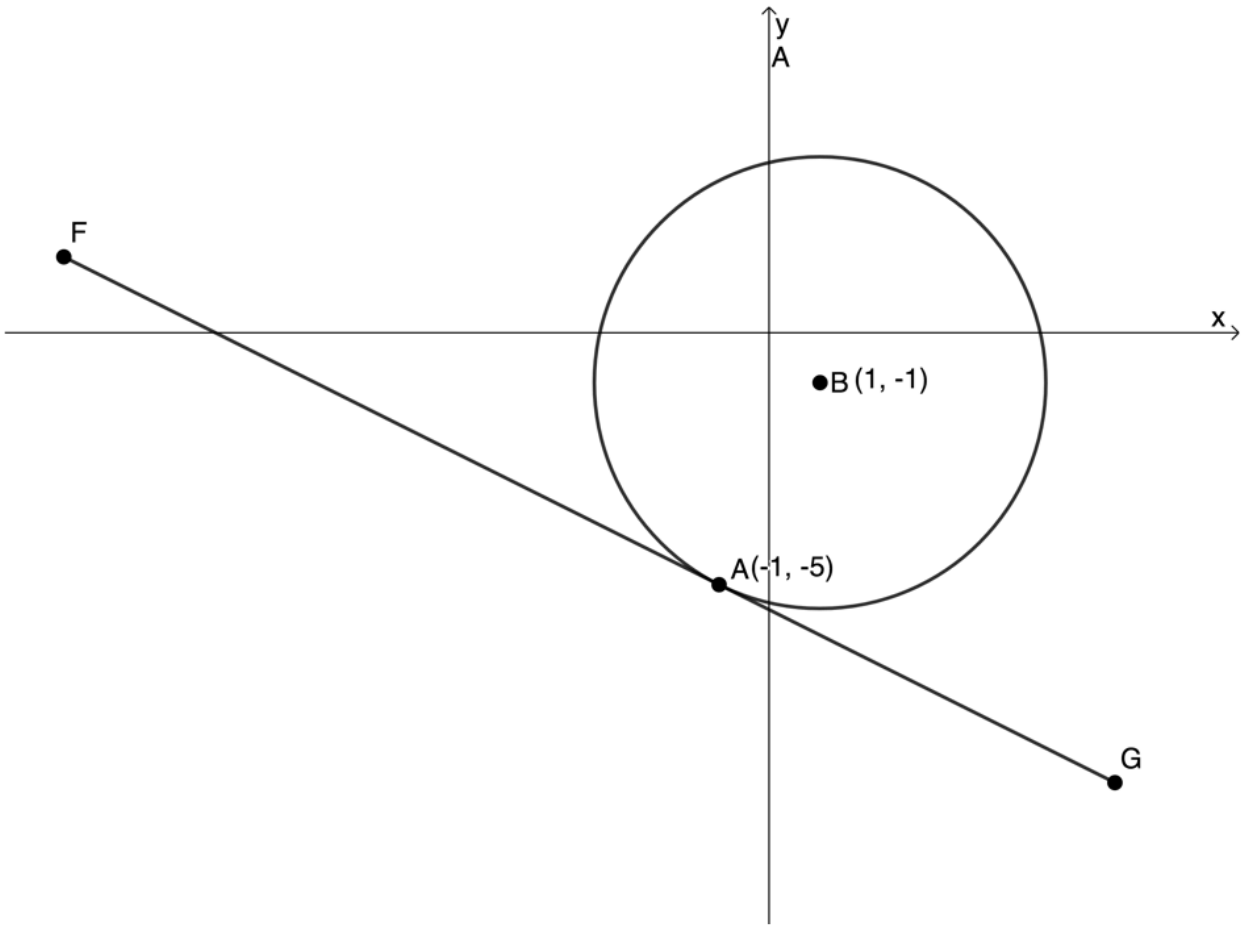
2. In the diagram below a circle with centre $M(3, 1)$ is given. A is a point on the y-axis. AB is the diameter of the circle and CD is a tangent to the circle at $B(6, -3)$.



a. Determine the coordinates of point A.
b. Determine the equation of the tangent to the circle at B.

Question 3 adapted from NC(V) Mathematics Level 4 Paper 2 November 2014 question 1.5

3. In the diagram below $A(-1,-5)$ is a point on the circle with centre $B(1,-1)$. The line FG is a tangent to the circle at point A.



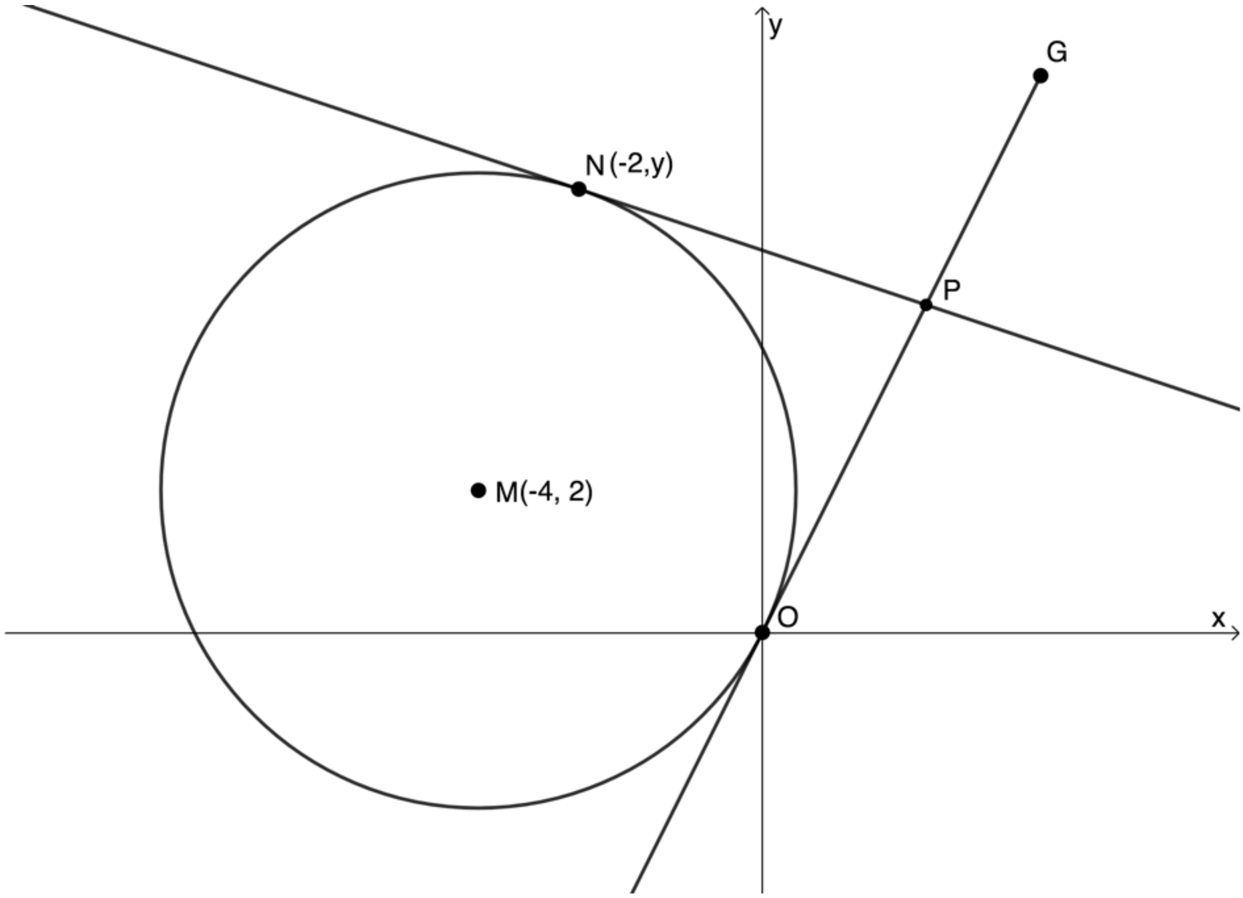
a. Determine the equation of FG.

b. Determine the equation of the circle.

c. If the distance from F to B is $\sqrt{69}\ \text{units}$, determine the length of FA.

Question 4 adapted from NC(V) Mathematics Level 4 Paper 2 November 2015 question 1.4

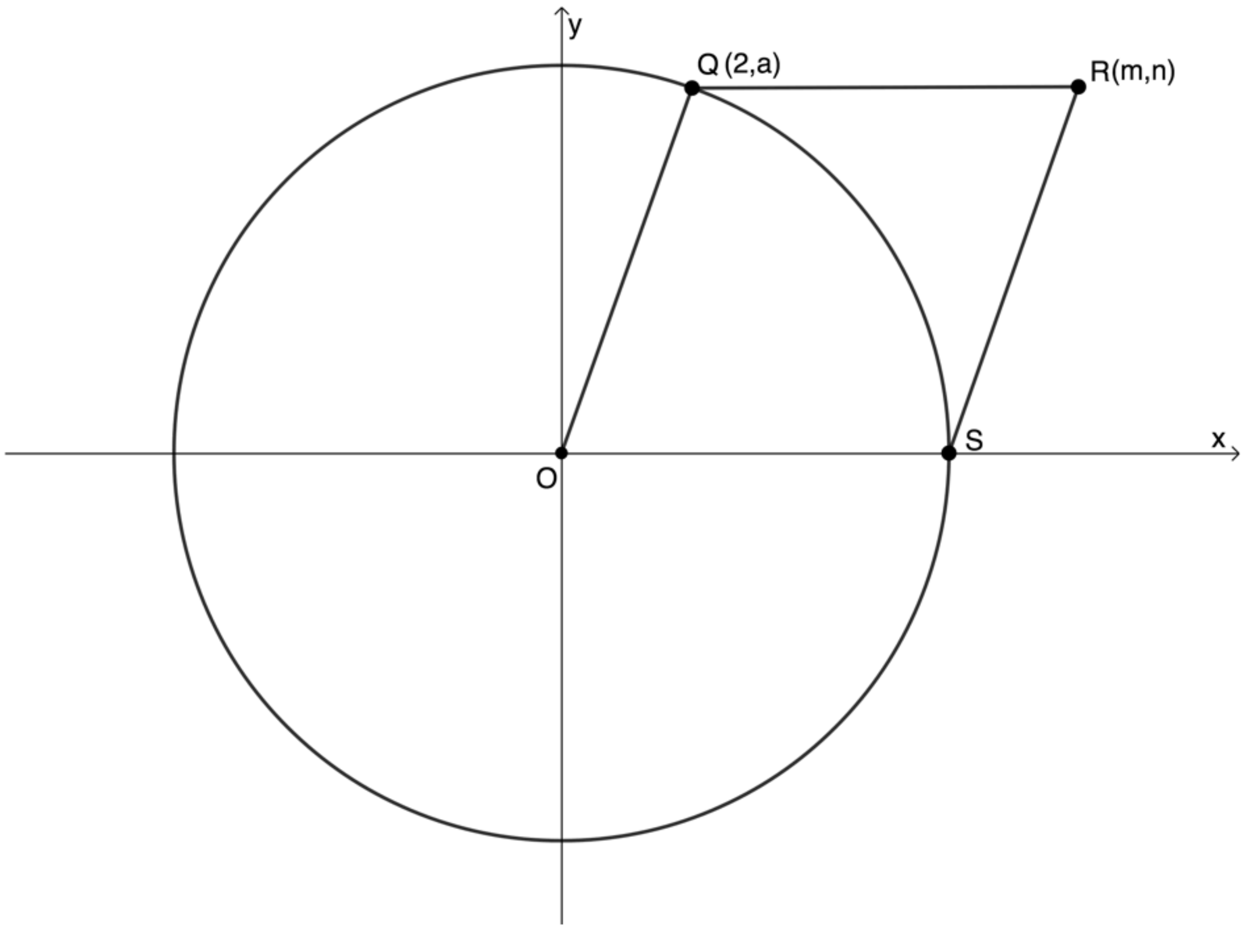
4. A circle with centre $M(-4, 2)$ has the points $O(0, 0)$ and $N(-2, y)$ on the circumference. The tangents at O and N meet at P.



a. Determine the equation of the circle.
b. Determine the value of y.
c. Determine the gradient of OP.

Question 5 adapted from NC(V) Mathematics Level 4 Paper 2 November 2011 question 1.4

5. The equation of the circle with centre O is $x^2 + y^2 = 36$. Circle O cuts the x-axis at S, and Q is the point $(2, a)$. $R(m, n)$ is a point in the first quadrant.



a. Write down the length of the radius of the circle.
b. Calculate the value of a and leave the answer in surd form.
c. Determine the coordinates of R if $OQRS$ is a parallelogram.

The full solutions are at the end of the unit.

Unit 2: Solutions

Exercise 2.1

The equation of the circle is $3x^2 + 6x - 36 = -3y^2 - 12y$.

$$3x^2 + 6x - 36 = -3y^2 - 12y$$
$$\therefore 3x^2 + 6x + 3y^2 + 12y = 36$$
$$\therefore x^2 + 2x + y^2 + 4y = 12 \quad \text{Complete the squares}$$
$$\therefore x^2 + 2x + 1 + y^2 + 4y + 4 = 12 + 1 + 4$$
$$\therefore (x+1)^2 + (y+2)^2 = 17$$

Centre O of the circle is $(-1, -2)$.

1.

$$\begin{aligned} m_{OA} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{-2-2}{-1-0} \\ &= \frac{-4}{-1} \\ &= 4 \\ m_{\text{tangent}} &= -\frac{1}{4} \end{aligned}$$

2.

$$\begin{aligned} m_{OB} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{-2-(-1)}{-1-(-3)} \\ &= -\frac{1}{2} \\ m_{\text{tangent}} &= 2 \end{aligned}$$

3.

$$\begin{aligned} m_{OC} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{-2-(-6)}{-1-2} \\ &= -\frac{4}{3} \\ m_{\text{tangent}} &= \frac{3}{4} \end{aligned}$$

4.

$$\begin{aligned} m_{OD} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{-2-(-3)}{-1-5} \\ &= -\frac{1}{6} \\ m_{\text{tangent}} &= 6 \end{aligned}$$

Back to Exercise 2.1

Exercise 2.2

1.

$$\begin{aligned} m_{CA} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{-1-(-2)}{-2-2} \\ &= -\frac{1}{4} \end{aligned}$$

Therefore, $m_{\text{tangent}} = 4$.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ \therefore y - (-1) &= 4\,(x - (-2)) \\ \therefore y + 1 &= 4x + 8 \\ \therefore y &= 4x + 7 \end{aligned}$$

2.

a.

$$
\begin{aligned}
r = d_{CS} &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\
&= \sqrt{(3 - (-2))^2 + (4 - 1)^2} \\
&= \sqrt{5^2 + 3^2} \\
&= \sqrt{25 + 9} \\
&= \sqrt{34}
\end{aligned}
$$

The equation of the circle is $(x - 3)^2 + (y - 4)^2 = 34$.

b. $T(6, p)$ lies on the circle. Therefore:

$$
\begin{aligned}
(6 - 3)^2 + (p - 4)^2 &= 34 \\
\therefore 3^2 + p^2 - 8p + 16 &= 34 \\
\therefore p^2 - 8p - 9 &= 0 \\
\therefore (p - 9)(p + 1) &= 0 \\
\therefore p = 9 \text{ or } p &= -1
\end{aligned}
$$

Therefore, $p = 9$. We choose this solution because we can see from the sketch that $T(6, p)$ is a point in the first quadrant where $p \geq 0$.

c.

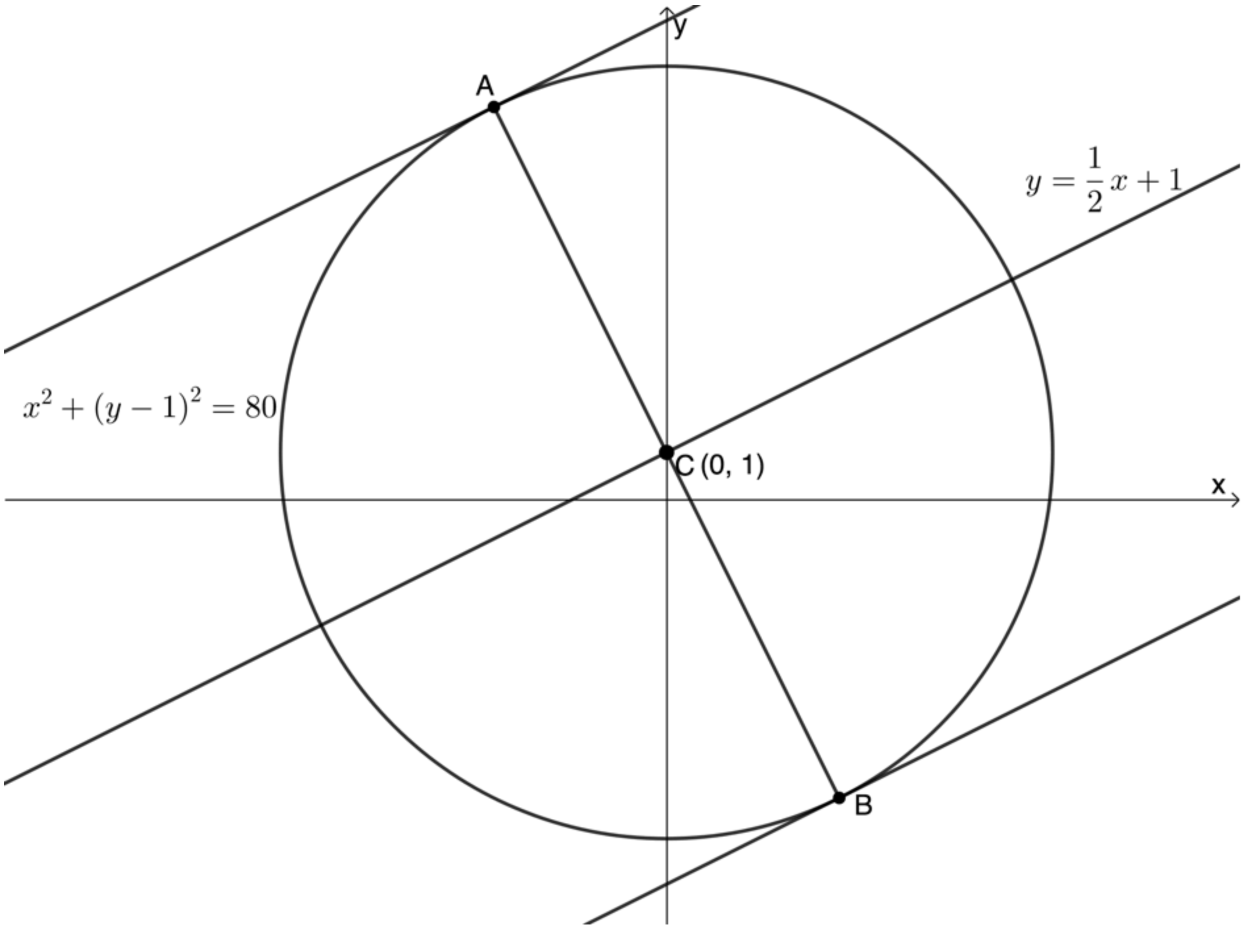
$$
\begin{aligned}
m_{CT} &= \frac{y_2 - y_1}{x_2 - x_1} \\
&= \frac{9 - 4}{6 - 3} \\
&= \frac{5}{3}
\end{aligned}
$$

Therefore, $m_{\text{tangent}} = -\frac{3}{5}$.

$$
\begin{aligned}
y - y_1 &= m(x - x_1) \\
\therefore y - 9 &= -\frac{3}{5}(x - 6) \\
\therefore y - 9 &= -\frac{3}{5}x + \frac{18}{5} \\
\therefore y &= -\frac{3}{5}x + \frac{18 + 45}{5} \\
&= -\frac{3}{5}x + \frac{63}{5}
\end{aligned}
$$

The equation of the tangent is $y = -\frac{3}{5}x + \frac{63}{5}$ or $5y = -3x + 63$.

3. Let the points of tangency be A and B. Both tangents have a gradient of $m = \frac{1}{2}$. Therefore, the gradients of the radii to A and B are $m = -2$. But both these lines pass through the centre. Therefore, AB is a diameter.



Equation of AB:

$y - y_1 = m(x - x_1)$

$\therefore y - 1 = -2(x - 0)$

$\therefore y - 1 = -2x$

$\therefore y = -2x + 1$

To find A and B, solve $y = -2x + 1$ and $x^2 + (y-1)^2 = 80$ simultaneously.

$y = -2x + 1 \qquad (1)$

$x^2 + (y-1)^2 = 80 \qquad (2)$

Substitute (1) into (2):

$x^2 + ((-2x + 1) - 1)^2 = 80$

$\therefore x^2 + 4x^2 = 80$

$\therefore 5x^2 = 80$

$\therefore x^2 = 16$

$\therefore x = \pm 4$

Substitute $x = -4$ into (1):

$y = -2(-4) + 1$

$\therefore y = 9$

$A(-4, 9)$

Substitute $x = 4$ into (1):

$y = -2(4) + 1$

$\therefore y = -7$

$B(4, -7)$

Tangent at A:

$$y - y_1 = m(x - x_1)$$
$$\therefore y - (9) = \frac{1}{2}(x - (-4))$$
$$\therefore y - 9 = \frac{1}{2}x + 2$$
$$\therefore y = \frac{1}{2}x + 11$$

Tangent at B:

$$y - y_1 = m(x - x_1)$$
$$\therefore y - (-7) = \frac{1}{2}(x - 4)$$
$$\therefore y + 7 = \frac{1}{2}x - 2$$
$$\therefore y = \frac{1}{2}x - 9$$

Back to Exercise 2.2

Unit 2: Assessment

1.

a.

$$\begin{aligned} d_{AB} &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(1 - (-3))^2 + (4 - (-2))^2} \\ &= \sqrt{4^2 + 6^2} \\ &= \sqrt{16 + 36} \\ &= \sqrt{52} \\ &= 2\sqrt{13} \end{aligned}$$

Therefore, radius of the circle is $\sqrt{13}$.

The midpoint of AB is the centre of the circle.

$$\begin{aligned} \text{midpoint}_{AB} &= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) \\ &= \left(\frac{1 - 3}{2}, \frac{4 - 2}{2}\right) \\ &= (-1, 1) \end{aligned}$$

The equation of the circle is $(x+1)^2 + (y-1)^2 = 13$.

b.

$$\begin{aligned} m_{\text{radius}} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{1 - (-2)}{-1 - (-3)} \\ &= \frac{3}{2} \end{aligned}$$

Therefore, $m_{\text{tangent}} = -\frac{2}{3}$.

$$y - y_1 = m(x - x_1)$$
$$\therefore y - (-2) = -\frac{2}{3}(x - (-3))$$
$$\therefore y + 2 = -\frac{2}{3}x - 2$$
$$\therefore y = -\frac{2}{3}x - 4$$

The equation of the tangent at A is $y = -\frac{2}{3}x - 4$.

2.

a.

$$\begin{aligned} m_{MB} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{1 - (-3)}{3 - 6} \\ &= -\frac{4}{3} \end{aligned}$$

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ \therefore y - 1 &= -\frac{4}{3}(x - 3) \\ \therefore y - 1 &= -\frac{4}{3}x + 4 \\ \therefore y &= -\frac{4}{3}x + 5 \end{aligned}$$

The equation of MB is $y = -\frac{4}{3}x + 5$. A is the y-intercept of MB. Therefore, the coordinates of A are $(0, 5)$.

b. $m_{DC} = \frac{3}{4}$ (tangent perpendicular to radius at point of tangency)

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ \therefore y - (-3) &= \frac{3}{4}(x - 6) \\ \therefore y + 3 &= \frac{3}{4}x - \frac{18}{4} \\ \therefore y &= \frac{3}{4}x - \left(\frac{18 + 12}{4}\right) \\ &= \frac{3}{4}x - \frac{30}{4} \end{aligned}$$

The equation of the tangent at B is $y = \frac{3}{4}x - \frac{30}{4}$.

3.

a.

$$\begin{aligned} m_{AB} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{-1 - (-5)}{1 - (-1)} \\ &= \frac{4}{2} \\ &= 2 \end{aligned}$$

Therefore, $m_{FG} = -\frac{1}{2}$.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ \therefore y - (-5) &= -\frac{1}{2}(x - (-1)) \\ \therefore y + 5 &= -\frac{1}{2}x - \frac{1}{2} \\ \therefore y &= -\frac{1}{2}x - \frac{11}{2} \end{aligned}$$

The equation of tangent FG is $y = -\frac{1}{2}x - \frac{11}{2}$.

b. Circle is of the form $(x - 1)^2 + (y + 1)^2 = r^2$.

$$
\begin{aligned}
r = d_{AB} &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\
&= \sqrt{(1 - (-1))^2 + (-1 - (-5))^2} \\
&= \sqrt{2^2 + (4)^2} \\
&= \sqrt{4 + 16} \\
&= \sqrt{20} \\
&= 2\sqrt{5}
\end{aligned}
$$

Therefore, the equation of the circle is $(x-1)^2 + (y+1)^2 = 20$.

c. $BA \perp FA$. Therefore ΔFAB is a right-angled triangle. Therefore:

$$
\begin{aligned}
FB^2 &= FA^2 + AB^2 \\
\therefore FA^2 &= FB^2 - AB^2 \\
\therefore FA^2 &= 69 - 20 = 49 \\
\therefore FA &= 7
\end{aligned}
$$

Note: Because FA is a length, we can ignore the negative root.

4.

a. Equation is of the form $(x+4)^2 + (y-2)^2 = r^2$.

$$
\begin{aligned}
r = d_{MO} &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\
&= \sqrt{(-4 - 0)^2 + (2 - 0)^2} \\
&= \sqrt{16 + 4} \\
&= \sqrt{20} \\
&= 2\sqrt{5}
\end{aligned}
$$

Therefore, the equation of the circle is $(x+4)^2 + (y-2)^2 = 20$.

b. $N(-2, y)$ lies on the circle.

$$
\begin{aligned}
((-2) + 4)^2 + (y-2)^2 &= 20 \\
\therefore 4 + y^2 - 4y + 4 &= 20 \\
\therefore y^2 - 4y - 12 &= 0 \\
\therefore (y-6)(y+2) &= 0 \\
\therefore y = 6 \text{ or } y &= -2
\end{aligned}
$$

Therefore, at N $y = 6$.

c.

$$
\begin{aligned}
m_{MO} &= \frac{y_2 - y_1}{x_2 - x_1} \\
&= \frac{2 - 0}{-4 - 0} \\
&= -\frac{1}{2}
\end{aligned}
$$

Therefore, $m_{OP} = 2$ (tangent perpendicular to radius as point of tangency).

5.

a.

$$
\begin{aligned}
r^2 &= 36 \\
\therefore r &= 6
\end{aligned}
$$

b. $(2, a)$ lies on the circle. Therefore:

$$
\begin{aligned}
(2)^2 + a^2 &= 36 \\
\therefore a^2 &= 32 \\
\therefore a &= \pm\sqrt{32} \\
&= \pm 4\sqrt{2}
\end{aligned}
$$

Therefore, $a = 4\sqrt{2}$.

c. $QR \parallel OS$ ($OQRS$ is a parallelogram, given)
But OS is on the x-axis. Therefore, y-coordinate of Q is equal to y-coordinate of R. Therefore, $n = 2$.
$QR = OS$ ($OQRS$ is a parallelogram, given)
But $OS = r = 6 = QR$. Therefore, $m = 2 + 6 = 8$.

Back to Unit 2: Assessment

Media Attributions

SUBJECT OUTCOME IX

SPACE, SHAPE AND MEASUREMENT: EXPLORE, INTERPRET AND JUSTIFY GEOMETRIC RELATIONSHIPS

Subject outcome

Subject outcome 3.2: Explore, interpret and justify geometric relationships

Learning outcomes

- Use geometry of straight lines and triangles to solve problems and to justify relationships in geometric figures. Concepts to include are:
  - angles of a triangle
  - exterior angles
  - straight lines
  - vertically opposite angles
  - corresponding angles
  - co-interior angles
  - alternate angles.
- State and apply the following theorems of circles:
  - If a line is drawn from the centre of a circle to the midpoint of a chord, then that line is perpendicular to the chord.
  - If a line is drawn from the centre of the circle perpendicular to the chord, then it bisects the chord.
  - If an arc subtends an angle at the centre of the circle and at any point on the circumference, then the angle at the centre is twice the measure of the angle at the circumference.
  - If the diameter of a circle subtends an angle at the circumference, then the angle subtended is a right-angle triangle.
  - If an angle subtended by a chord at a point on the circumference is a right angle, then the chord is a diameter.
  - Angles in the same segment of a circle are equal.
  - The opposite angles of a cyclic quadrilateral are supplementary.
  - An exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.
  - If the exterior angle of a quadrilateral is equal to the interior opposite angle the quadrilateral will be a cyclic quadrilateral.
  - The four vertices of a quadrilateral in which the opposite angles are supplementary will be a cyclic quadrilateral.
  - If a tangent to a circle is drawn, then it is perpendicular to the radius at the point of contact.
  - If a line is drawn perpendicular to a radius at the point where the radius meets the circle, then it is a tangent to the circle.
  - If two tangents are drawn from the same point outside a circle then they are equal in length.
  - The angle between a tangent to a circle and a chord drawn from the point of contact is equal

to an angle in the alternate segment (tan-chord theorem).
Note: Proofs of the above theorems are excluded

Unit 1 outcomes

By the end of this unit you will be able to:

- Use properties of angles on straight lines and angles in a triangle to find unknown angles including:
    - angles of a triangle
    - exterior angles
    - straight lines
    - vertically opposite angles
    - corresponding angles
    - co-interior angles
    - alternate angles

Unit 2 outcomes

By the end of this unit you will be able to:

- Apply the following theorems relating to circles:
    - Line drawn perpendicular to a chord from the centre of the circle bisects the chord, and its converse (line drawn from the circle centre to the mid-point of chord is perpendicular to the chord).
    - Angle at the centre of a circle is twice the size of the angle at the circumference.
    - Diameter of a circle subtends a right angle at the circumference, and its converse (if an angle subtended by a chord at a point on the circumference is a right angle, then the chord is a diameter).
    - Angles in the same segment of a circle are equal.

Unit 3 outcomes

By the end of this unit you will be able to:

- Define a cyclic quadrilateral.
- Apply the theorem opposite angles of a cyclic quadrilateral are supplementary.
- Apply the theorem exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.

- Apply the converses of equality between opposite angles, between angles in the same segment, and between exterior angles and interior opposite angles, to prove a quadrilateral is cyclic.

Unit 4 outcomes

By the end of this unit you will be able to:

- Apply the theorem tangent perpendicular to radius at point of contact.
- Apply the converse theorem of tangent perpendicular to radius.
- Apply the theorem two tangents drawn from the same point outside a circle are equal.
- Apply the tangent-chord theorem.

Unit 1: Finding angles of straight lines and triangles

DYLAN BUSA

Unit outcomes

By the end of this unit you will be able to:

- Use properties of angles on straight lines and angles in a triangle to find unknown angles including:
  - angles of a triangle
  - exterior angles
  - straight lines
  - vertically opposite angles
  - corresponding angles
  - co-interior angles
  - alternate angles

What you should know

Before you start this unit, make sure you can:

- Work with triangles and the measurement of angles. Level 2 subject outcome 3.2 will remind you of these basics.

Introduction

The word 'geometry' comes from two Greek words 'geo' meaning earth and 'metria' meaning measure. It is the area of mathematics that deals with objects and spatial arrangements. Analytical geometry, as we studied in subject outcome 3.1, deals with these objects and their spatial arrangements by using the Cartesian coordinate system and algebra.

Euclidean geometry, on the other hand, deals with these objects and their spatial arrangement using a system of reasoning and logical deductions.

Euclidean geometry was first developed to solve surveying problems and is still used widely for this purpose today. Euclidean geometry principles are also used in architectural design and art. Some of the most interesting applications of Euclidean geometry are used to help us determine the best packing arrangement for various types of objects.

Did you know?

Euclid, the 'Father of Geometry' was a Greek mathematician from the third century BC. He wrote a book on geometry called the Elements which is still considered one of the most important and influential books on mathematics. The Elements also dealt with number theory, factorisation and common divisors.

Euclidean geometry is not the only kind of geometry. Spherical geometry, for example, deals with the geometry of spheres. In spherical geometry, the angles inside a triangle do not add up to 180°. They add up to more than 180°. How much more depends on the size and shape of the triangle. These alternative geometries are called non-Euclidean geometries.

Watch the video called "5-Sided Square" if you are interested in learning more about spherical geometry.

5-Sided Square (Duration: 09.14)

In this subject outcome, we will focus on a section of Euclidean geometry called **circle geometry** which considers the relationships between circles, triangles (three-sided figures) and quadrilaterals (four-sided figures).

Studying geometry in general and circle geometry in particular is an excellent way to help us develop and hone our logic, reasoning and argumentation skills. Being able to reason logically and express our reasons and arguments clearly and concisely is an important and valuable life skill.

Angles and lines

Many of the properties of angles and lines you will most likely be familiar with already, but we will list them all in this section for completeness.

We know that an angle is formed when two lines meet or intersect at a point (called a **vertex**). We measure the size of the angle between the two lines in degrees. Here is some of the basic terminology used to describe and work with angles.

- Angles smaller than 90° are called **acute** angles.
- Angles of 90° are called **right** angles
- Angles between 90° and 180° are called **obtuse** angles.
- Angles of 180° are called **straight** angles and are on a straight line.
- Angles between 180° and 360° are called **reflex** angles.

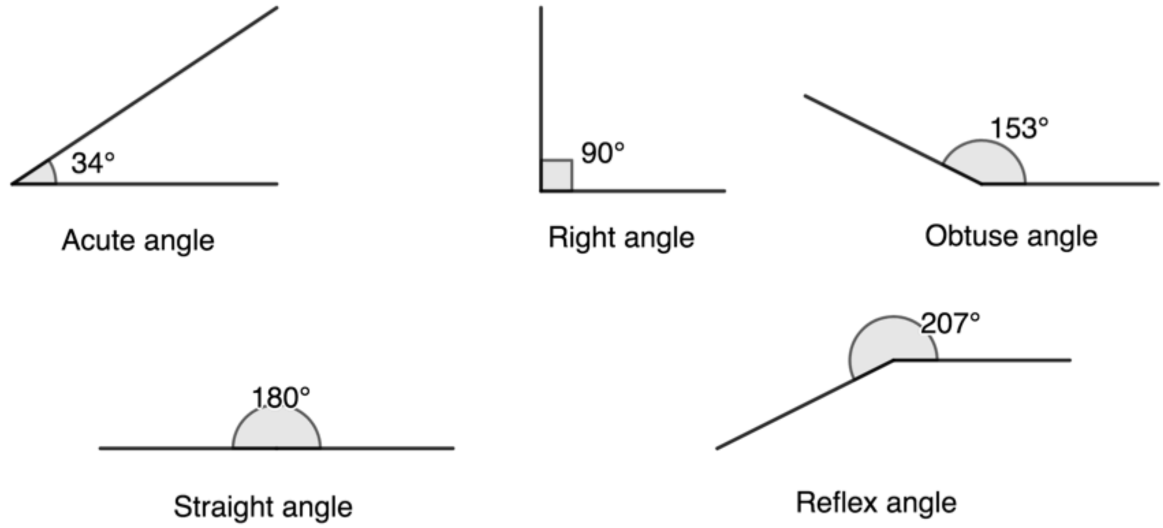


Figure 1: Different types of angles

- Angles that add up to 90° are called **complementary** angles.
- Angles that add up to 180° are called **supplementary** angles.

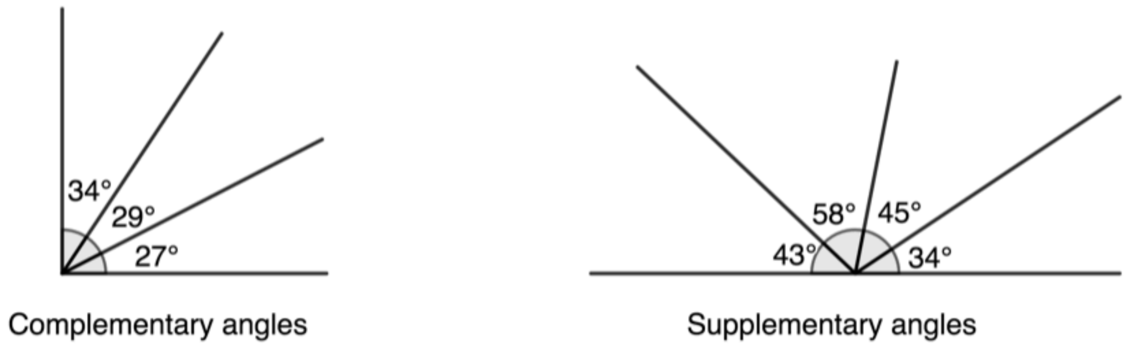


Figure 2: Sum of angles

If two straight lines intersect each other, four angles are created. The vertically opposite angles in this case are equal. In other words, $\hat{A}_1 = \hat{A}_3$ and $\hat{A}_2 = \hat{A}_4$ (see figure 3).

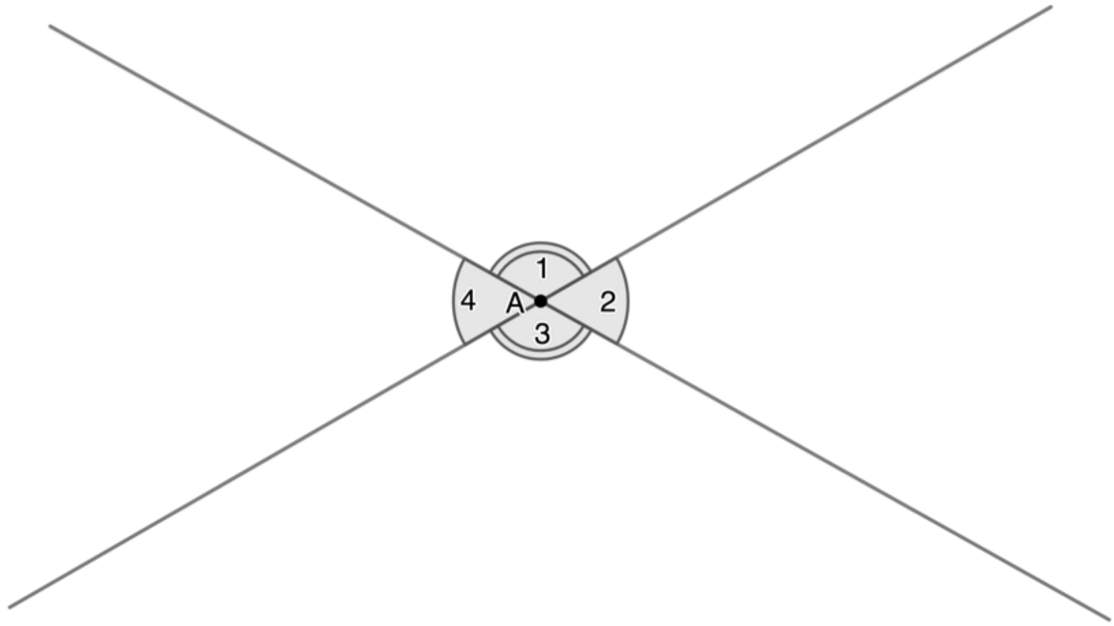


Figure 3: Vertically opposite angles are equal

Angles and parallel lines

Often, we need to work with the angles formed by a transversal line (EF) cutting two or more other lines (AB and BC) (see figure 4). This situation creates a number of important angles.

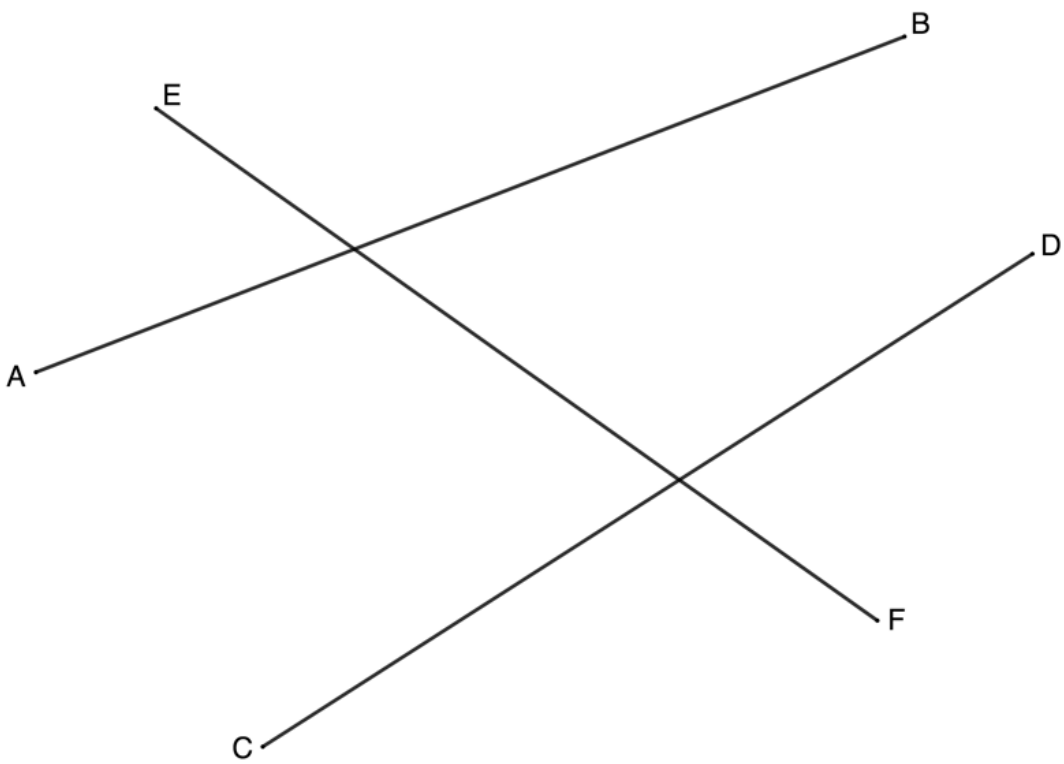


Figure 4: Two lines being cut by a transversal

Corresponding angles are formed such as $\hat{G}_1$ and $\hat{H}_1$ (see figure 5). Corresponding angles are on the corresponding side of the lines and the same side of the transversal.

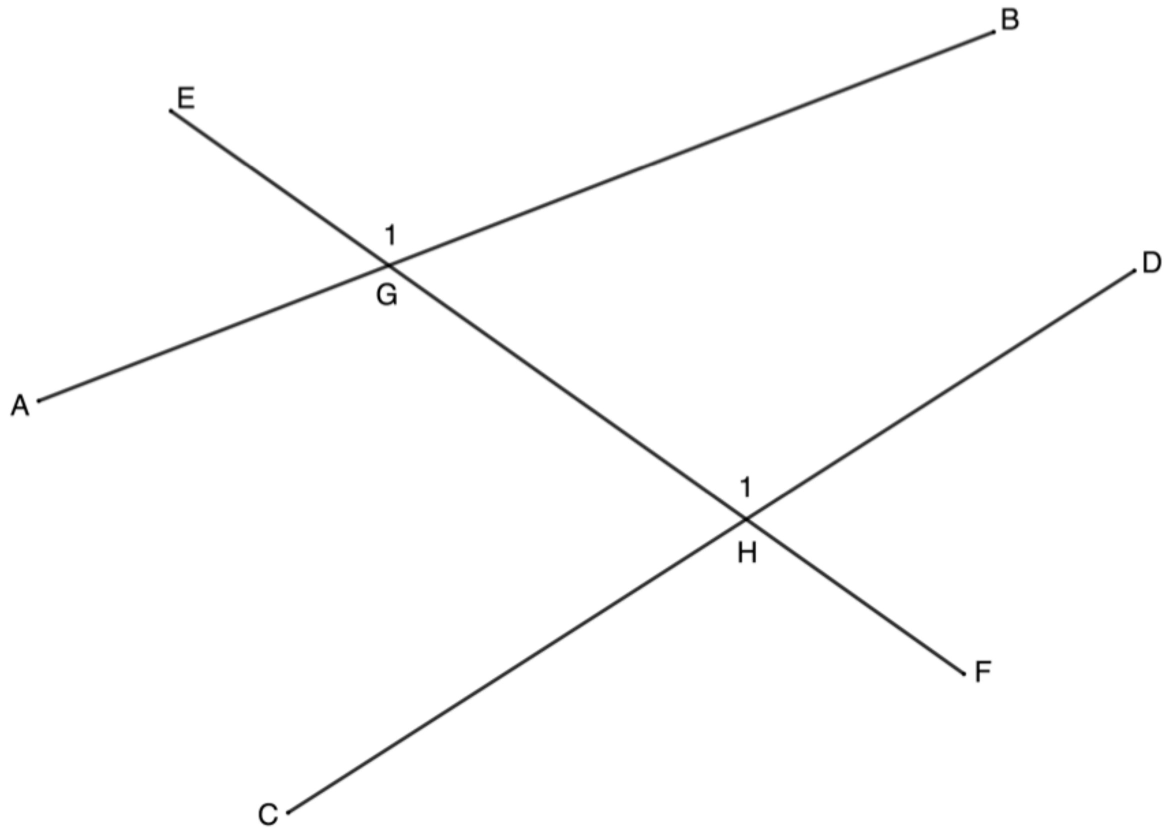


Figure 5: Corresponding angles

Alternate angles are formed such as $\hat{G}_2$ and $\hat{H}_1$ (see figure 6). Alternate angles lie between the lines and on opposite sides of the transversal.

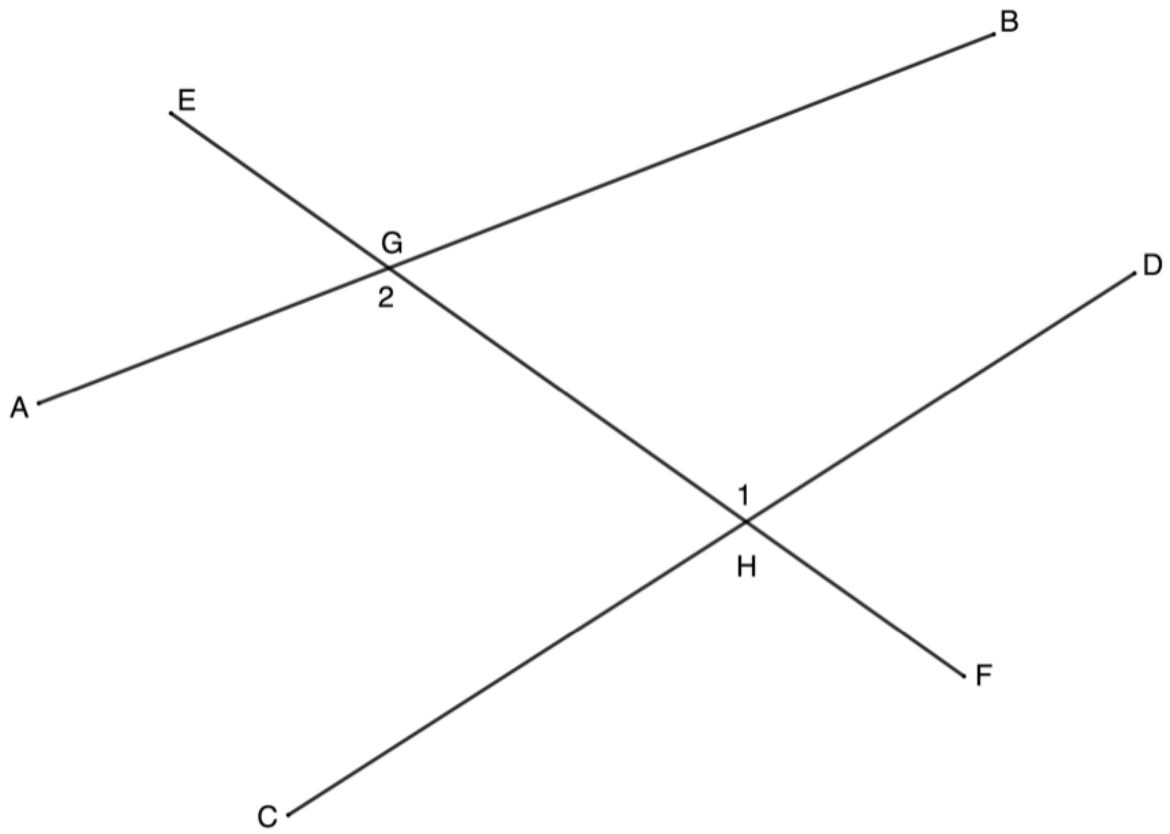


Figure 6: Alternate angles

Co-interior angles are formed such as $\hat{G}_3$ and $\hat{H}_1$ (see figure 7). Co-interior angles lie between the lines and on the same side of the transversal.

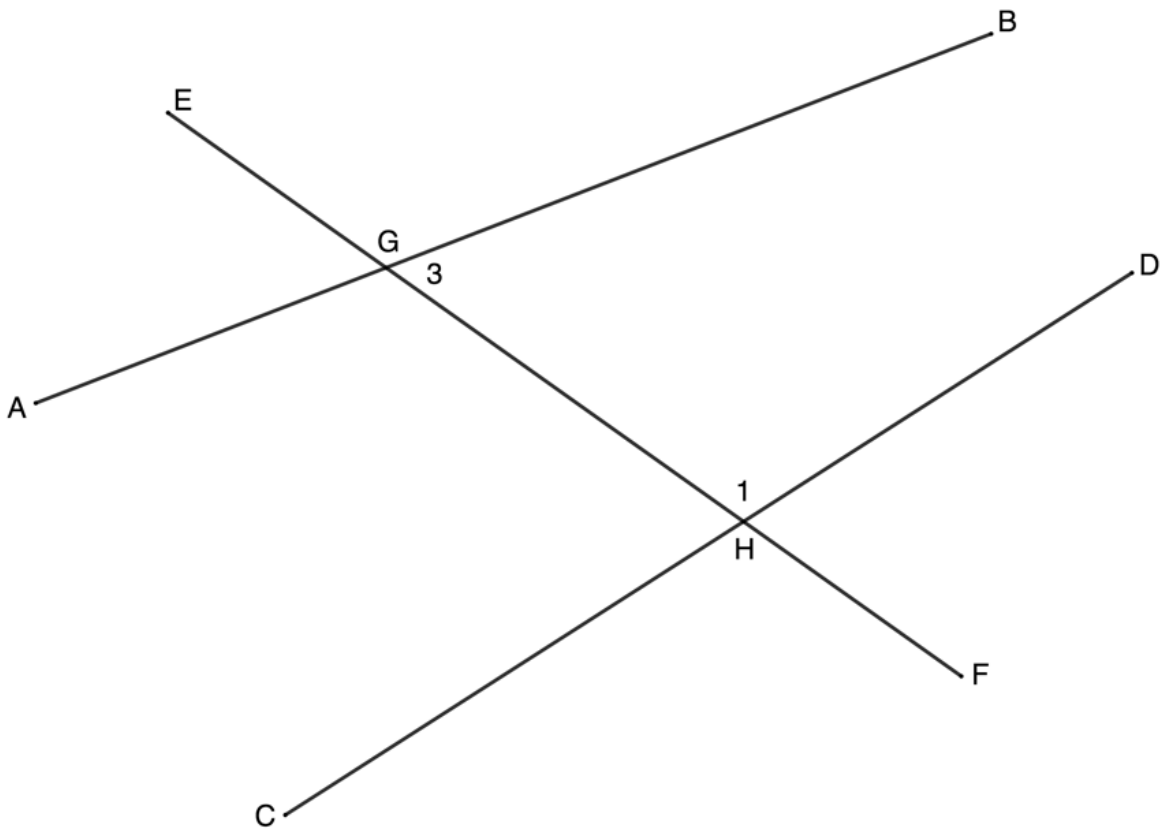


Figure 7: Co-interior angles

When those two (or more) lines cut by the transversal are parallel, these three angle pairs have special properties. In figure 8, $AB \parallel CD$ and so:

- the corresponding angles are **equal** ($\hat{G}_1 = \hat{H}_1$)
- the alternate angles are **equal** ($\hat{G}_2 = \hat{H}_1$)
- the co-interior angles are **supplementary** ($\hat{G}_3 + \hat{H}_1 = 180^\circ$).

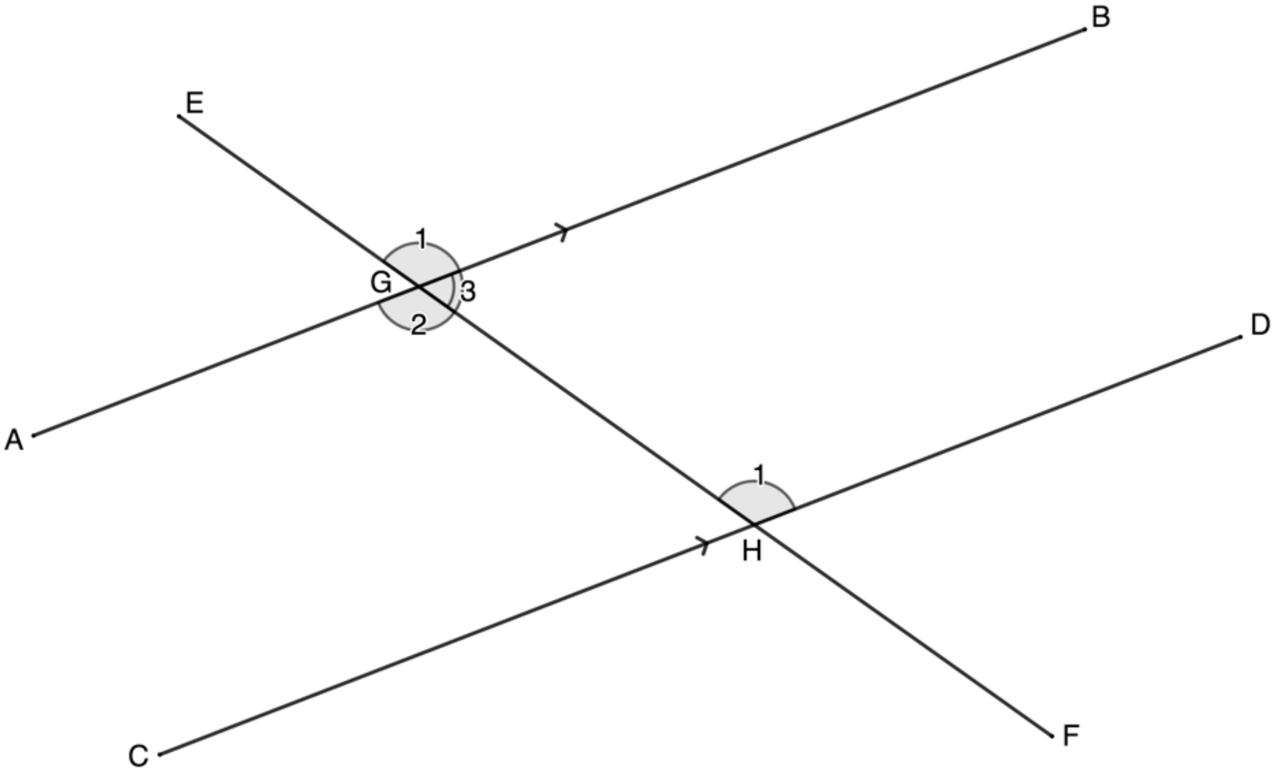


Figure 8: Properties of angles when parallel lines are cut by a transversal

Note in figure 8 that we know that $\hat{G}_1 = \hat{G}_2$ because these angles are also vertically opposite. Also note that $\hat{G}_1 + \hat{G}_3 = 180°$ because they are angles on the same straight line (EF). Therefore, it makes sense that if $\hat{G}_1 + \hat{G}_3 = 180°$ and $\hat{G}_1 = \hat{H}_1$, then $\hat{G}_3 + \hat{H}_1 = 180°$.

Take note!

When two lines are cut by a transversal, corresponding, alternate and co-interior angles are always created.

If lines being cut by the transversal are **parallel**, *then* the corresponding angles are equal, the alternate angles are equal and the co-interior angles are supplementary.

By the same logic, *if* the corresponding angles are **equal**, or the alternate angles are **equal** or the co-interior angles are **supplementary**, *then* the lines are parallel.

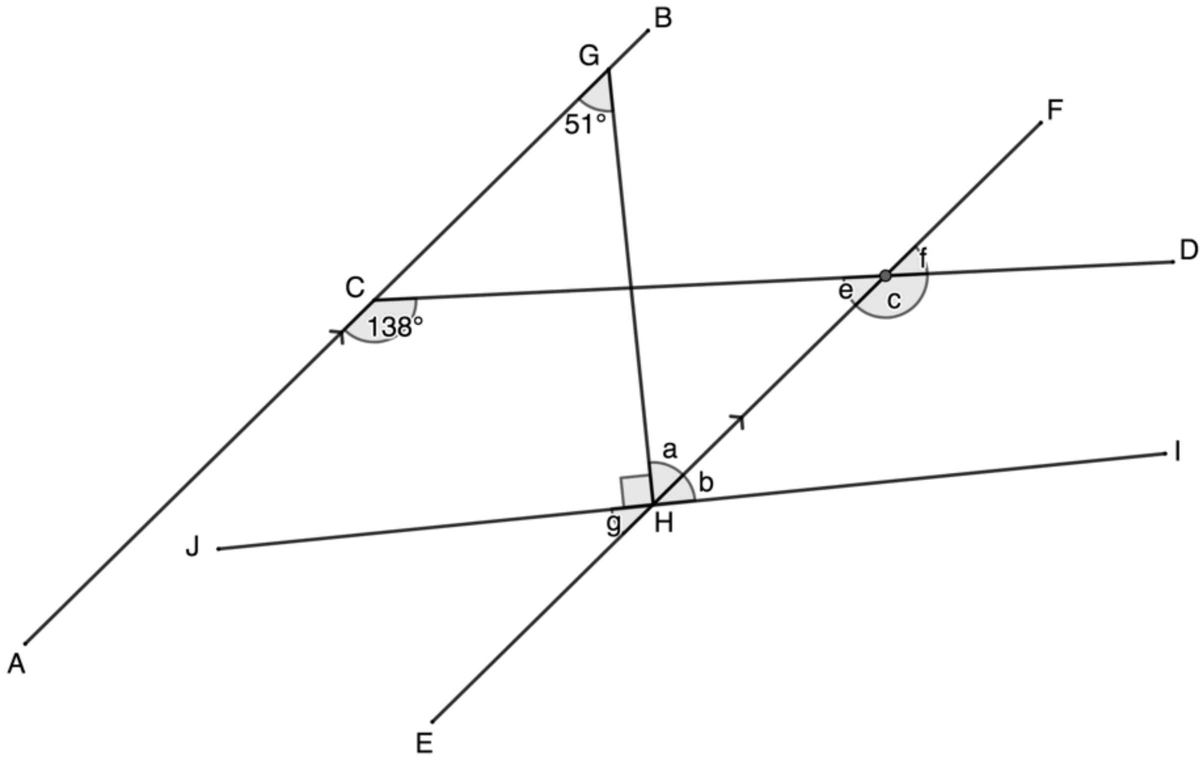
Note

If you would like an excellent summary of the angles created by parallel lines, watch the video called "Angles In Parallel Lines".

[Angles In Parallel Lines](#) (Duration: 2.55)

Example 1.1

Find all the unknown angles if $AB \parallel EF$.



Solution

The most important thing when answering Euclidean geometry questions is to provide reasons for each statement you make. Remember, you are constructing a logical argument. Each step in your argument needs to have a logical reason so that the entire argument is true. If just one step is false, the entire argument falls down. Pay special attention to how the reasons for each assertion made below are provided.

We have been told that $AB \parallel EF$. Therefore, we need to look for corresponding, alternate and co-interior angles.

$\hat{a} = 51^\circ \quad \text{alt angles are equal; } AB \parallel EF$

You must state which parallel lines make the alternate angles equal.

$$G\hat{H}J + \hat{a} + \hat{b} = 180^\circ \quad \text{angles on a str line suppl}$$
$$\text{But } G\hat{H}J = 90^\circ \quad \text{given}$$
$$\therefore \hat{a} + \hat{b} = 90^\circ$$
$$\therefore \hat{b} = 90^\circ - 51^\circ = 39^\circ$$

$$A\hat{C}D + \hat{e} = 180^\circ \quad \text{co-int angles are supplementary; } AB \parallel EF$$
$$\therefore \hat{e} = 180^\circ - 138^\circ = 42^\circ$$

$$ACD = \hat{c} \quad \text{corresp angles } =; AB \parallel EF$$
$$\therefore \hat{c} = 138^\circ$$

$$\hat{e} = \hat{f} \quad \text{vert opp angles } =$$
$$\therefore \hat{f} = 42^\circ$$

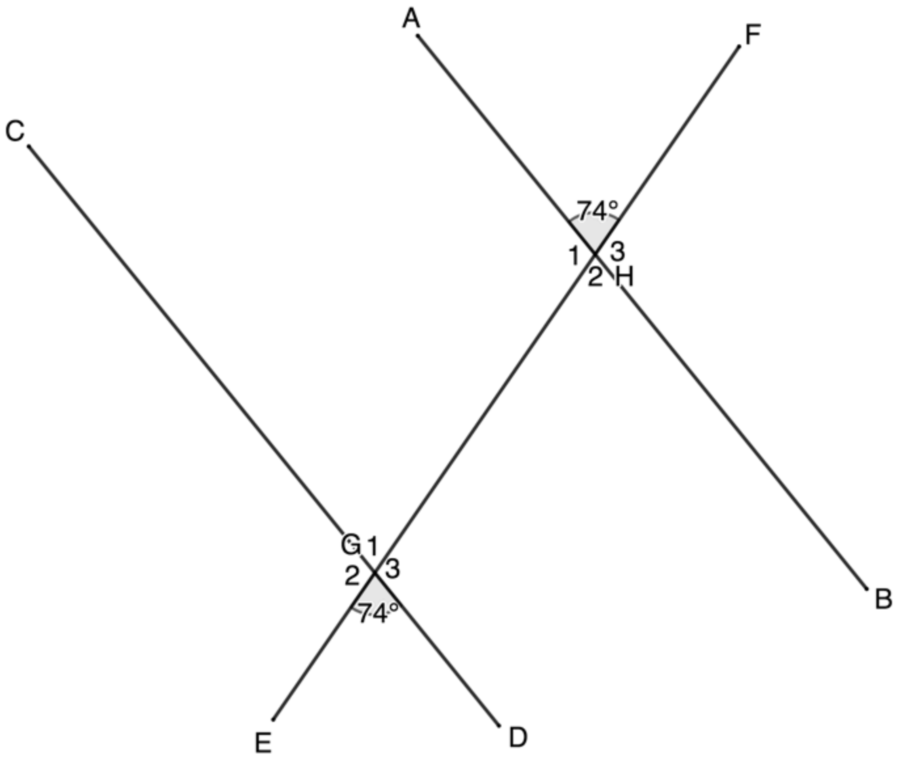
$$\hat{b} = \hat{g} \quad \text{vert opp angles } =$$
$$\therefore \hat{g} = 39^\circ$$

Note: You can use abbreviated words (like 'alt' for alternate) or the full word.

Note: Be careful. $\hat{b}$ and $\hat{c}$ are co-interior angles but CD is not parallel to JI, therefore these co-interior angles are not supplementary.

Example 1.2

Determine if AB is parallel to CD.



Solution

In this case, we need to prove that the lines are parallel by showing that the corresponding angles are equal, or the alternate angles are equal, or the co-interior angles are supplementary.

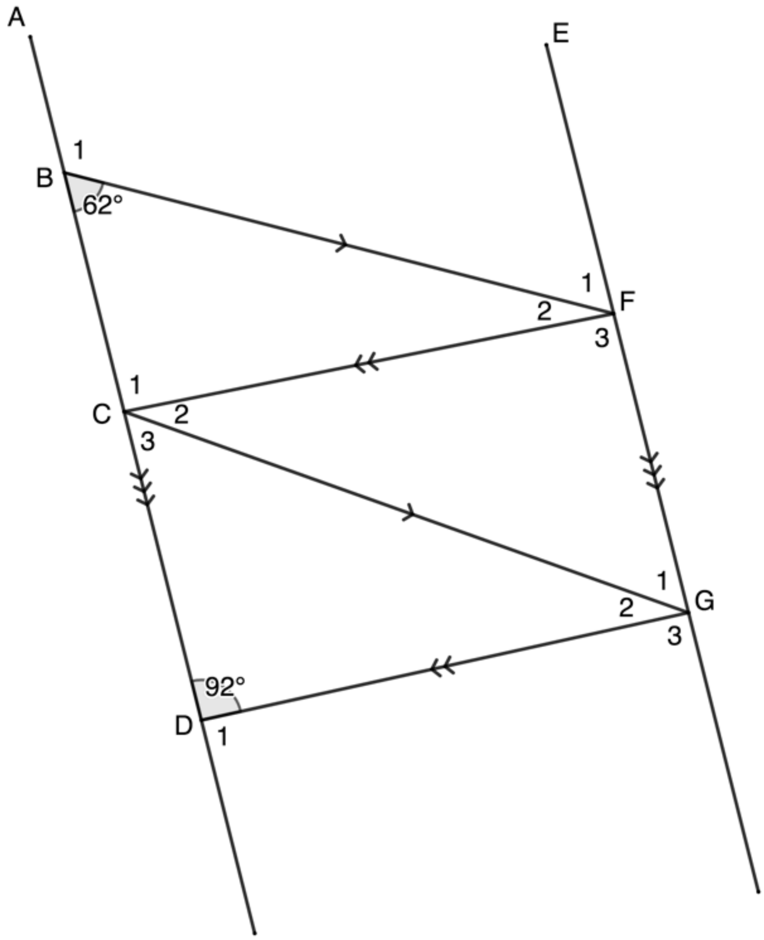
$$\hat{G}_1 = 74° \quad \text{vert opp angles} =$$
$$\therefore \hat{G}_1 = A\hat{H}F$$
$$\therefore AB \parallel CD \quad \text{corresp angles } =$$

Note: This is just one way to prove that $AB \parallel CD$. We could also have shown that, for example, the alternate angles $\hat{G}_1$ and $\hat{H}_2$ are equal or that the co-interior angles $\hat{G}_1$ and $\hat{H}_1$ are supplementary.

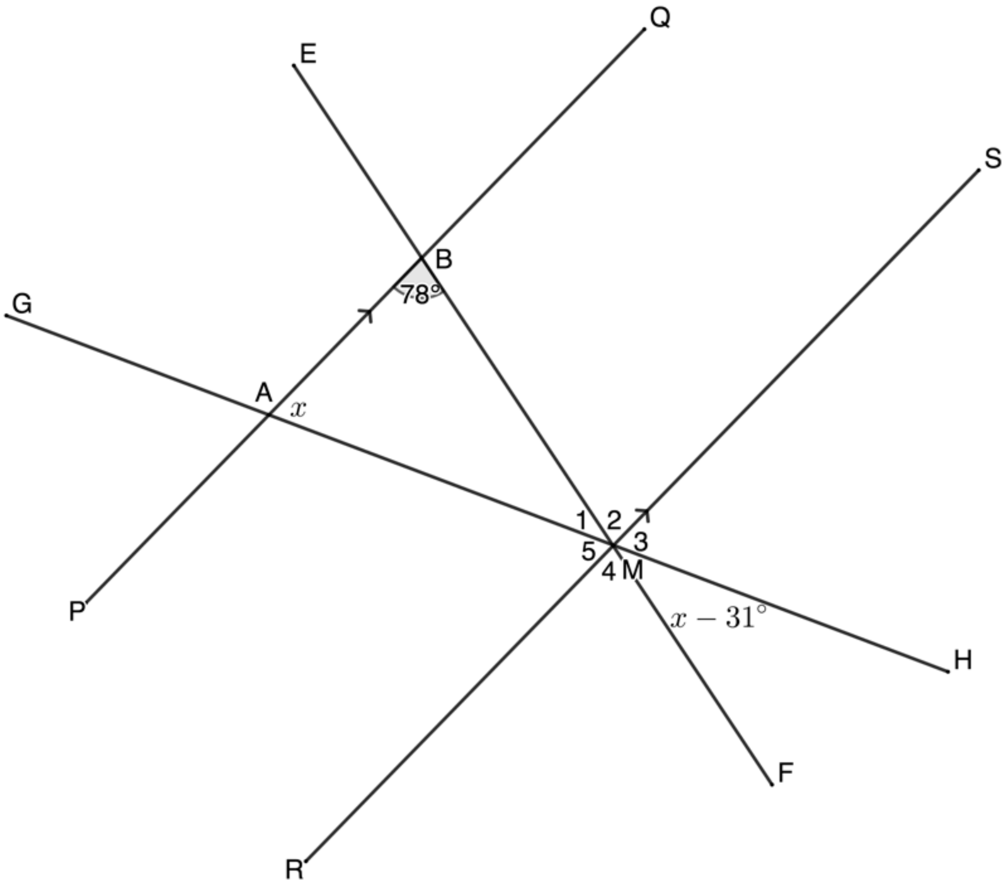
Exercise 1.1

Question 1 adapted from Everything Maths Grade 10 Exercise 7 – 1 question 2

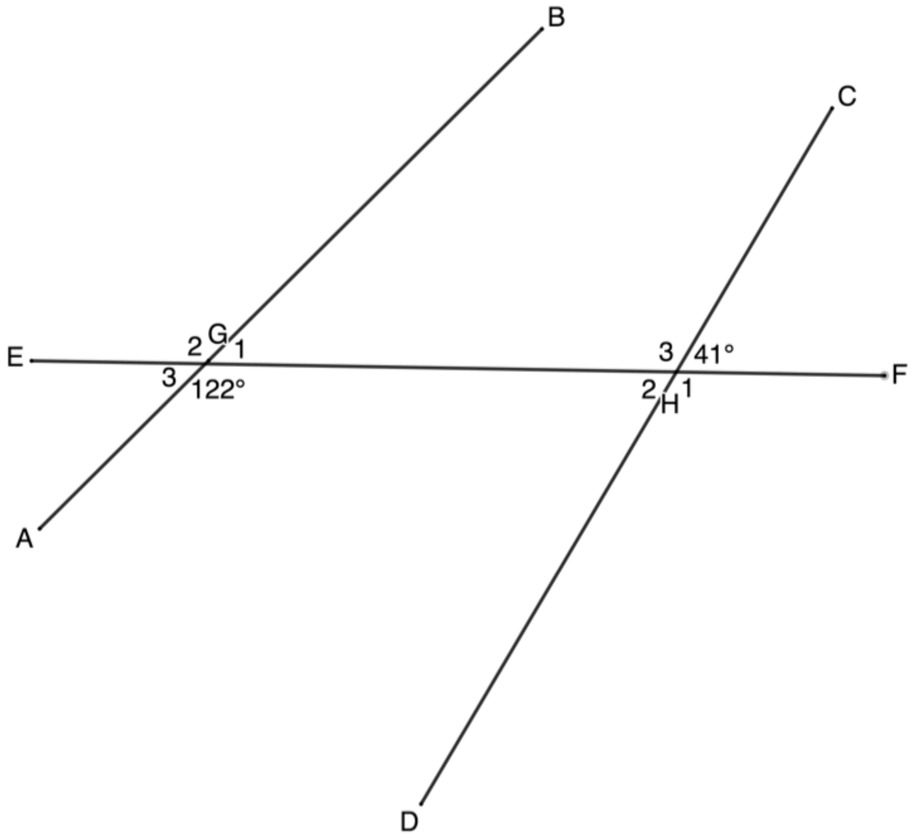
1. Find all the unknown angles in the figure below:



2. Determine the value of x in the following diagram:



3. Determine if AB and CD are parallel:



The full solutions are at the end of the unit.

Angles in triangles

By this stage, you should be very familiar with triangles. There are four types of triangles. These and their characteristics are summarised in figure 9.

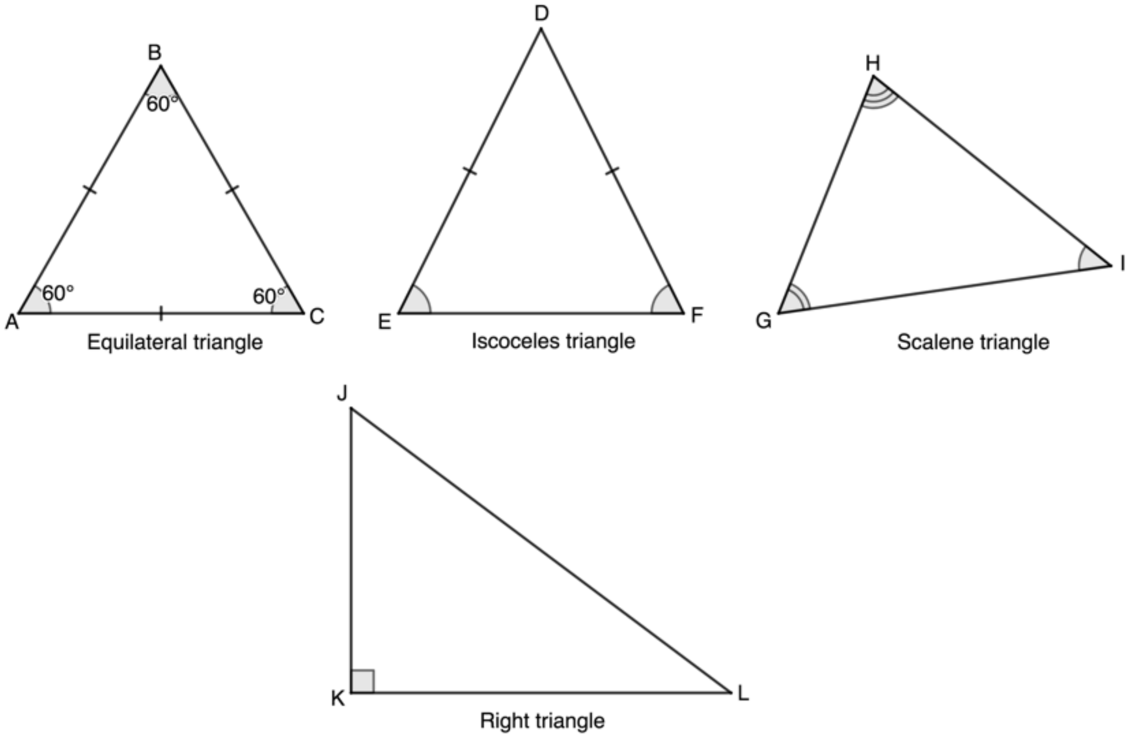


Figure 9: The four types of triangles

We know that the sum of the interior angles of a triangle is $180°$. In other words, in figure 10 $\hat{A}+\hat{B}+\hat{C}=180°$. The interior angles of a triangle are supplementary.

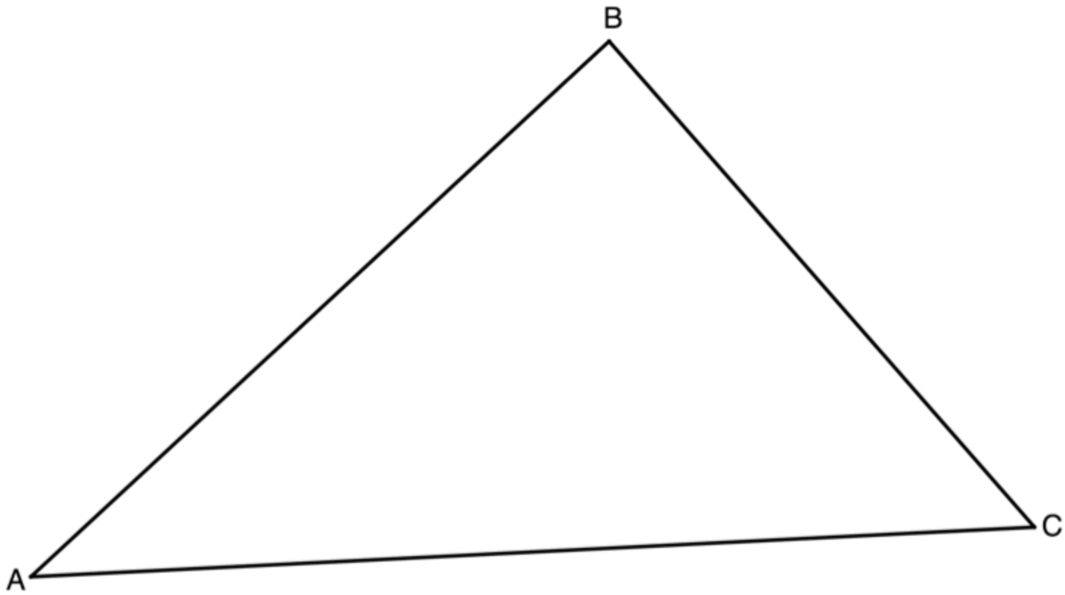


Figure 10: The interior angles of a triangle are supplementary

This is why all the angles in an equilateral triangle are $60°$ ($\frac{180°}{3}=60°$).

But we also know that the angles on a straight line are supplementary. Therefore, what can we say about angles $\hat{A}$, $\hat{B}$ and $\hat{C}_2$ in figure 11?

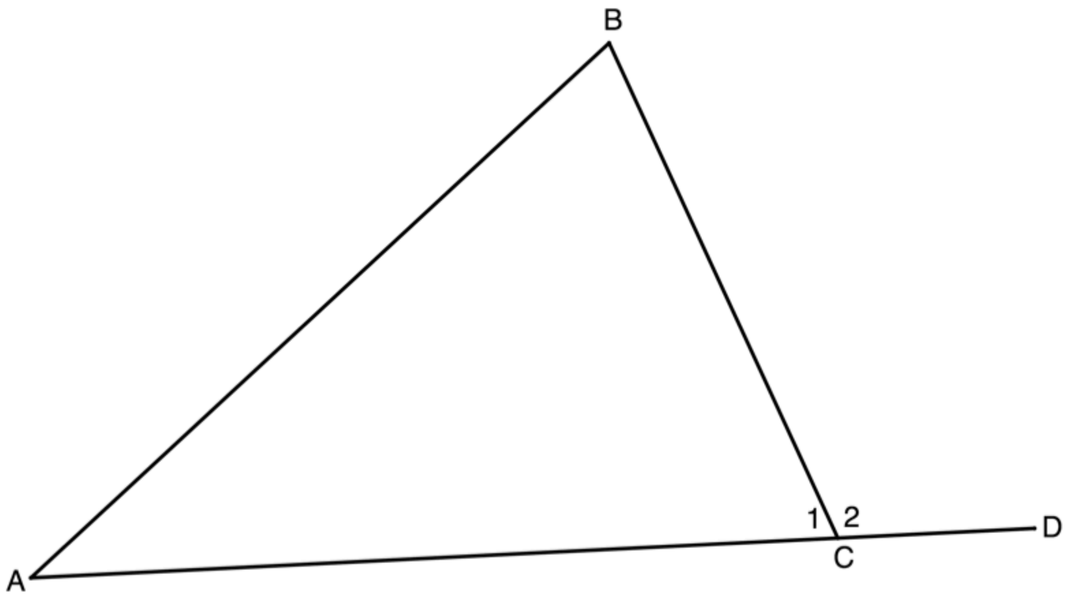


Figure 11: The exterior angle of a triangle is equal to the opposite interior angles

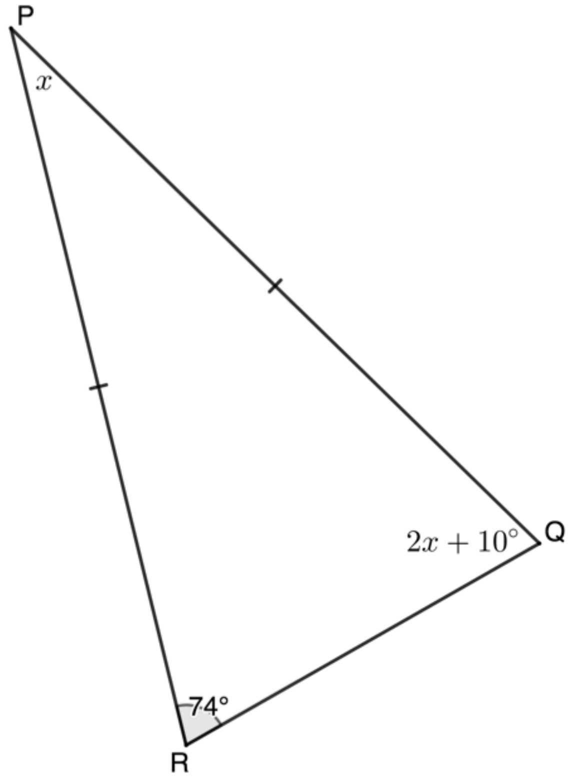
$$\begin{aligned} \hat{A} + \hat{B} + \hat{C}_1 &= 180^\circ \quad \text{int angles of } \Delta \text{ suppl} \\ \text{But } \hat{C}_1 + \hat{C}_2 &= 180^\circ \quad \text{angles on str line suppl} \\ \therefore \hat{C}_2 &= \hat{A} + \hat{B} \end{aligned}$$

We say that the exterior angle of a triangle (the angle made by extending any of the sides of the triangle) is equal to the sum of the two opposite interior angles (ext angle of $\Delta =$ opp int angles).

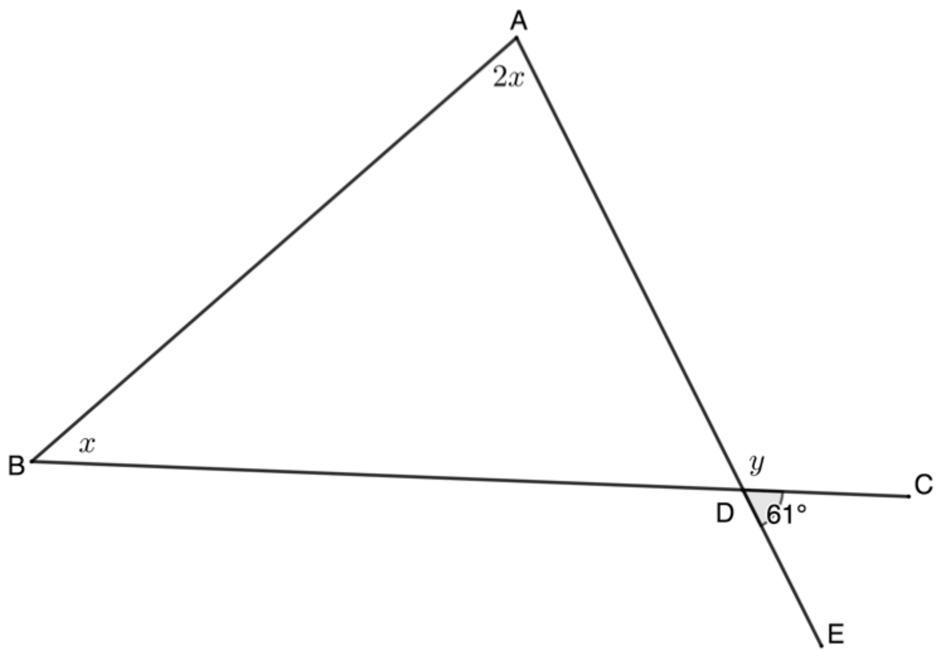
Exercise 1.2

Calculate the unknown variables in each of the following figures:

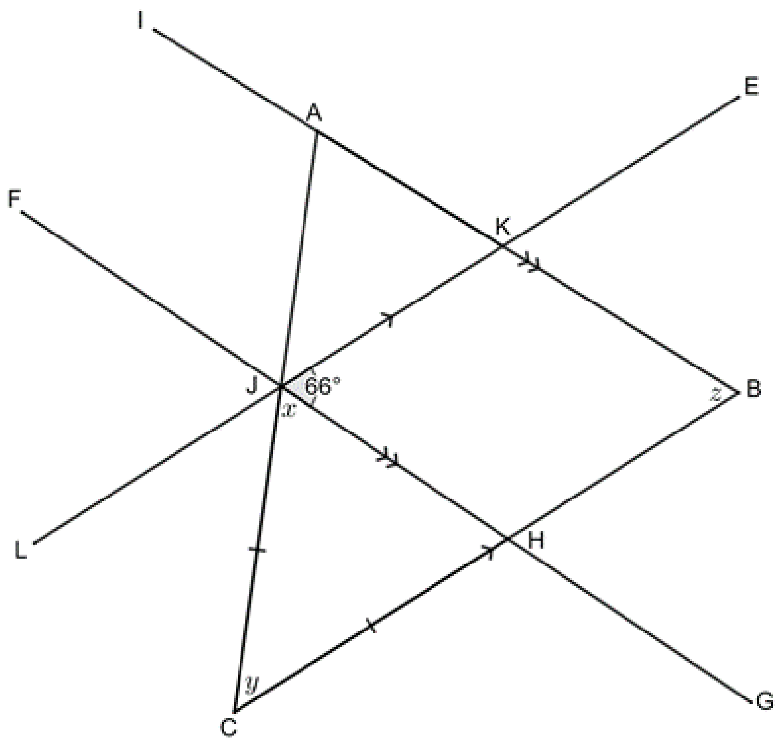
1.



2.



3.



The full solutions are at the end of the unit.

Summary

In this unit you have learnt the following:

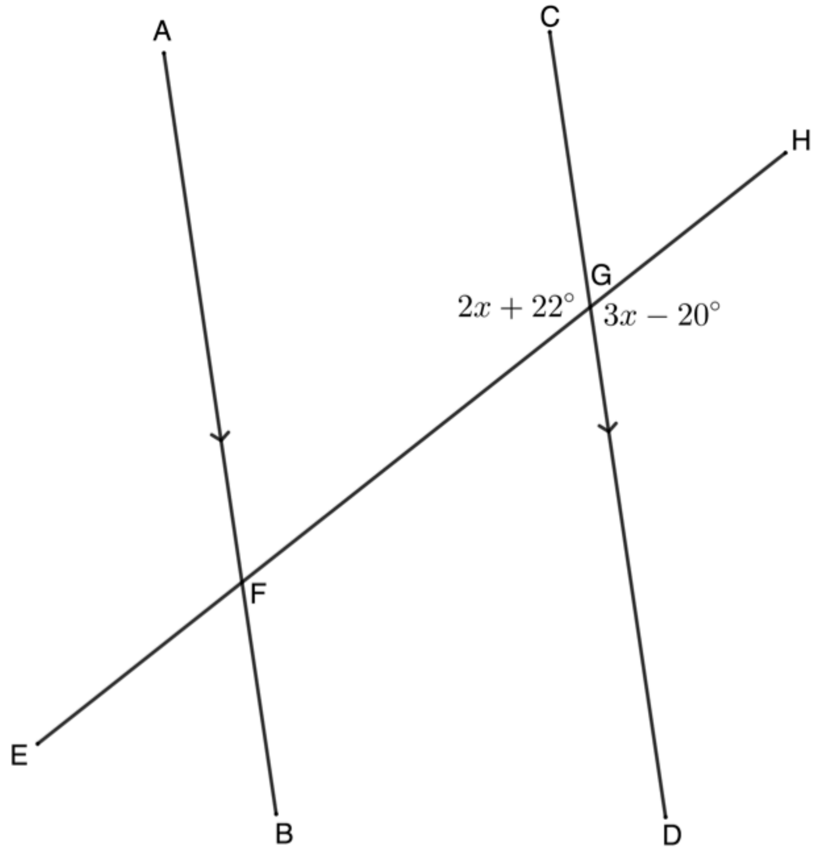
- The different kinds of angles are acute, right, obtuse, straight, and reflex.
- Complementary angles add up to 90°.
- Supplementary angles add up to 180°.
- That when lines cut by a transversal are parallel, the alternate angles are equal, the corresponding angles are equal and the co-interior angles are supplementary.
- The different kinds of triangles are equilateral, isosceles, scalene and right.
- That the interior angles of a triangle add up to 180°.
- That the exterior angle of a triangle is equal to the sum of the opposite interior angles.

Unit 1: Assessment

Suggested time to complete: 25 minutes

Question 1 adapted from NC(V) Mathematics Level 4 Paper 2 November 2016 question 1.1

1. In the diagram given below, $AB \parallel CD$ and $EFGH$ is a transversal with F on AB and G on CD. $F\hat{G}C = 2x + 22^\circ$ and $H\hat{G}D = 3x - 20^\circ$.

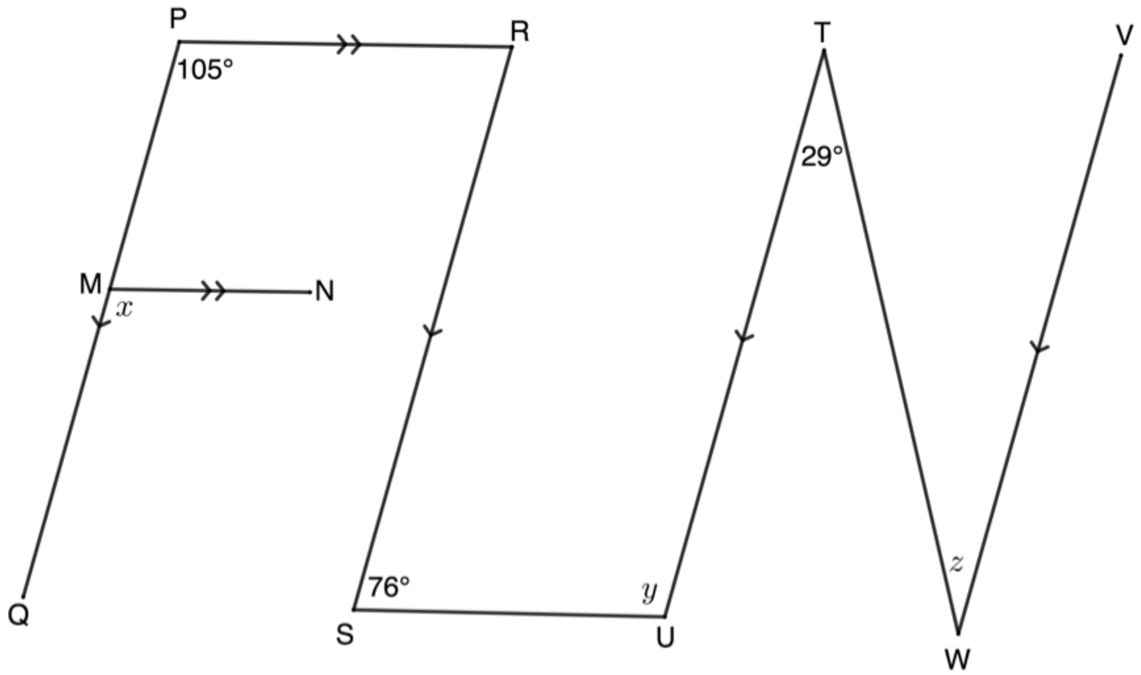


a. Determine the value of x.

b. Determine the magnitude of $E\hat{F}B$.

Question 2 adapted from NC(V) Mathematics Level 4 Paper 2 November 2015 question 1.1

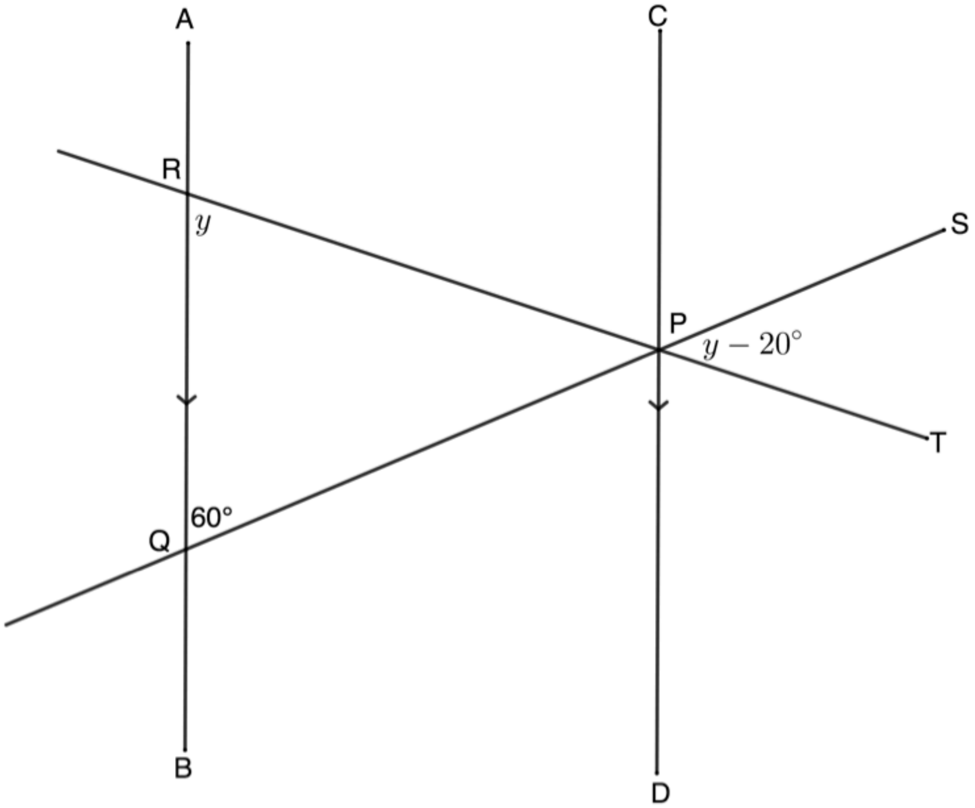
2. In the diagram below $PQ \parallel RS \parallel TU \parallel VW$. PR, SU and TW are drawn and $MN \parallel PR$. Also, $\hat{P} = 105°$, $\hat{S} = 76°$ and $\hat{T} = 29°$.



Determine, with reasons, the sizes of x, y and z.

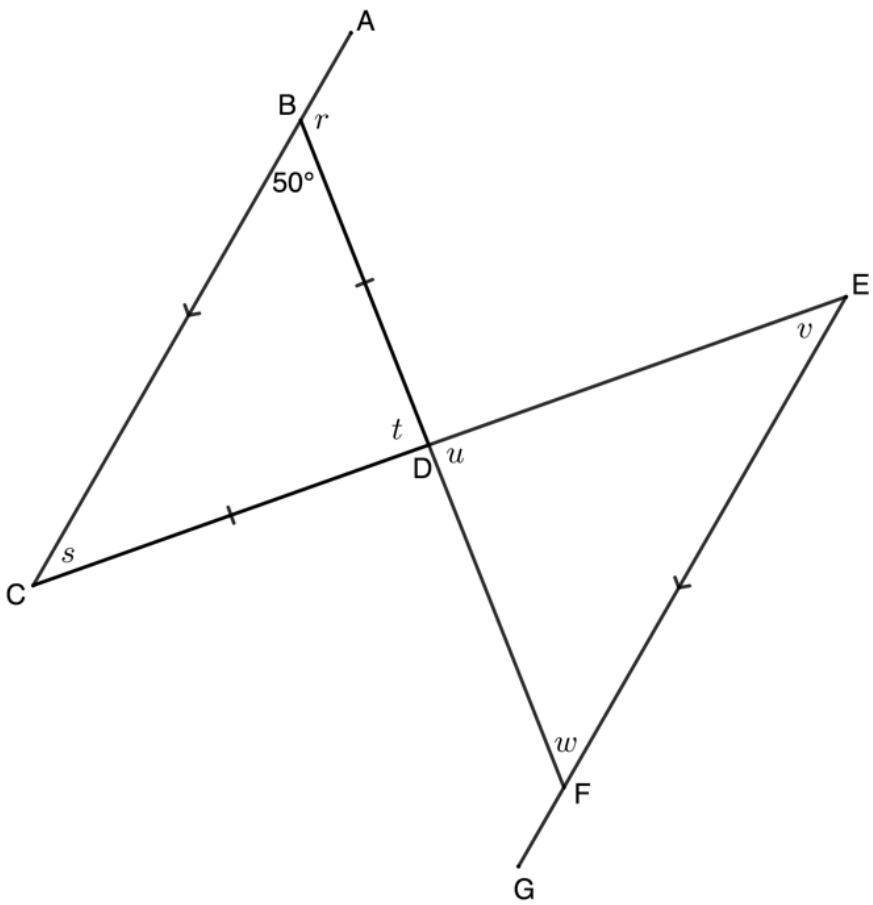
Question 3 adapted from NC(V) Mathematics Level 4 Paper 2 November 2014 question 1.2

3. In the diagram below, parallel lines AB and CD are cut by two transversals SPQ and TPR. Both transversals intersect CD at P. If $P\hat{R}Q = y$, $P\hat{Q}R = 60^\circ$ and $S\hat{P}T = y - 20^\circ$, determine the value of y.



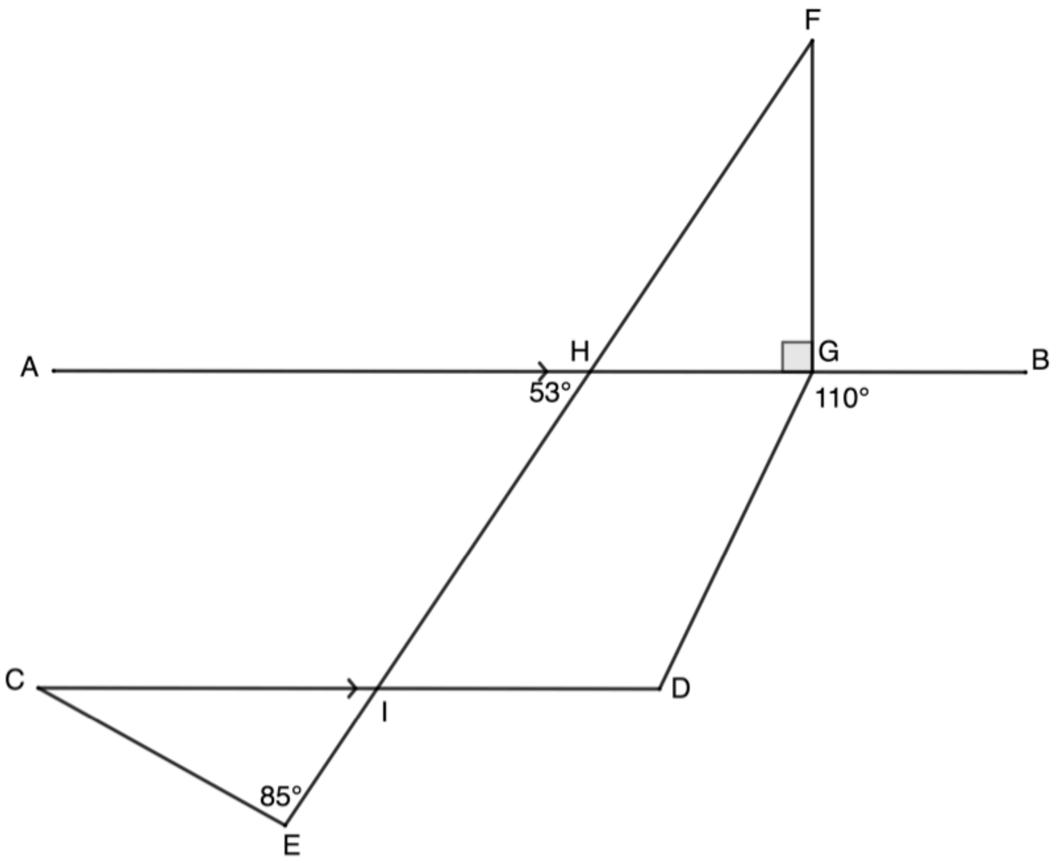
Question 4 adapted from NC(V) Mathematics Level 4 Paper 2 November 2013 question 1.1

4. In the figure below, $AC \parallel EG$, $BD = CD$ and $C\hat{B}D = 50^\circ$.



Determine, giving reasons, the value of r, s, t, u, v and w.

5. In the diagram, $AB \parallel CD$. Calculate the sizes of $F\hat{H}G$, $\hat{F}$, $\hat{C}$ and $\hat{D}$. Give reasons for your answers.



The full solutions are at the end of the unit.

Unit 1: Solutions

Exercise 1.1

1.

$$\hat{D}_1 = 180^\circ - 92^\circ = 88^\circ \quad \text{angles on a str line suppl}$$
$$\hat{B}_1 = 180^\circ - 62^\circ = 118^\circ \quad \text{angles on a str line suppl}$$
$$\hat{F}_1 = 62^\circ \quad \text{alt angles } =; AD \parallel EG$$
$$\hat{F}_1 = \hat{G}_1 \quad \text{corresp angles } =; BF \parallel CG$$
$$\therefore \hat{G}_1 = 62^\circ$$
$$\hat{C}_1 = 92^\circ \quad \text{corresp angles } =; CF \parallel DG$$
$$\hat{F}_3 = \hat{C}_1 \quad \text{alt angles } =; AD \parallel EG$$
$$\therefore \hat{F}_3 = 92^\circ$$
$$\hat{C}_1 + \hat{F}_1 + \hat{F}_2 = 180^\circ \quad \text{co-int angles suppl; } AD \parallel EG$$
$$\therefore \hat{F}_2 = 180^\circ - 62^\circ - 92^\circ = 26^\circ$$
$$\hat{C}_2 = \hat{F}_2 \quad \text{alt angles } =; BF \parallel CG$$
$$\therefore \hat{C}_2 = 26^\circ$$
$$\hat{G}_2 = \hat{C}_2 \quad \text{alt angles } =; BF \parallel CG$$
$$\therefore \hat{G}_2 = 26^\circ$$
$$\hat{C}_3 = C\hat{B}F = 62^\circ \quad \text{corresp angles } =; BF \parallel CG$$

$$\hat{G}_3 = 92^\circ \quad \text{alt angles } =; AD \parallel EG$$

2.

$$\begin{aligned} \hat{M}_5 &= x \quad \text{alt angles } =; PQ \parallel RS \\ \hat{M}_2 &= 78^\circ \quad \text{alt angles } =; PQ \parallel RS \\ \hat{M}_1 &= x - 31^\circ \quad \text{vert opp angles } = \\ \hat{M}_1 + \hat{M}_2 + \hat{M}_5 &= 180^\circ \quad \text{angles on str line suppl} \\ \therefore x - 31^\circ + 78^\circ + x &= 180^\circ \\ \therefore 2x &= 133^\circ \\ \therefore x &= 66.5^\circ \end{aligned}$$

3.

Note: You cannot make a conclusion based on what the diagram looks like. You must work with the angles to see if they provide a condition for the lines being parallel or not.

$$\begin{aligned} \hat{G}_1 + 122^\circ &= 180^\circ \quad \text{angles on str line suppl} \\ \therefore \hat{G}_1 &= 180^\circ - 122^\circ = 58^\circ \\ \therefore \hat{G}_1 &\neq 41^\circ \end{aligned}$$

$$\therefore \text{corresp angles not equal}$$

$$\therefore AB \text{ not parallel to } CD$$

Back to Exercise 1.1

Exercise 1.2

1.

$$\begin{aligned} 2x + 10^\circ &= 74^\circ \quad = \text{ sides } = \text{ angles} \\ \therefore 2x &= 64^\circ \\ \therefore x &= 32^\circ \end{aligned}$$

2.

$$\begin{aligned} y &= 180^\circ - 61^\circ \quad \text{angles on str line suppl} \\ \therefore y &= 119^\circ \\ x + 2x = y &= 119^\circ \quad \text{ext angle of } \Delta = \text{ opp int angles} \\ \therefore 3x &= 119^\circ \\ \therefore x &= 39.67^\circ \end{aligned}$$

3.

$$\begin{aligned} x + C\hat{H}J + y &= 180^\circ \quad \text{angles in } \Delta \text{ suppl} \\ \therefore y &= 180^\circ - 2 \times 66^\circ \\ \therefore y &= 48^\circ \end{aligned}$$

$$\begin{aligned} F\hat{J}A = x &= 66^\circ \quad \text{vert opp angles } = \\ J\hat{A}K &= 66^\circ \quad \text{alt angles } =; FG \parallel HB \\ J\hat{A}K + z + y &= 180^\circ \quad \text{angles in } \Delta\text{ABC suppl} \\ \therefore z &= 180^\circ - 66^\circ - 48^\circ = 66^\circ \end{aligned}$$

Back to Exercise 1.2

Unit 1: Assessment

1.

a.

$$C\hat{G}F = H\hat{G}D \quad \text{vert opp angles } =$$
$$\therefore 2x + 22^\circ = 3x - 20^\circ$$
$$\therefore x = 42^\circ$$

b.

$$CGF + AFG = 180^\circ \qquad \text{co-int angles suppl; } AB \parallel CD$$
$$\therefore AFG = 180^\circ - (2 \times 42^\circ + 22^\circ)$$
$$\therefore AFG = 180^\circ - 106^\circ = 74^\circ$$
$$\text{But } AFG = EFB \qquad \text{vert opp angles } =$$
$$\therefore EFB = 74^\circ$$

2. $x = 105^\circ \quad \text{corresp angles } =; PR \parallel MN$
$y = 180^\circ - 76^\circ = 104^\circ \quad \text{co-int angles suppl}; RS \parallel TU$
$z = 29^\circ \quad \text{alt angles } =; TU \parallel VW$

3.

$$Q\hat{P}R = y - 20^\circ \quad \text{vert opp angles } =$$
$$Q\hat{P}R + P\hat{R}Q + R\hat{Q}P = 180^\circ \quad \text{angles in } \Delta \text{ suppl}$$
$$\therefore y - 20^\circ + y + 60^\circ = 180^\circ$$
$$\therefore 2y = 140^\circ$$
$$\therefore y = 70^\circ$$

4. $r = 180^\circ - 50^\circ = 130^\circ \quad \text{angles in str line suppl}$
$s = 50^\circ \quad \text{equal sides equal angles}$
$t + s + 50^\circ = 180^\circ \quad \text{angles in } \Delta \text{ suppl}$
$\therefore t = 180^\circ - 50^\circ - 50^\circ = 80^\circ$
OR
$r = s + t \quad \text{ext angle of } \Delta \ = \text{ opp int angles}$
$\therefore t = 130^\circ - 50^\circ = 80^\circ$
$u = 80^\circ \quad \text{vert opp angles } =$
$v = s \quad \text{alt angles } =; AC \parallel EG$
$\therefore v = 50^\circ$
$w = C\hat{B}D \quad \text{alt angles } =; AC \parallel EG$
$\therefore w = 50^\circ$

5. $F\hat{H}G = 53^\circ \quad \text{vert opp angles } =$
$\hat{F} + F\hat{H}G + F\hat{G}H = 180^\circ \quad \text{angles in } \Delta \text{ suppl}$
$\therefore \hat{F} = 180^\circ - 53^\circ - 90^\circ = 37^\circ$
$H\hat{I}D = 53^\circ \quad \text{alt angles } =; AB \parallel CD$
$\therefore C\hat{I}E = 53^\circ \quad \text{vert opp angles } =$
$\therefore \hat{C} = 180^\circ - 85^\circ - 53^\circ \quad \text{angles in } \Delta \text{ suppl}$
$\therefore \hat{C} = 42^\circ$
$D = 110^\circ \quad \text{alt angles } =; AB \parallel CD$

Back to Unit 1: Assessment

Media Attributions

Unit 2: Circle geometry theorems

DYLAN BUSA

Unit outcomes

By the end of this unit you will be able to:

- Apply the following theorems relating to circles:
    - Line drawn perpendicular to a chord from the centre of the circle bisects the chord, and its converse (line drawn from the circle centre to the mid-point of chord is perpendicular to the chord).
    - Angle at the centre of a circle is twice the size of the angle at the circumference.
    - Diameter of a circle subtends a right angle at the circumference, and its converse (if an angle subtended by a chord at a point on the circumference is a right angle, then the chord is a diameter).
    - Angles in the same segment of a circle are equal.

What you should know

Before you start this unit, make sure you can:

- Describe the characteristics of the different types of triangles.
- Work with vertically opposite angles and angles on a straight line.
- Work with the alternate, corresponding, and co-interior angles between parallel lines.

Refer to unit 1 of this subject outcome if you need any help with any of these.

Introduction

In the previous unit we learnt about and used the properties of angles on straight lines and in triangles. We noted that studying geometry in general, and circle geometry in particular, is an excellent way to help us develop and hone our logic, reasoning and argumentation skills. Being able to reason logically and express our reasons and arguments clearly and concisely is an important and valuable life skill.

In that unit, we used the basic 'facts' of angles on straight lines and in triangles to construct arguments to prove relationships between angles that were not immediately obvious. We used **deductive reasoning**.

These basic 'facts' (such as that angles on a straight line add up to $180°$) are called axioms – statements that are assumed to be true for the purposes of building further arguments. Axioms are sometimes called self-evident truths or **postulates**. The whole of mathematics is based on several crucial axioms that, if ever proven untrue, would undo entire branches of mathematics.

One such axiom is called the **reflexive axiom** and states that a number is equal to itself. In other words, $a = a$. This seems completely obvious (a self-evident truth or an axiom), but it underpins all of mathematics. Another, that we rely upon in many geometry arguments, is the **transitive axiom** (if $a = b$ and $b = c$ then $a = c$).

Did you know?

There are six fundamental axioms for algebra. You can read clear and simple explanations of these axioms of algebra at this link.

Many of these are mirrored in the seven axioms that Euclid stated for geometry. Visit an Introduction to Euclidean Geometry for more details.

Some basics

Sometimes, we use the fundamental axioms to build reusable arguments called **theorems**. These are **non-self-evident statements of truth** that we prove once and then keep using to build other arguments without having to prove them each time. Theorems can also be built upon by other theorems. If ever the underlying axioms (or theorems) were proven untrue, the theorem could no longer be taken to be true. This is why mathematicians spend so much time carefully and meticulously proving that every statement they make is true.

Did you know?

There is a famous theorem called **Fermat's last theorem**. It says that there are **no** solutions to the equation $a^n + b^n = c^n$ for any value of n greater than 2. You should recognise the case when $n = 2$ as the theorem of Pythagoras.

This theorem was assumed to be true for hundreds of years without any formal or official proof. Whole areas of mathematics were built on this theorem that were at risk! Finally, in 1993, a mathematician called Andrew Wiles presented the first proof. It took him almost six years to develop and three days to present!

Unfortunately, there was an error in his proof which took him another year to correct. Finally, in 1995, the final theorem was published in the form of two papers.


« Fermat . veritable invente
« du calcul differentiel »
1812
Xⁿ+Yⁿ≷Zⁿ
Si n est superieur à 2 .

Figure 1: Andrew Wiles in front of a statue of Pierre de Fermat

Watch the excellent video called "Fermat's Last Theorem" if you want to learn more.

[Fermat's Last Theorem](#) (Duration: 09.30)

There are a number of theorems that relate to circles. We will learn about four of them in this unit. In order to understand what they mean, we need to make sure that we know what the different parts of a circle are called. Study figure 2 to learn about the parts of a circle if you don't know them already.

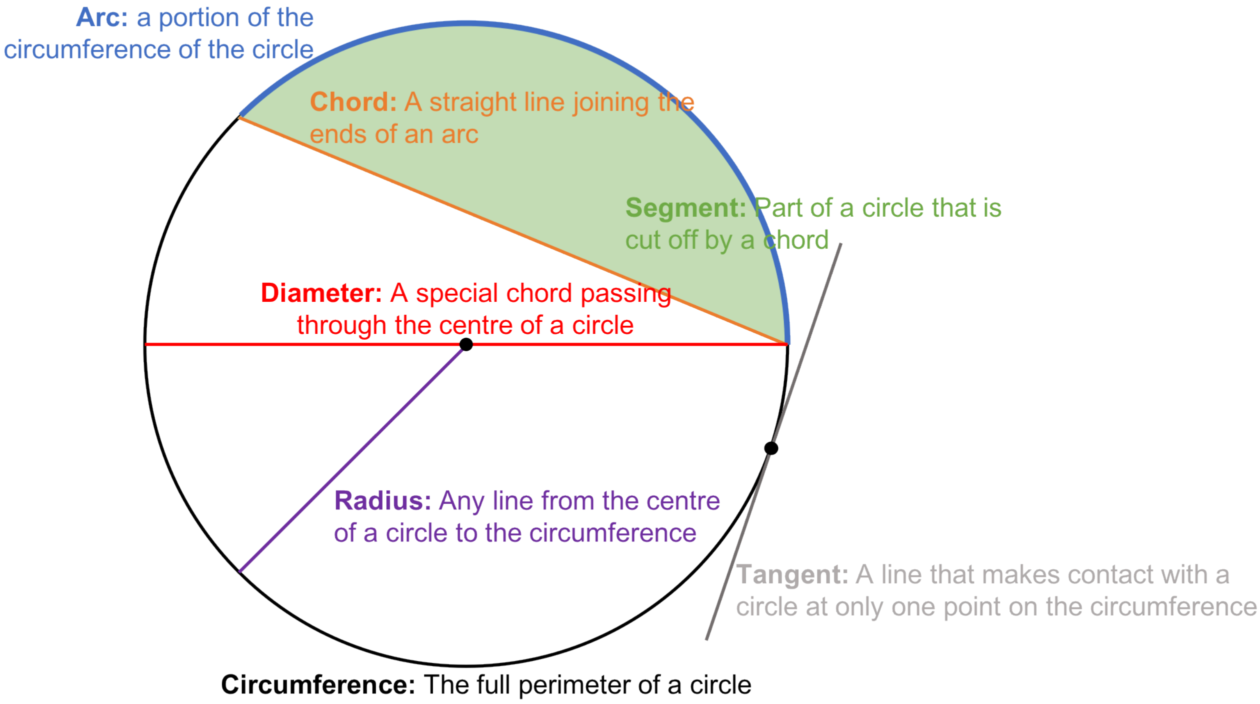


Figure 2: The parts of a circle

Circle theorems

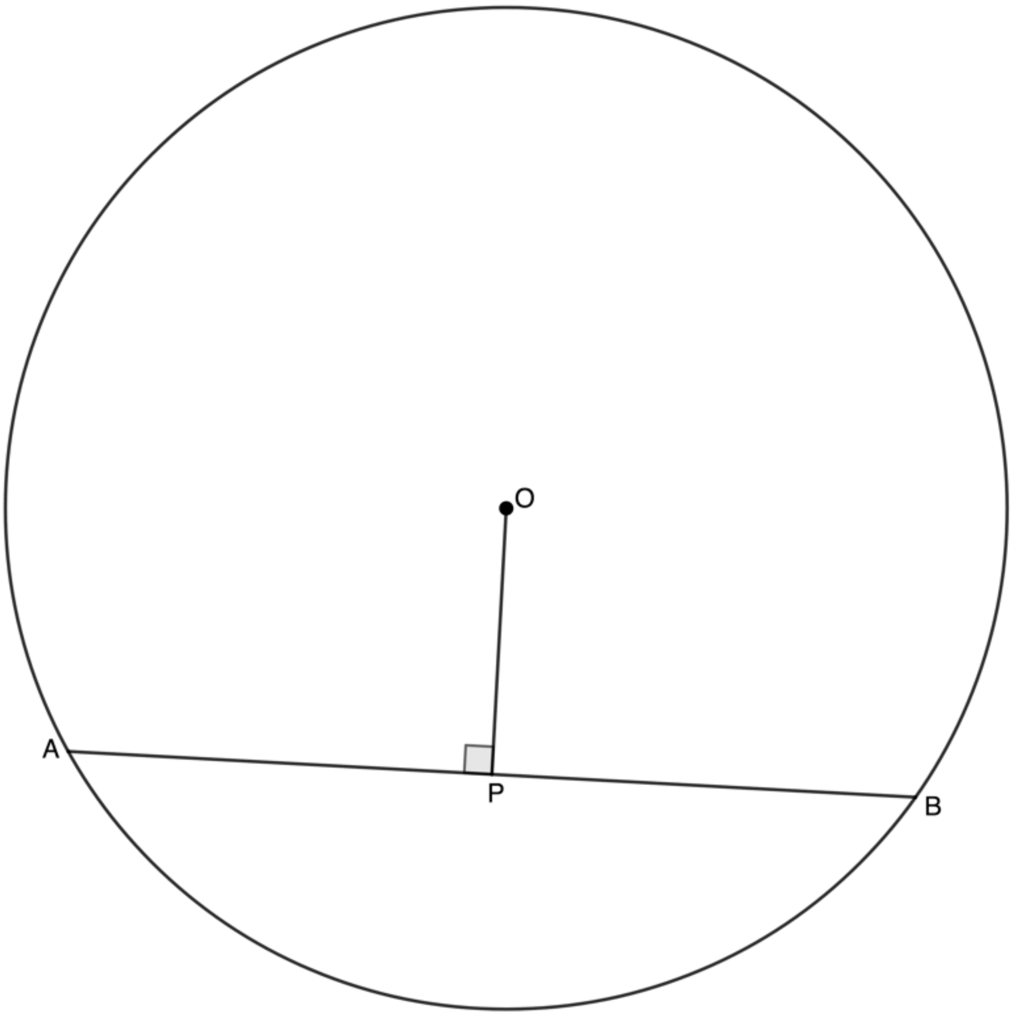
You do not need to be able to prove any of the circle theorems yourself. You can simply assume that they are true. The following sections explain the theorems that you need to be able to state and use. Note that they are numbered only for reference purposes. These theorems do not have official numbers.

Theorem 1

This circle theorem deals with chords.

Theorem 1: Perpendicular line from circle centre bisects chord

If a line is drawn from the centre of a circle perpendicular (at right angles) to a chord, then it bisects the chord.

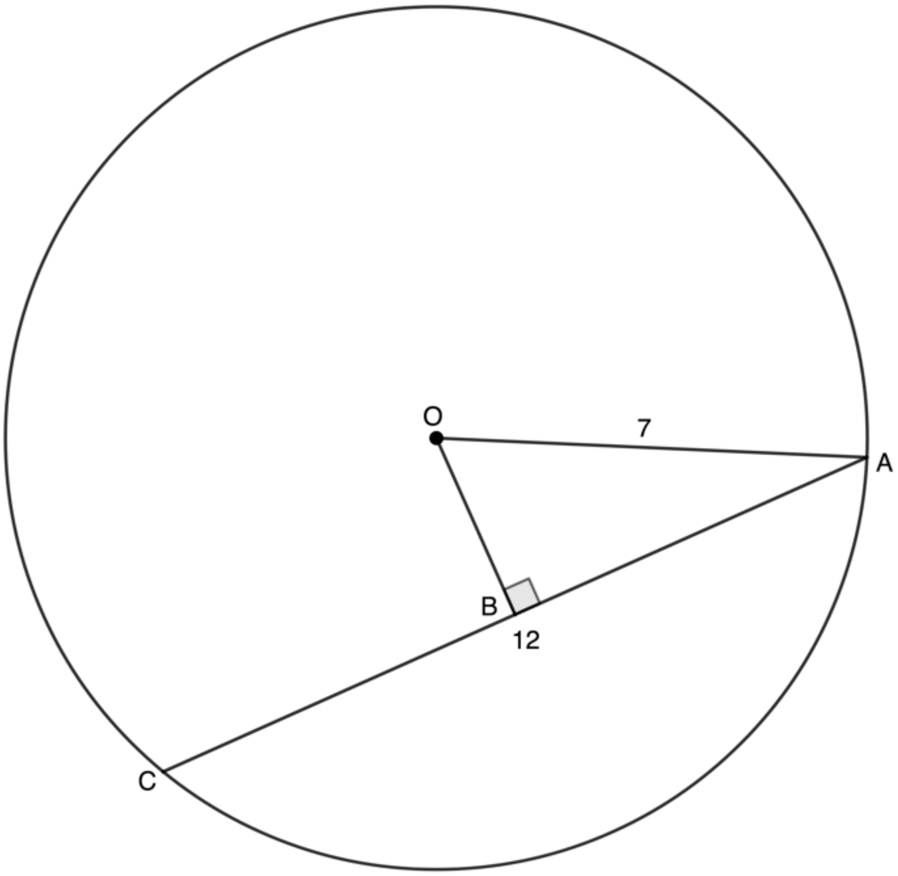


If OP is perpendicular to AB, then OP bisects AB (i.e. $AP = BP$).

Reason: $\perp$ (or perp) from centre bisects chord

Example 2.1

Given $OB \perp AC$, $OA = 7 \text{ units}$ and $AC = 12 \text{ units}$, determine the length of OB.



Solution

We need to recognise that the line from the centre meets the chord AB at right angles and bisects the chord or cuts it in half.

$AB = \frac{1}{2}AC = 6 \text{ units} \quad (\perp \text{ from centre bisects chord})$

In ΔABO:

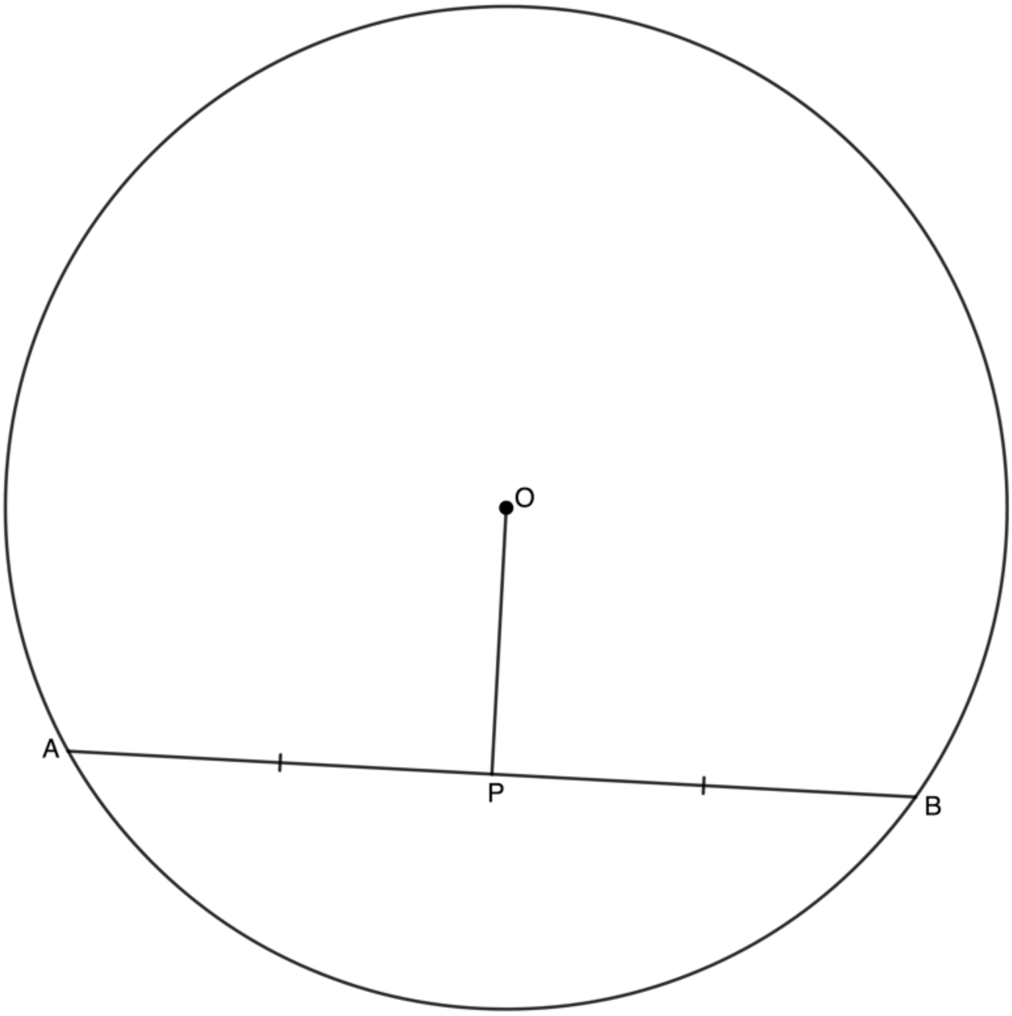
$\hat{B} = 90^\circ \text{ Given}$

$$\begin{aligned} OA^2 &= OB^2 + AB^2 \quad \text{(Pythagoras)} \\ \therefore OB^2 &= OA^2 - AB^2 \\ \therefore OB^2 &= 7^2 - 6^2 \\ &= 49 - 36 \\ &= 13 \\ \therefore OB &= \sqrt{13} \end{aligned}$$

A converse of a theorem can be thought of as the reverse of a theorem. If a theorem states that 'if A is true then B is true' then the converse of the theorem says that 'if B is true then$A$ is true'. Not all theorems have converses that are also true, but many do.

Converse to theorem 1: Line from circle centre to mid-point of chord is perpendicular to the chord

If a line is drawn from the centre of a circle to the mid-point of a chord, then the line is perpendicular to the chord.

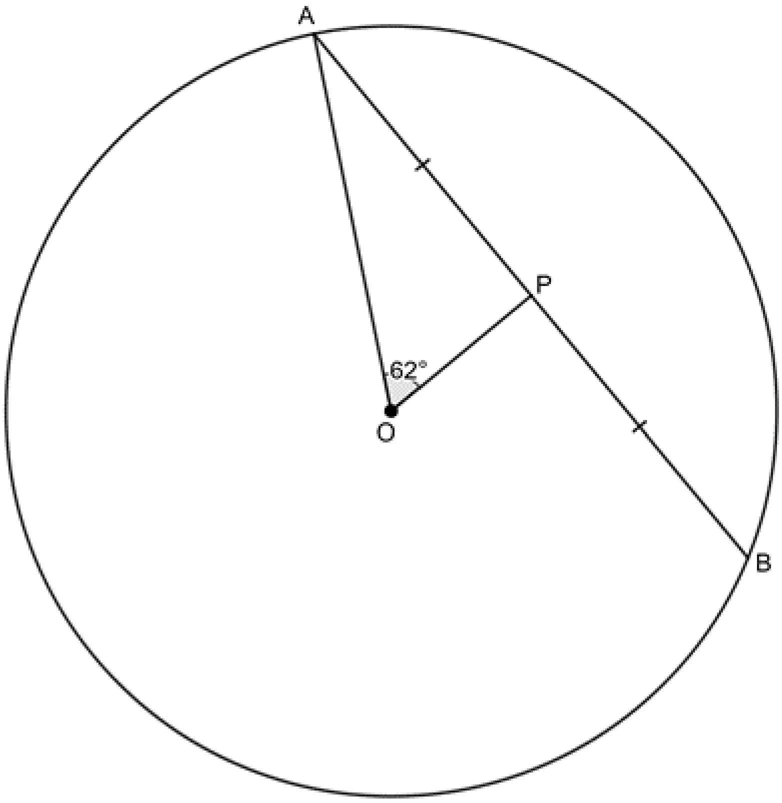


If OP bisects AB (i.e. $AP = BP$), then OP is perpendicular to AB.

Reason: Line from centre to mid-point $\perp$ (or perp)

Example 2.2

Given $AP = BP$ and $P\hat{O}A = 62°$, determine the $O\hat{A}P$.



Solution

We need to recognise that we are dealing with the converse of theorem 1. If a line from the centre bisects a chord, then it meets the chord at right-angles.

$O\hat{P}A = 90^\circ \quad \text{(Line from centre to mid-point } \perp\text{)}$
In ΔAOP:
$O\hat{A}P = 180^\circ - 90^\circ - 62^\circ \quad (\angle\text{s in}\Delta\text{suppl})$
$\therefore O\hat{A}P = 28^\circ$

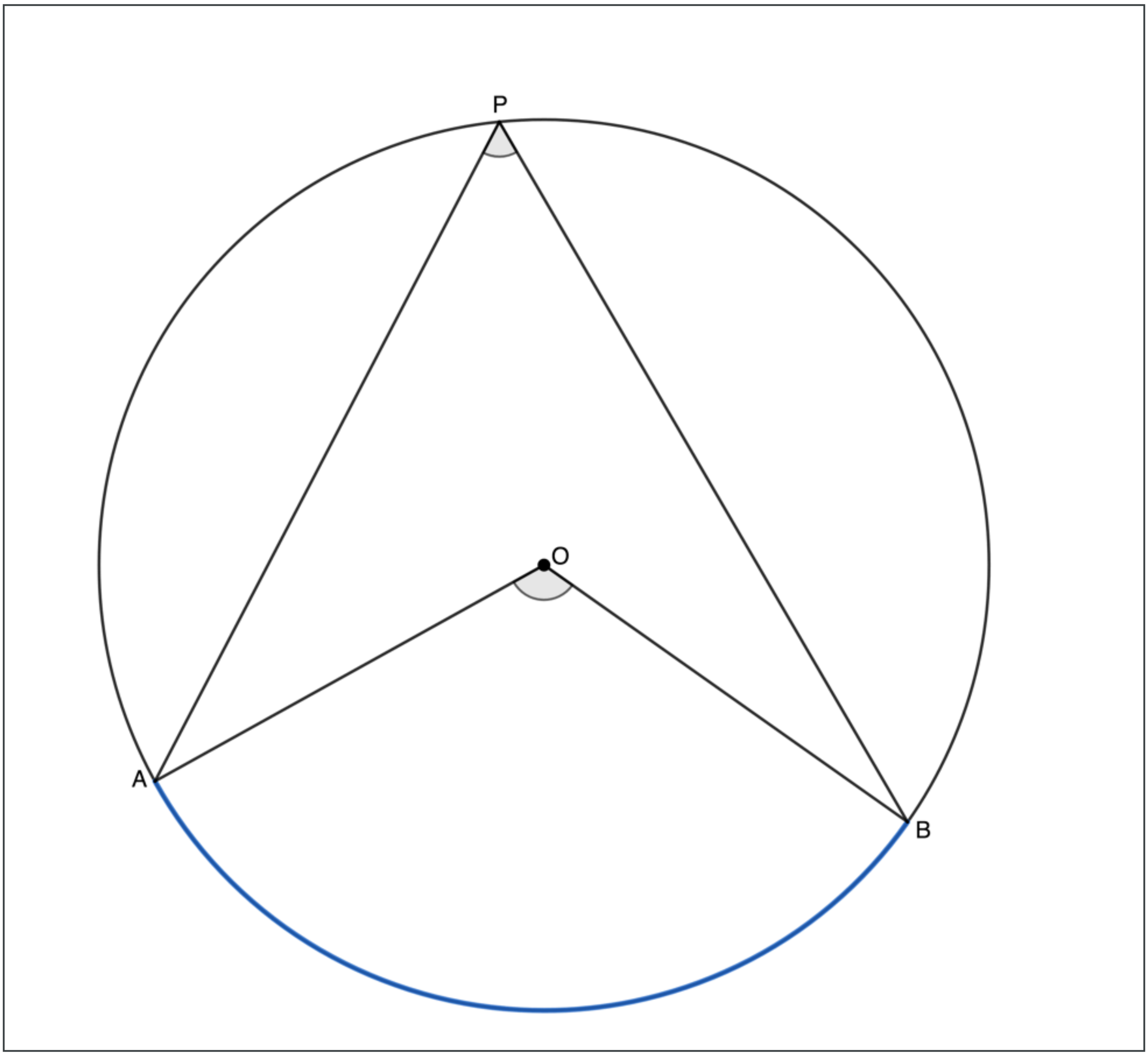
Theorems 2 and 3

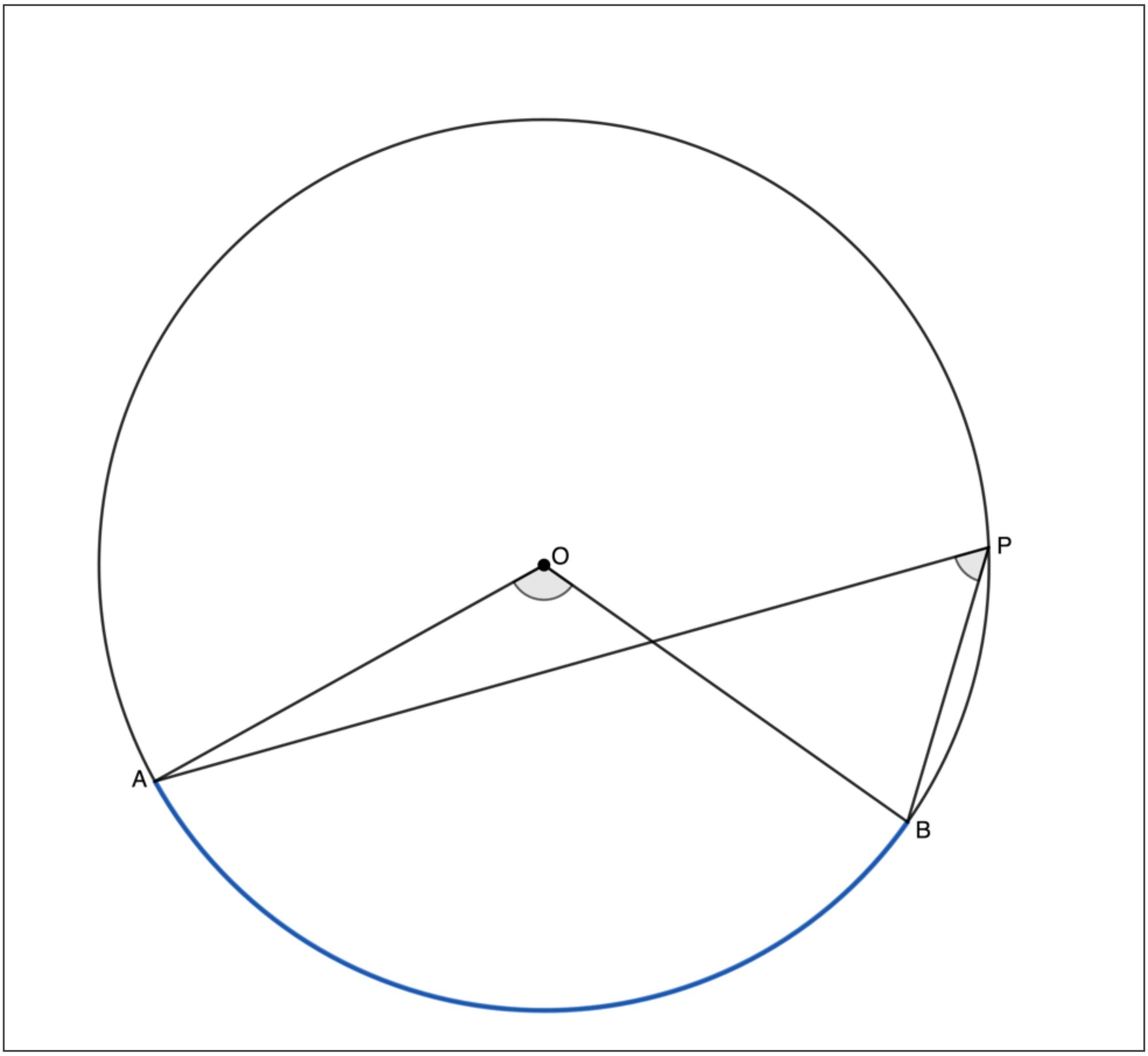
Let's now look at two more circle theorems.

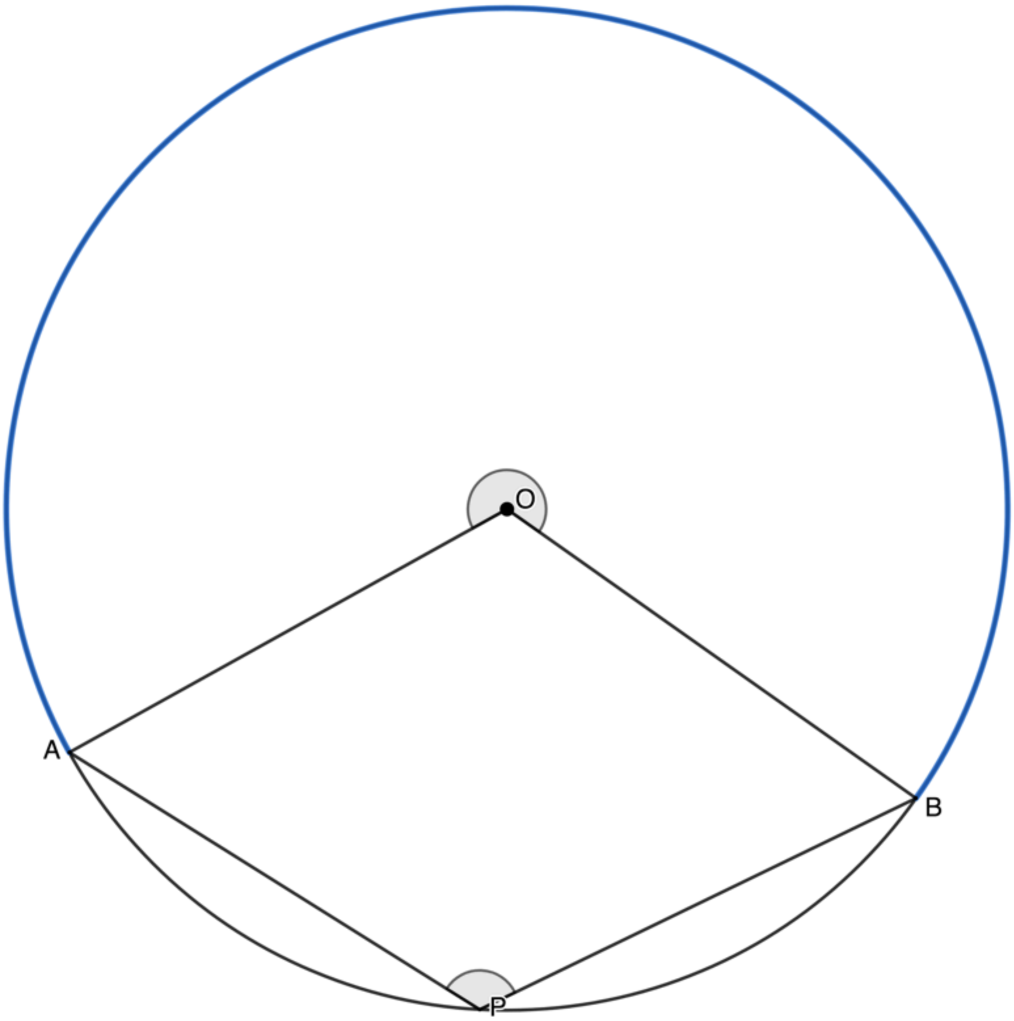
Theorem 2: Angle at the centre of a circle is twice the size of the angle at the circumference

If an arc subtends an angle at the centre of a circle and at the circumference, then the angle at the centre is twice the size of the angle at the circumference.

Note: We say an arc subtends an angle when the lines creating that angle start at either end of the arc. We can also say that the angle is subtended by the relevant chord. Three different possibilities are illustrated.


P
O
A
B


O
P
A
B



If arc AB (or chord AB) subtends an angle at the centre and an angle at the circumference, then the angle at the centre ($A\hat{O}B$) is twice the size of the angle at the circumference ($A\hat{P}B$).

Reason: $\angle$ at centre = $2\angle$ at circumference.

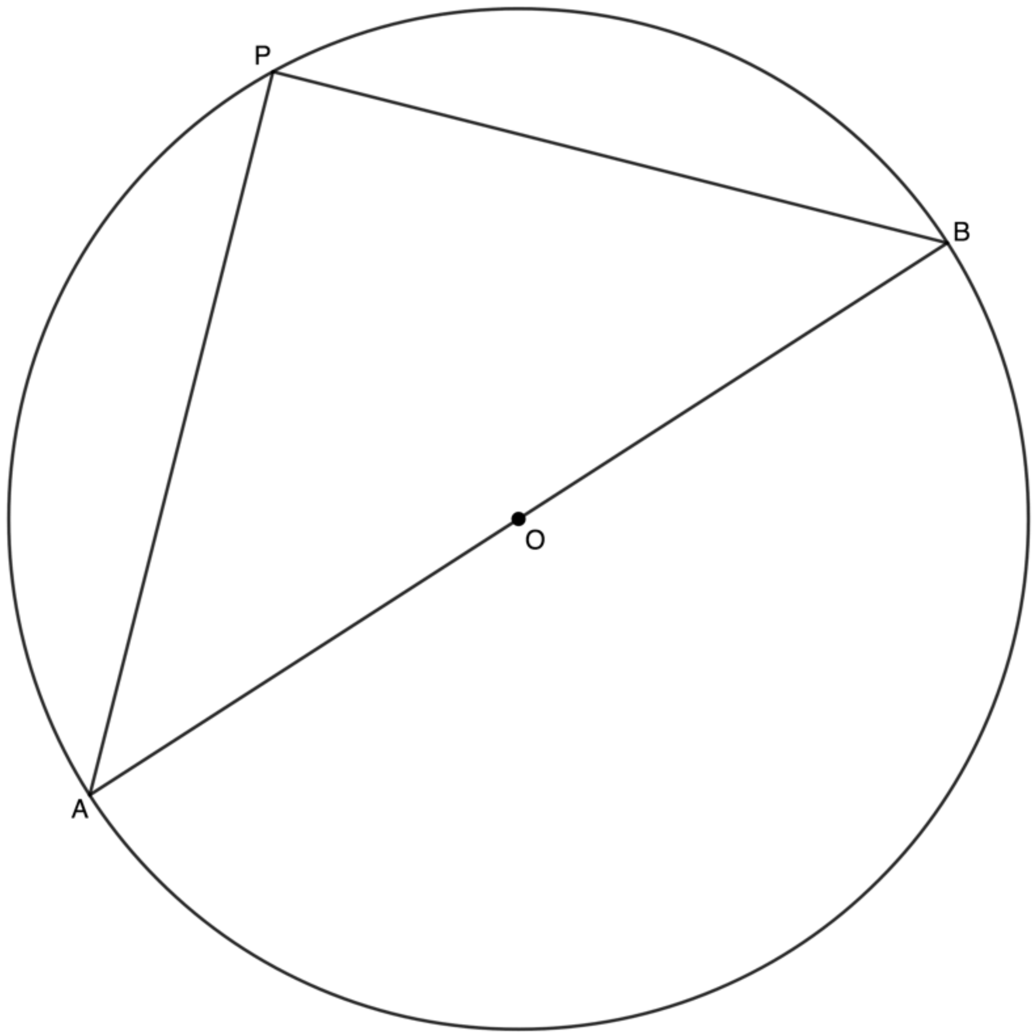
Visit this activity to play with an interactive circle to explore theorem 2 in more detail.

Move point C to change the size of the arc (or chord) and hence the angles created and then drag point A around the circumference of the circle. What happens if you make the arc a diameter?

You should have seen that if the arc subtending an angle is a diameter then the angle at the centre will be 180°. This means that the angle at the circumference will be 90°. Therefore, a diameter will subtend an angle of 90° at the circumference. This is theorem 3.

Theorem 3: Diameter of a circle subtends a right angle at the circumference

If a diameter subtends an angle at the circumference, then this angle is a right angle.

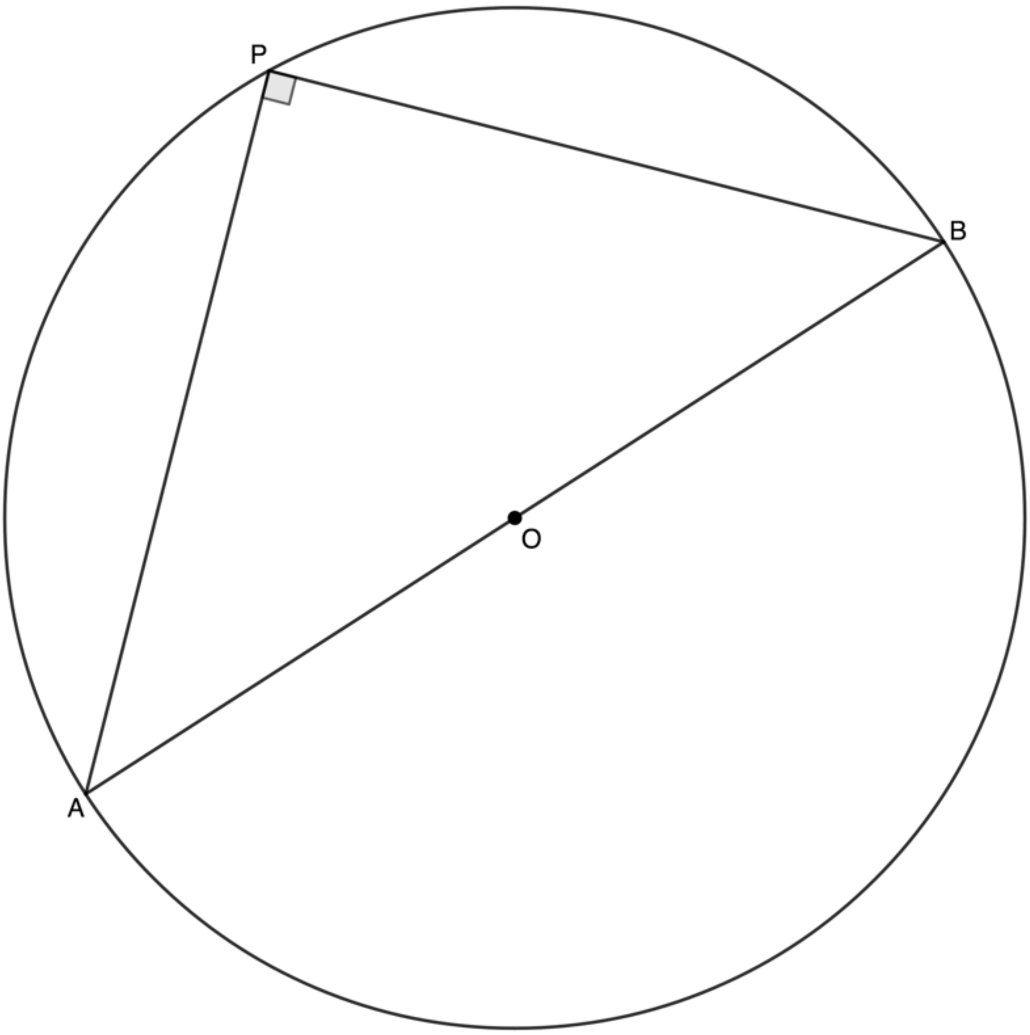


If AB is a diameter, then the angle subtended at the circumference ($A\hat{P}B$) is a right angle.

Reason: Angles in a semi-circle

Converse to theorem 3: If a chord subtends a right angle at the circumference then the chord is a diameter

If an angle subtended by a chord at a point on the circumference is a right angle, then the chord is a diameter.



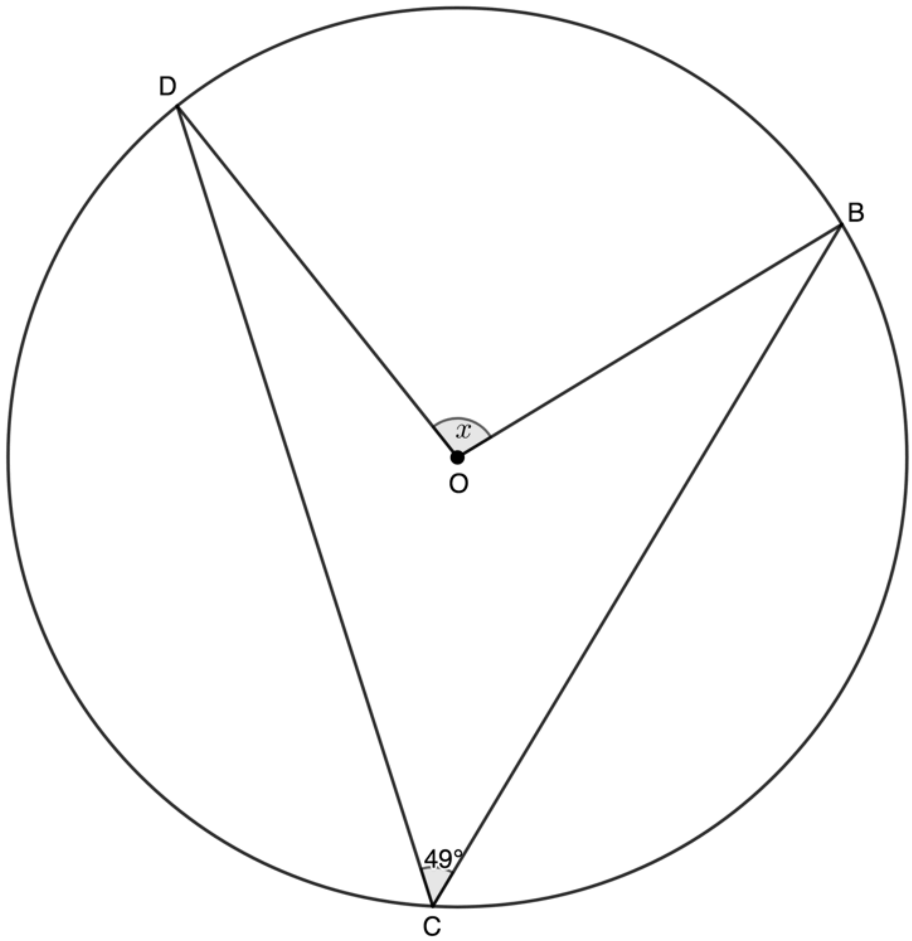
If AB subtends a right-angle at the circumference, then AB is a diameter.

Reason: Chord subtends right-angle

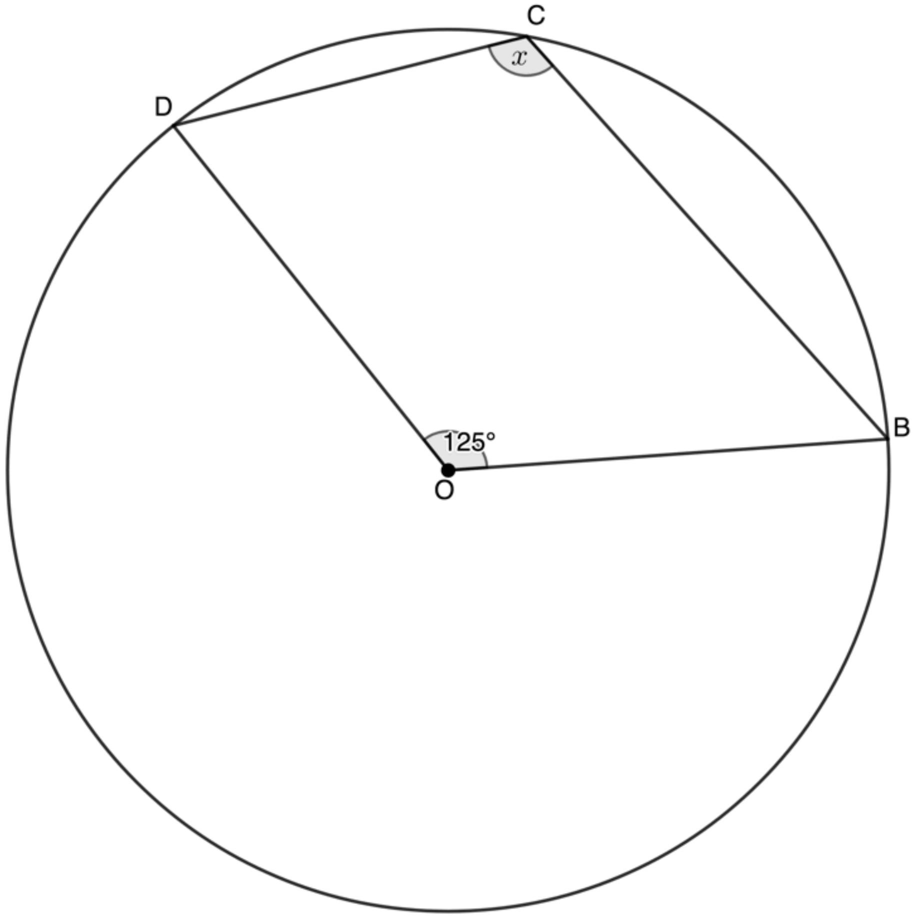
Example 2.3

Given O the centre of the circle, determine the value of x in each case.

1.



2.

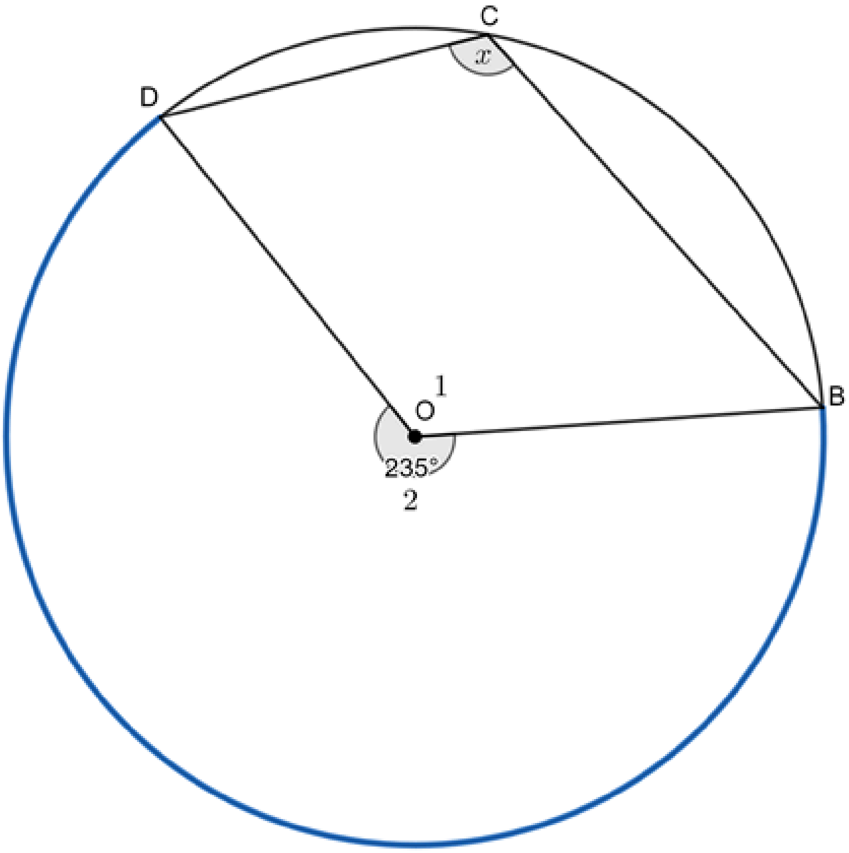


Solution

1. Arc DB (or chord DB) subtends an angle of 49° at the circumference and angle $D\hat{O}B$ at the centre.

   $D\hat{O}B = x = 2 \times 49^\circ = 98^\circ \quad (\angle \text{ at centre } 2\angle \text{ at circumference})$

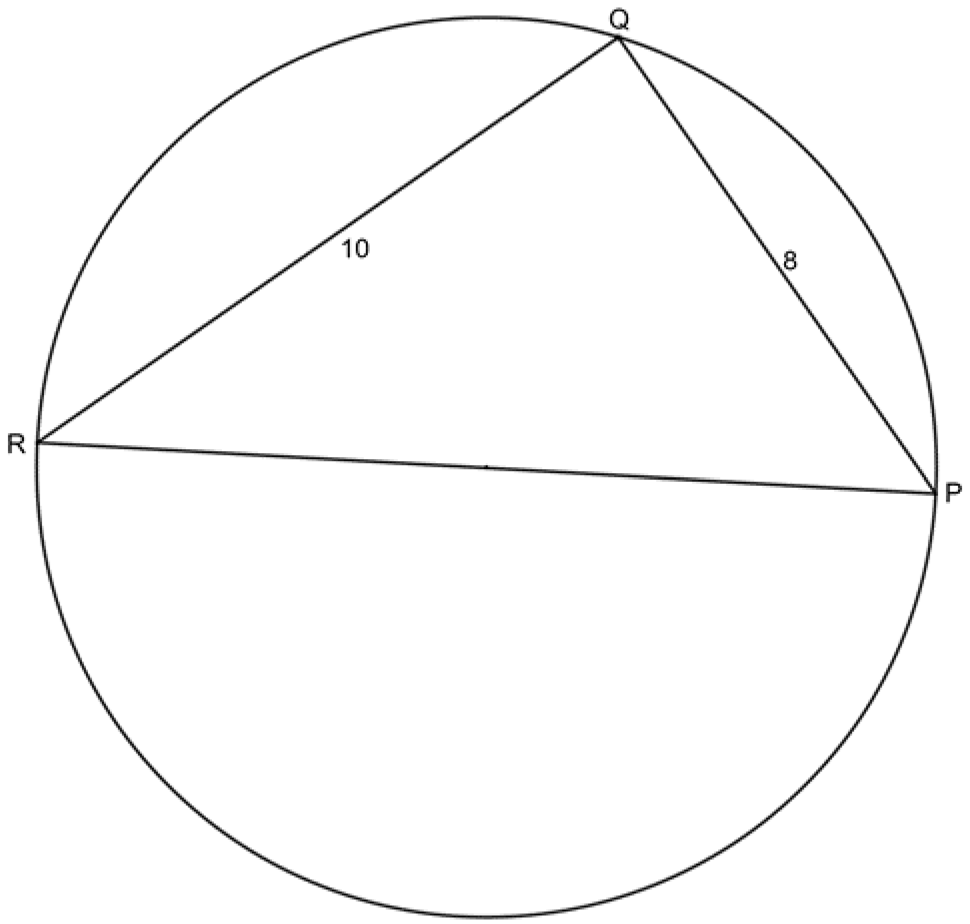
2. We need to be careful in this case because the angle at the circumference and the given angle at the centre are not on the same side of the chord DB. Instead, we need to consider the larger arc DB as indicated below. Therefore, the angle at the centre is actually 235°.



$$\hat{O}_2 = 360° - \hat{O}_1 \quad \text{(angles around a point} = 360°)$$
$$= 360° - 125°$$
$$= 235°$$
$$B\hat{C}D = x = \frac{235°}{2} = 117.5° \quad (\angle \text{at centre} = 2\angle \text{ at circumference})$$

Example 2.4

Given PR a diameter of the circle PQR, $PQ = 8$ unitsand $QR = 10$ units, determine the radius of the circle PQR.



Solution

We are told that PR is a diameter. Therefore, it subtends an angle of 90° at the circumference. Therefore, ΔPQR is a right-angled triangle and we can use Pythagoras to determine the length of the diameter and, hence, the length of the radius.

PR is a diameter (given)
Therefore $P\hat{Q}R = 90^\circ \quad (\angle\text{s in a semi-circle})$
In ΔPQR:

$$\begin{aligned} PR^2 &= PQ^2 + QR^2 \quad \text{(Pythagoras)} \\ &= 8^2 + 10^2 \\ &= 164 \end{aligned}$$

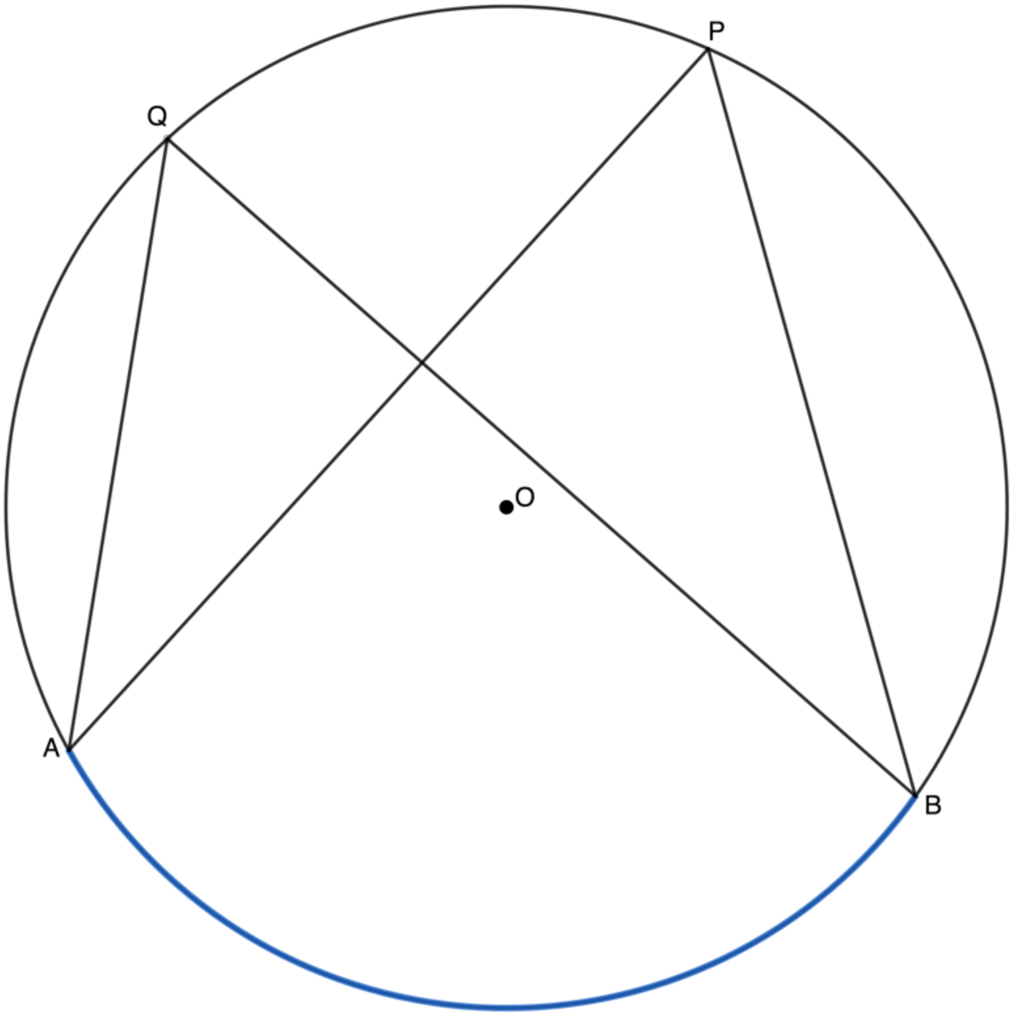
$\therefore PR = 2\sqrt{41} \text{ units}$
A radius is equal to half the diameter. Therefore, the radius is $\sqrt{41} \text{ units}$.

Theorem 4

This next theorem deals with segments of a circle.

Theorem 4: Angles in the same segment of a circle are equal

If the angles subtended by a chord of the circle are on the same side of the chord, then the angles are equal.

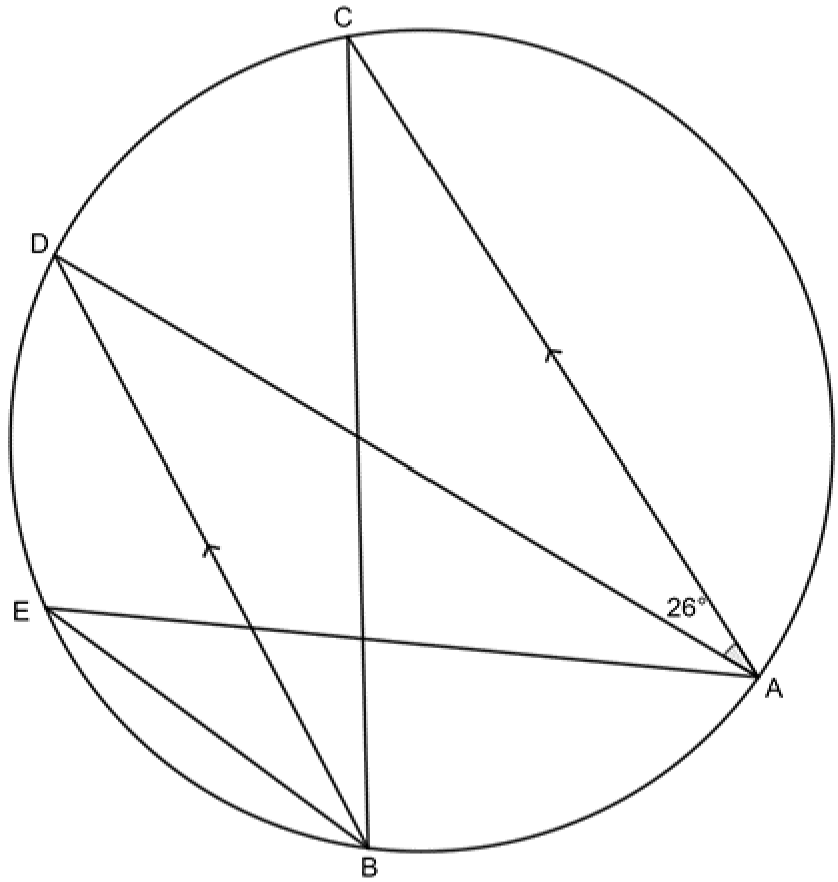


If AB subtends angles $A\hat{P}B$ and $A\hat{Q}B$ at the circumference and $A\hat{P}B$ and are both on the same side of the chord AB, then $A\hat{P}B = A\hat{Q}B$.

Reason: $\angle$s in same seg

Example 2.5

Given $C\hat{A}D = 26^{\circ}$, determine the sizes of angles $\hat{C}$, $\hat{D}$ and $\hat{E}$.

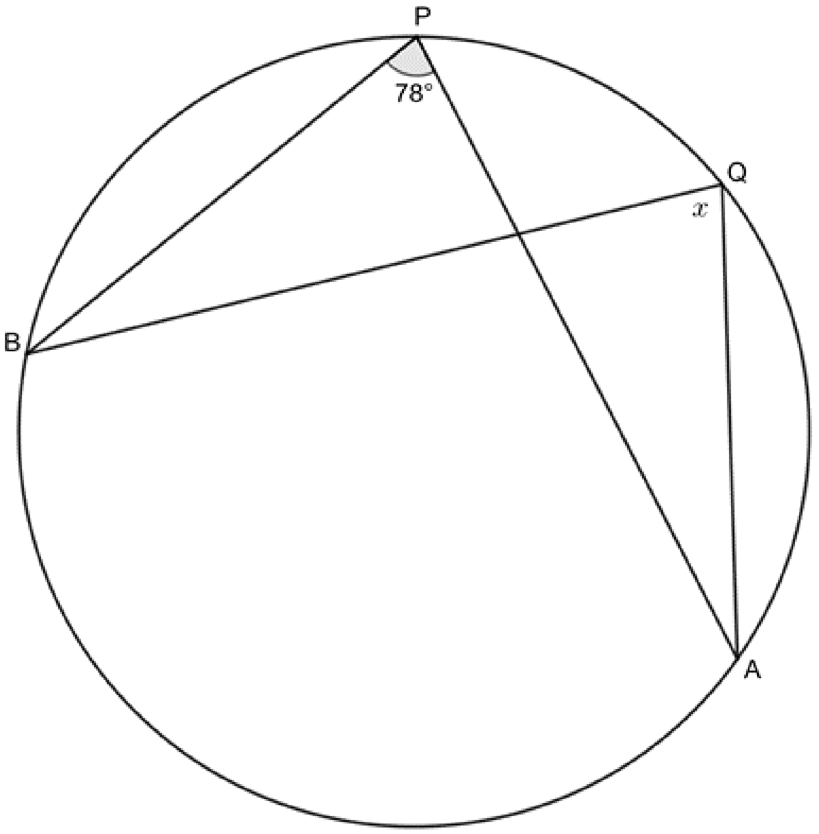


Solution

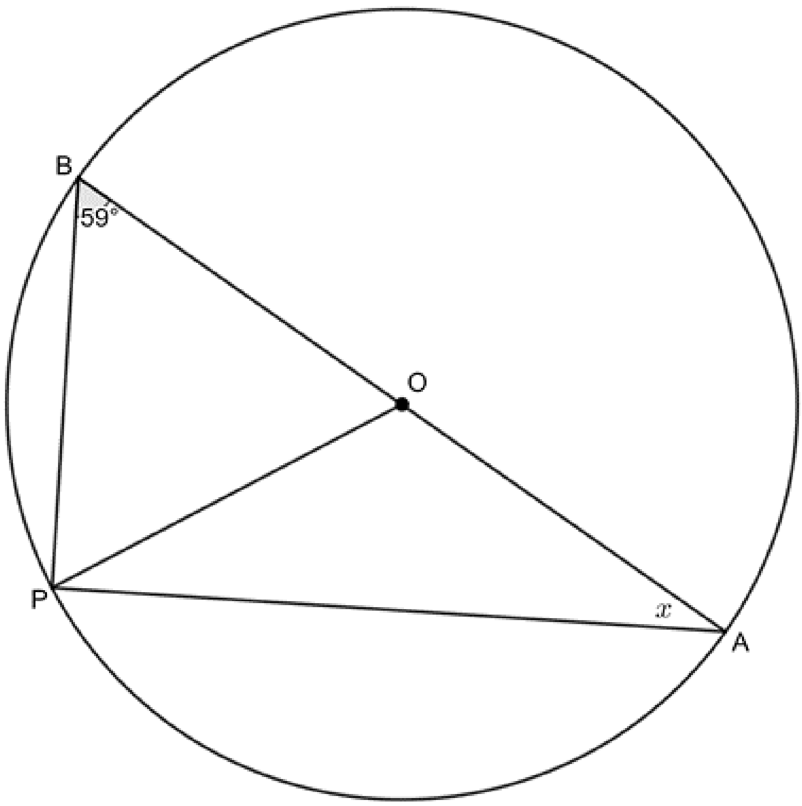
$\hat{D} = 26^\circ$ (alt $\angle$s =;AC$\|$BD)
$\hat{C} = \hat{D} = 26^\circ$ ($\angle$s in same seg)
$\hat{E} = \hat{D} = 26^\circ$ ($\angle$s in same seg)

Exercise 2.1

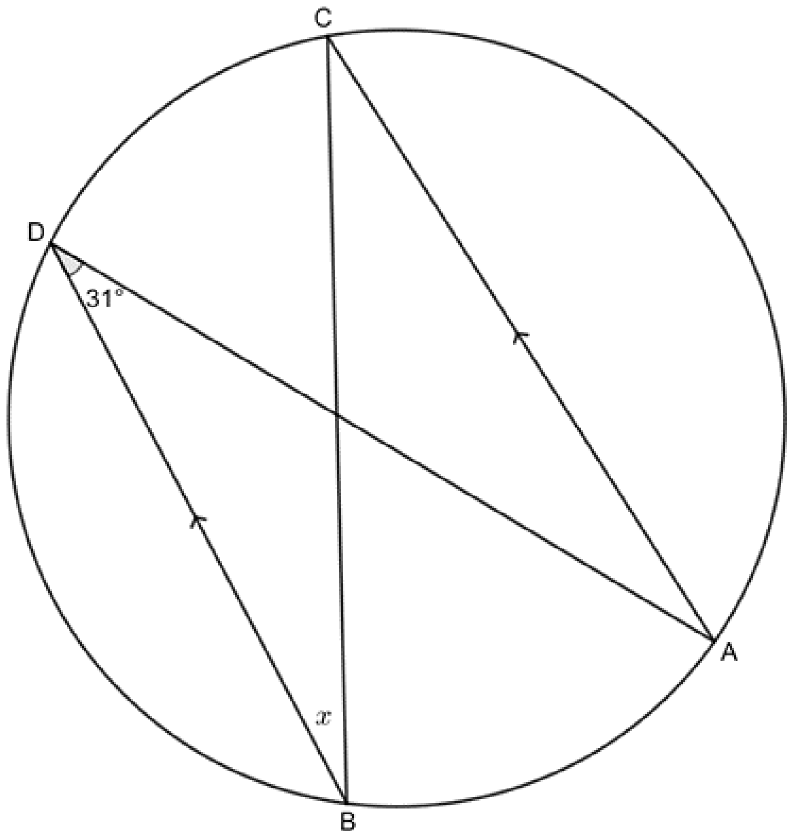
1. Find the values of the unknown angles in each case. Note that where O appears, it is the centre of the circle.
   a.



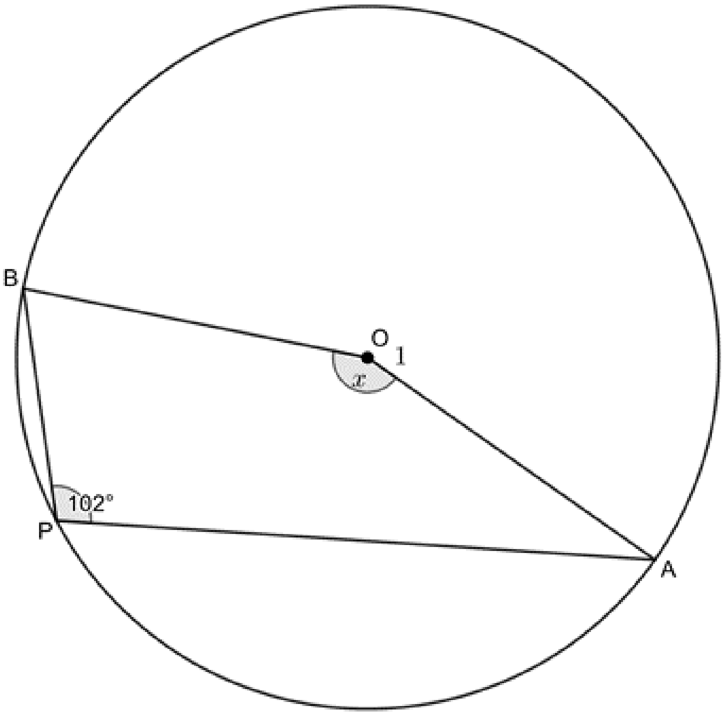
b.



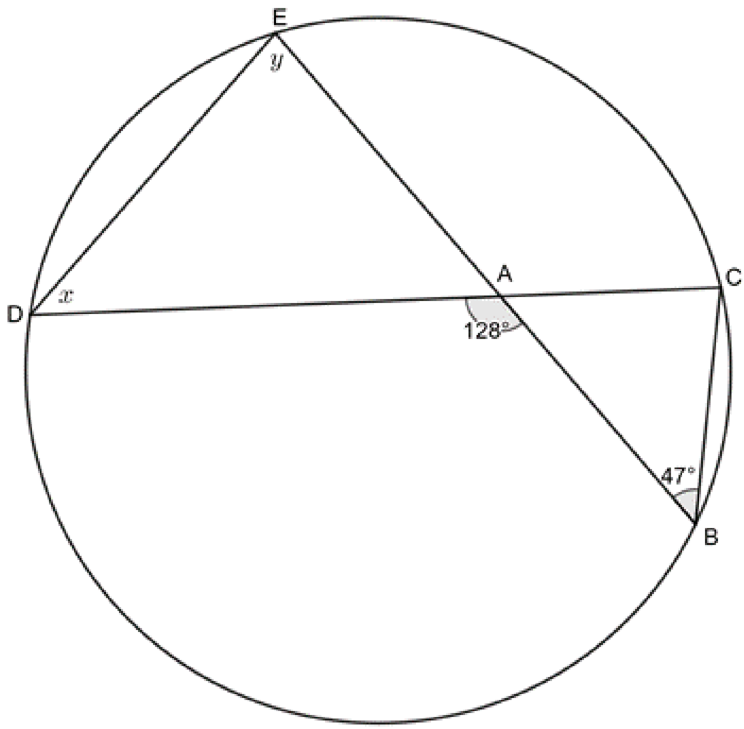
c.



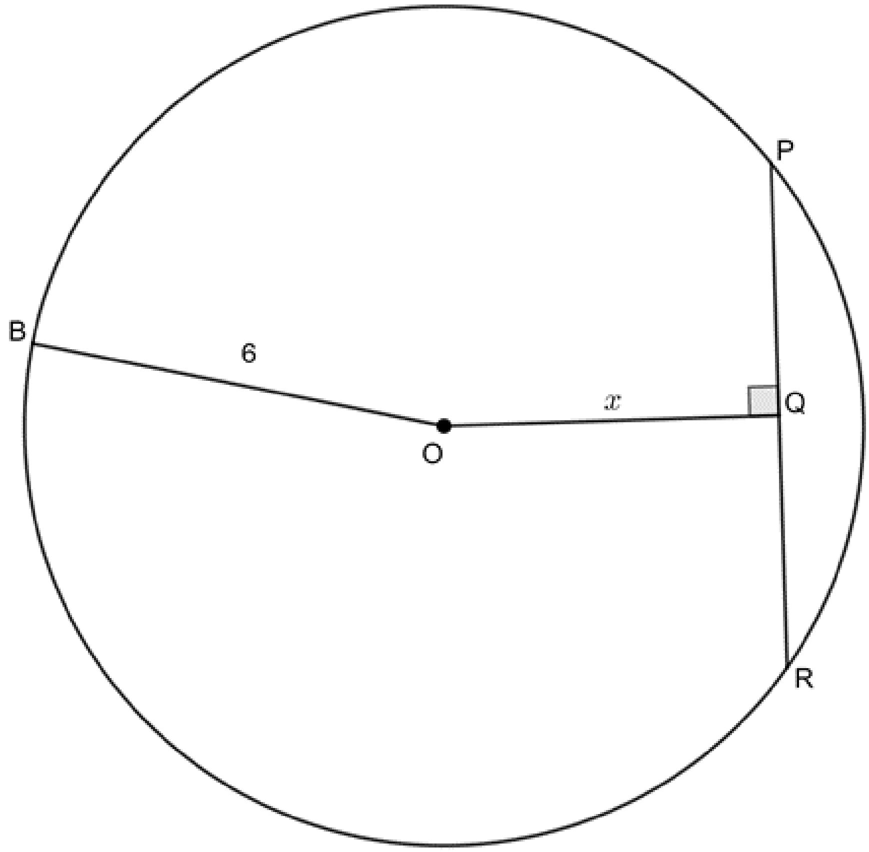
d.



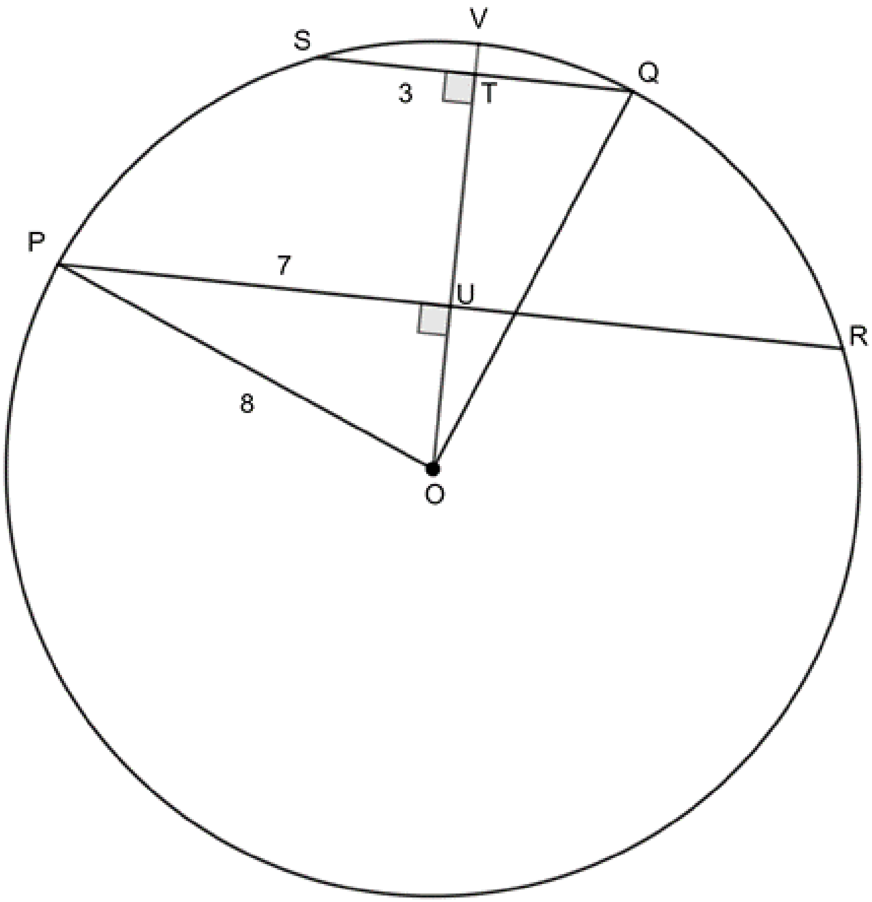
e.



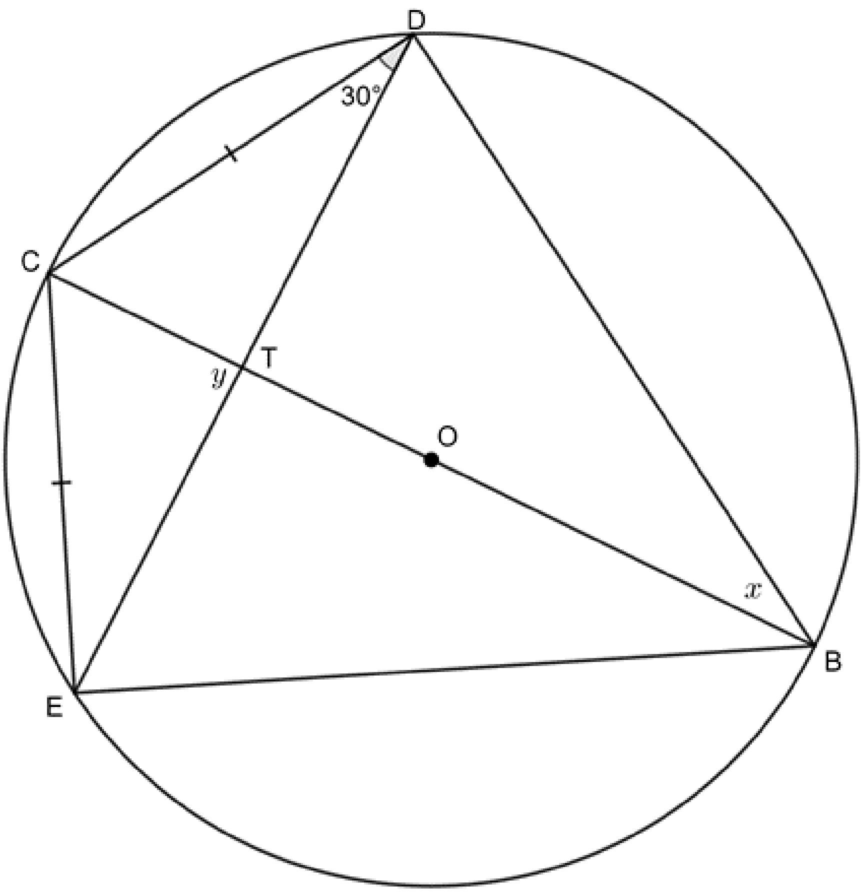
2. In the circle with centre O, the radius is 6 units $OQ \perp PR$ and $PR = 8 \text{ units}$. Determine x.



3. In the circle with centre O, $OT \perp SQ$, $OT \perp PR$, $OP = 8 \text{ units}$, $ST = 3 \text{ units}$ and $PU = 7 \text{ units}$. Determine TU.



4. Given circle with centre O, $CD = CE$ and $C\hat{D}T = 30^{\circ}$. Determine x and y. Does OC bisect DE?



The full solutions are at the end of the unit.

Summary

In this unit you have learnt the following:

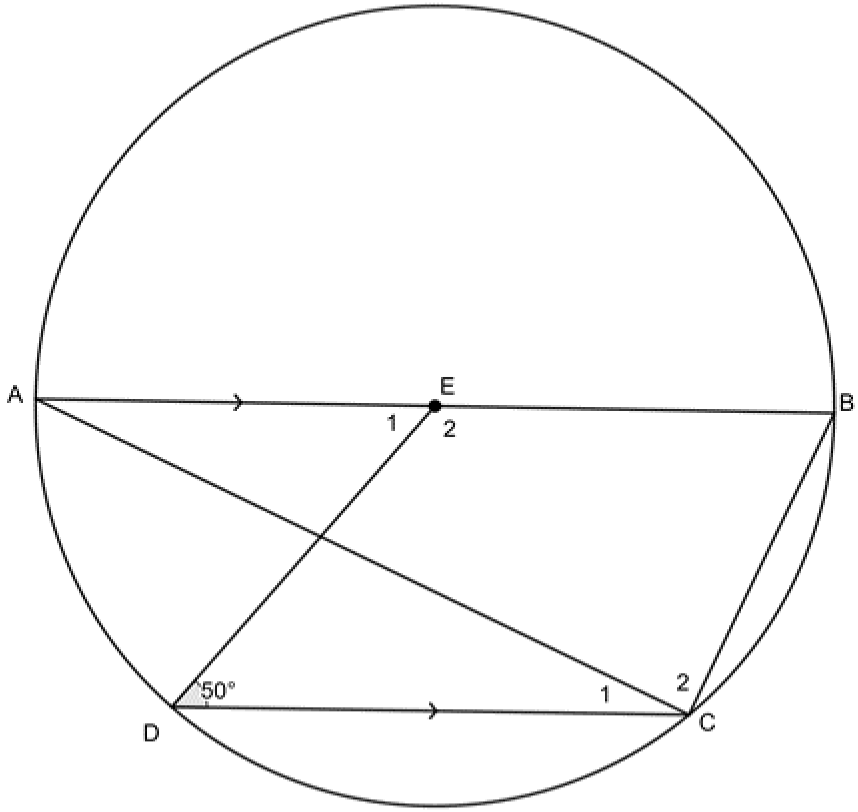
- A line drawn perpendicular to a chord from the centre of the circle bisects the chord.
- A line drawn from the circle centre to the mid-point of chord is perpendicular to the chord.
- The angle subtended by an arc or chord at the centre of a circle is twice the size of the angle subtended at the circumference.
- The diameter of a circle subtends a right angle at the circumference.
- If an angle subtended by a chord at a point on the circumference is a right angle, then the chord is a diameter.
- Angles subtended by the same arc or chord in the same segment of a circle (on the same side of the chord) are equal.

Unit 2: Assessment

Suggested time to complete: 40 minutes

Question 1 adapted from NC(V) Mathematics Level 4 Paper 2 November 2011 question 1.2

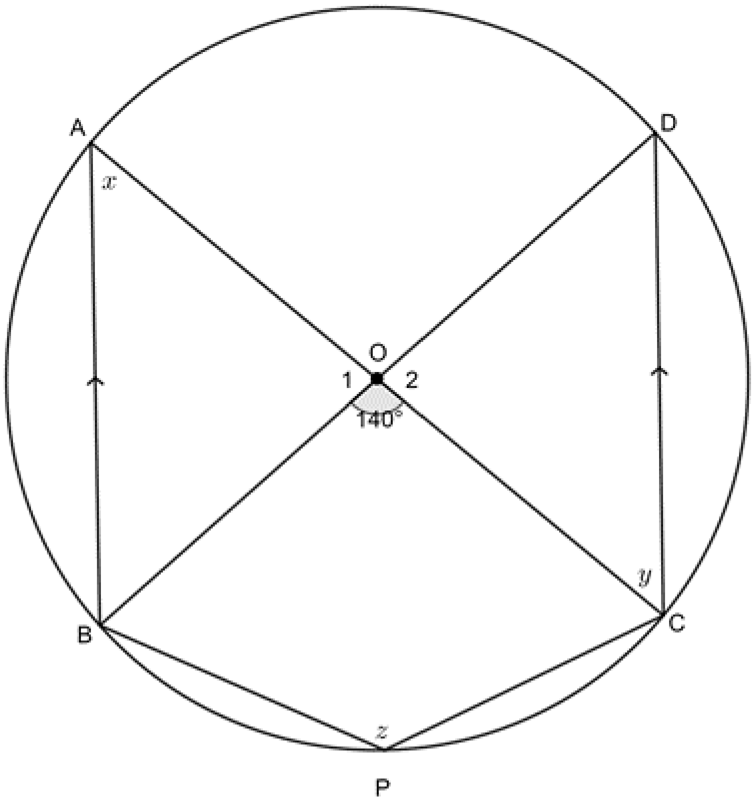
1. AB is a diameter of the circle with centre E. $C\hat{D}E = 50^\circ$ and $AB \parallel DC$.



Determine, with reasons, the values of $\hat{E}_1$, $\hat{C}_1$ and $\hat{B}$.

Question 2 adapted from NC(V) Mathematics Level 4 Paper 2 November 2013 question 1.2

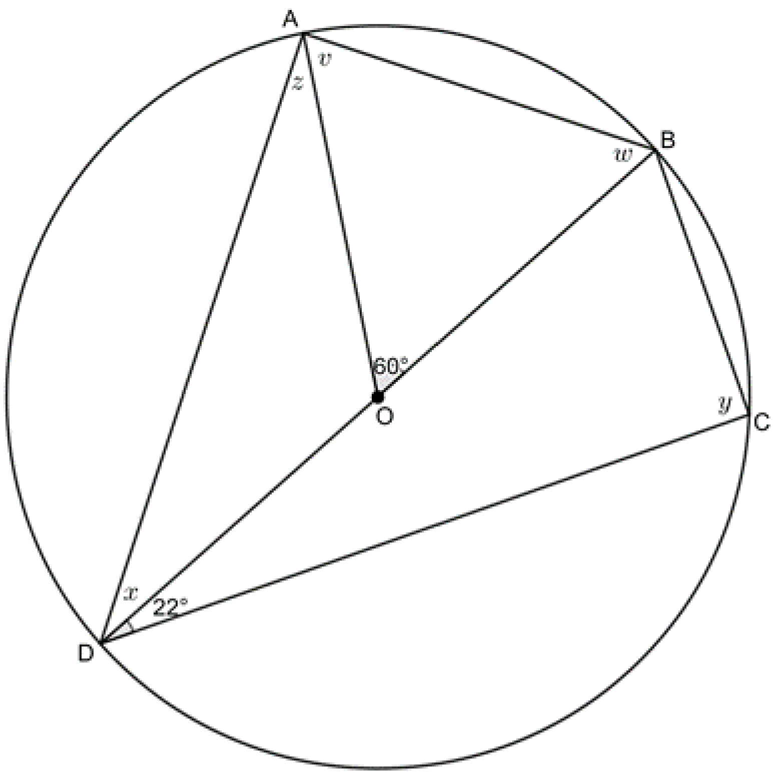
2. In the diagram below, O is the centre of the circle with diameters AC and BD. $AB \parallel DC$ and AC meets BD at O. P is any point on the minor arc BC. PB and PC are joined.



Calculate, with reasons, the values of x, y and z.

Question 3 adapted from NC(V) Mathematics Level 4 Paper 2 November 2014 question 1.3

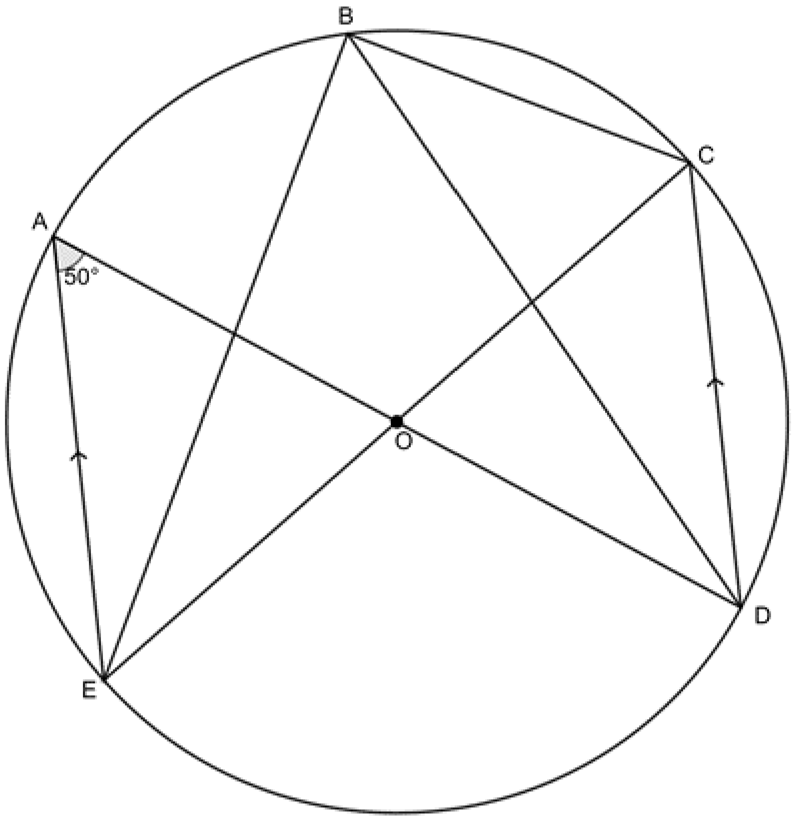
3. In the figure given below, O is the centre of the circle with diameter BD. $A\hat{O}B = 60^\circ$ and $B\hat{D}C = 22^\circ$ with A and C any two points on the circle on either side of BD.



Determine, with reasons, the values of v, w, x, y and z.

Question 4 adapted from NC(V) Mathematics Level 4 Paper 2 November 2015 question 1.3

4. In the figure below O is the centre of the circle, AOD is a diameter and $AE \parallel CD$, with E, C and B on the circumference of the circle. Also, $\hat{A} = 50^{\circ}$.



a. Why is $A\hat{D}C = 50°$?
b. Determine the size of $C\hat{B}D$.
c. Prove that EOC is a straight line.

The full solutions are at the end of the unit.

Unit 2: Solutions

Exercise 2.1

1.
    a. $x = 78° \quad (\angle\text{s in same seg})$
    b. AB is a diameter (straight line passing through the centre)
    Therefore $A\hat{P}B = 90° \quad (\angle\text{s in semi-circle})$
    $x = 180° - 90° - 59° \quad (\angle\text{s in a } \Delta)$
    $= 31°$
    c. .
    $C\hat{A}D = 31° \quad (\text{alt } \angle\text{s} =;\ AC \parallel BD)$
    $x = C\hat{A}D = 31° \quad (\angle\text{s in same seg})$
    d.
    $\hat{O}_1 = 204° \quad (\angle \text{ at centre } = 2\angle \text{ at circumference})$
    $\therefore x = 360° - 204° \quad (\angle\text{s round a point})$
    $= 156°$

e. $x = 47^\circ$ $(\angle\text{s in same seg})$

$D\hat{A}E = 180^\circ - 128^\circ$ $(\angle\text{s on str line suppl})$

$= 52^\circ$

$\therefore y = 180^\circ - 52^\circ - 47^\circ$ $(\angle\text{s in } \Delta \text{ suppl})$

$= 81^\circ$

2. $OP = OB = 6 \text{ units}$ (both radii)

$PR = 8 \text{ units}$ (given)

$\therefore PQ = 4 \text{ units}$ $(\perp \text{ from centre bisects chord})$

$x^2 = OP^2 - PQ^2$ (Pythagoras)

$\therefore x^2 = 6^2 - 4^2$

$= 36 - 16$

$= 20$

$\therefore x = 2\sqrt{5}$

3.

$OU^2 = OP^2 - PU^2$ (Pythagoras)

$\therefore OU^2 = 8^2 - 7^2$

$= 64 - 49$

$= 15$

$\therefore OU = \sqrt{15}$

$OP = OQ = 8 \text{ units}$ (both radii)

$QT = ST = 3 \text{ units}$ $(\perp \text{ from centre bisects chord})$

$OT^2 = OQ^2 - QT^2$ (Pythagoras)

$\therefore OT^2 = 8^2 - 3^2$

$= 64 - 9$

$= 55$

$\therefore OT = \sqrt{55}$

$TU = OT - OU$

$= \sqrt{55} - \sqrt{15}$

4.

$C\hat{E}D = 30^\circ$ $(\text{isos } \Delta)$

$x = C\hat{E}D = 30^\circ$ $(\angle\text{s in same seg})$

$B\hat{D}C = 90^\circ$ $(CB \text{ passes through centre O; } \angle\text{s in semi-circle})$

$\therefore E\hat{D}B = 60^\circ$

$E\hat{C}T = 60^\circ$ $(\angle\text{s in same seg})$

$\therefore y = 180^\circ - 30^\circ - 60^\circ$ $(\angle\text{s in } \Delta \text{ suppl})$

$= 90^\circ$

Therefore OC bisects DE ($\perp$ from centre bisects chord).

Back to Exercise 2.1

Unit 2: Assessment

1.

$\hat{E}_1 = 50^\circ$ $(\text{alt } \angle\text{s } =; AB \parallel CD)$

$\hat{C}_1 = \frac{1}{2}\hat{E}_1 = 25^\circ$ $(\angle \text{ at centre } = 2\angle \text{ at circumference})$

$\hat{C}_2 = 90^\circ$ $(\angle\text{s in semi-circle})$

$\hat{B} + \hat{C}_1 + \hat{C}_2 = 180^\circ$ $(\text{co-int } \angle\text{s suppl; } AB \parallel CD)$

$\therefore \hat{B} = 180^\circ - 90^\circ - 25^\circ = 65^\circ$

2.

$x = 70^\circ \quad (\angle \text{ at centre } = 2\angle \text{ at circumference})$
$y = x = 70^\circ \quad (\text{alt } \angle\text{s } =;\ AB \parallel CD)$
$\hat{O}_1 + \hat{O}_2 + A\hat{O}D = 360^\circ - 140^\circ = 220^\circ \quad (\angle\text{s around a point})$
$\therefore z = 110^\circ \quad (\angle \text{ at centre } = 2\angle \text{ at circumference})$

3. In ΔABO:
$OA = OB \quad (\text{both radii})$
$\therefore v = w \quad (\text{isosc } \Delta)$
$\therefore 2v = 180^\circ - 60^\circ = 120^\circ \quad (\angle\text{s in a } \Delta \text{ suppl})$
$\therefore v = 60^\circ$
$\therefore w = 60^\circ$
$x = 30^\circ \quad (\angle \text{ at centre} = 2\angle \text{ at circumference})$
$y = 90^\circ \quad (\angle\text{s in semi-circle})$
$z + v = 90^\circ \quad (\angle\text{s in semi-circle})$
$\therefore z = 30^\circ$

4.
1. $A\hat{D}C = 50^\circ \quad (\text{alt } \angle\text{s } =;\ AE \parallel CD)$
2. In ΔAOE:
$OA = OE \quad (\text{both radii})$
$\therefore \hat{A} = A\hat{E}O \quad (\text{isosc } \Delta)$
$\therefore A\hat{O}E = 180^\circ - 50^\circ - 50^\circ = 80^\circ \quad (\angle\text{s in a } \Delta \text{ suppl})$
$A\hat{O}E = C\hat{O}D = 80^\circ \ (\text{vert opp } \angle\text{s})$
$\therefore C\hat{B}D = 40^\circ \quad (\angle \text{ at centre } = 2\angle \text{ at circumference})$
3.
$E\hat{B}D = 50^\circ \quad (\angle\text{s in same seg})$
$E\hat{B}C = E\hat{B}D + C\hat{B}D$
$= 40^\circ + 50^\circ$
$= 90^\circ$
$\therefore EOC \text{ is a diameter (chord subtend right-angle)}$
Therefore EOC is a straight line.

Back to Unit 2: Assessment

Media Attributions

Unit 3: Properties of cyclic quadrilaterals

DYLAN BUSA

Unit 3 outcomes

By the end of this unit you will be able to:

- Define a cyclic quadrilateral.
- Apply the theorem opposite angles of a cyclic quadrilateral are supplementary.
- Apply the theorem exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.
- Apply the converses of equality between opposite angles, between angles in the same segment, and between exterior angles and interior opposite angles, to prove a quadrilateral is cyclic.

What you should know

Before you start this unit, make sure you can:

1. State and use all the circle theorems covered in unit 2:
   1. A line drawn perpendicular to a chord from the centre of the circle bisects the chord.
   2. A line drawn from the circle centre to the mid-point of chord is perpendicular to the chord.
   3. The angle subtended by an arc or chord at the centre of a circle is twice the size of the angle subtended at the circumference.
   4. The diameter of a circle subtends a right angle at the circumference.
   5. If an angle subtended by a chord at a point on the circumference is a right angle, then the chord is a diameter.
   6. Angles subtended by the same arc or chord in the same segment of a circle (on the same side of the chord) are equal.

Introduction

A quadrilateral is any flat four-sided figure. Each of the four sides must be straight. A square is an example of a quadrilateral, as is a parallelogram. These are very special kinds of quadrilaterals with special properties.

Most quadrilaterals have no special characteristics other than that they have four straight sides and, therefore, four interior angles. We call these irregular quadrilaterals. Figure 1 shows various examples of quadrilaterals.

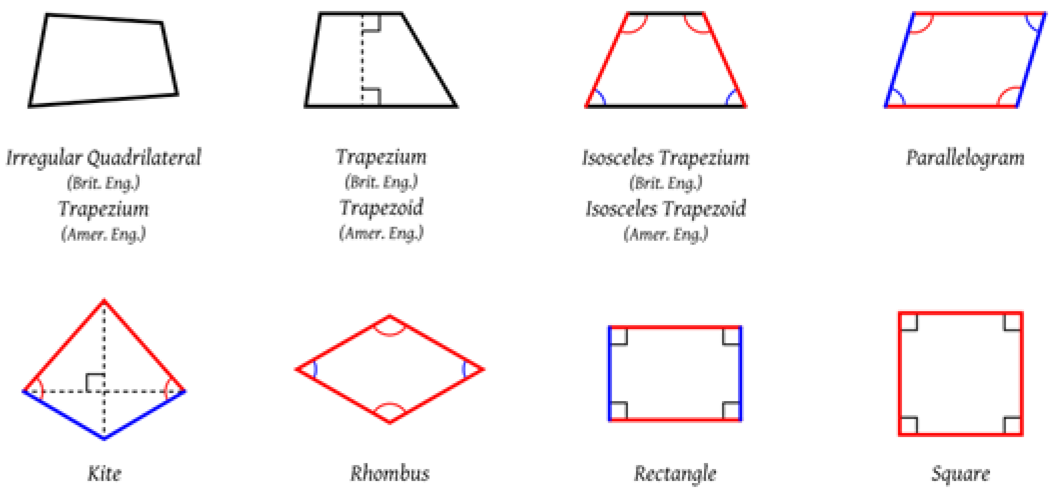


Figure 1: Examples of quadrilaterals

But some quadrilaterals are just the right shape that their four corners all lie on the circumference of the same circle. These are called **cyclic quadrilaterals** (see figure 2). Some special quadrilaterals such as squares, rectangles and parallelograms are always cyclic but many irregular quadrilaterals are cyclic as well.

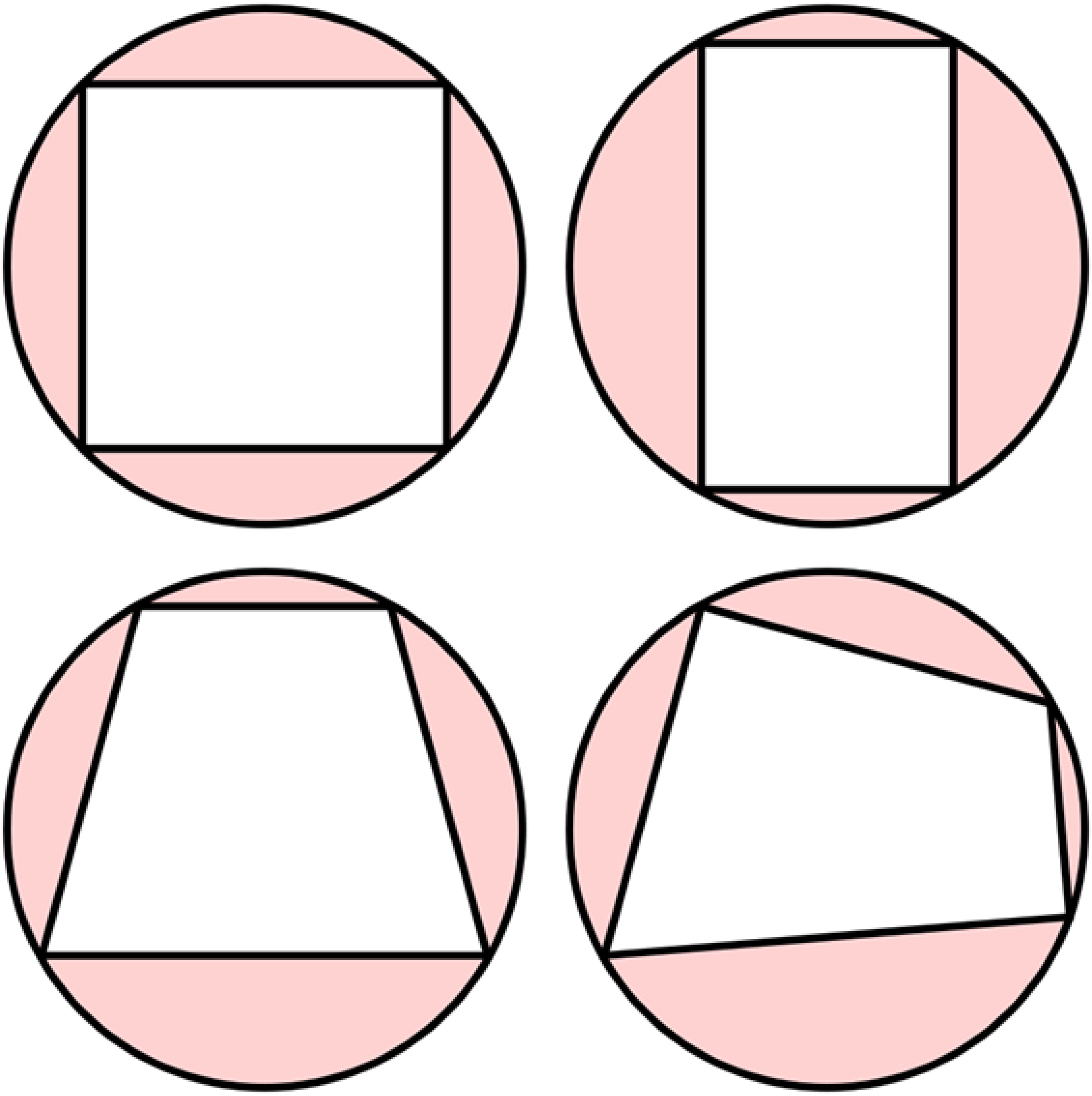


Figure 2: Examples of cyclic quadrilaterals

Take note!

For a quadrilateral to be cyclic, all four **vertices** (corners) of the quadrilateral must lie on the circumference of the **same** circle.

Cyclic quadrilateral theorems

You do not need to be able to prove any of the cyclic quadrilateral theorems yourself. You can simply assume that they are true. The following sections explain the theorems that you need to be able to state and use.

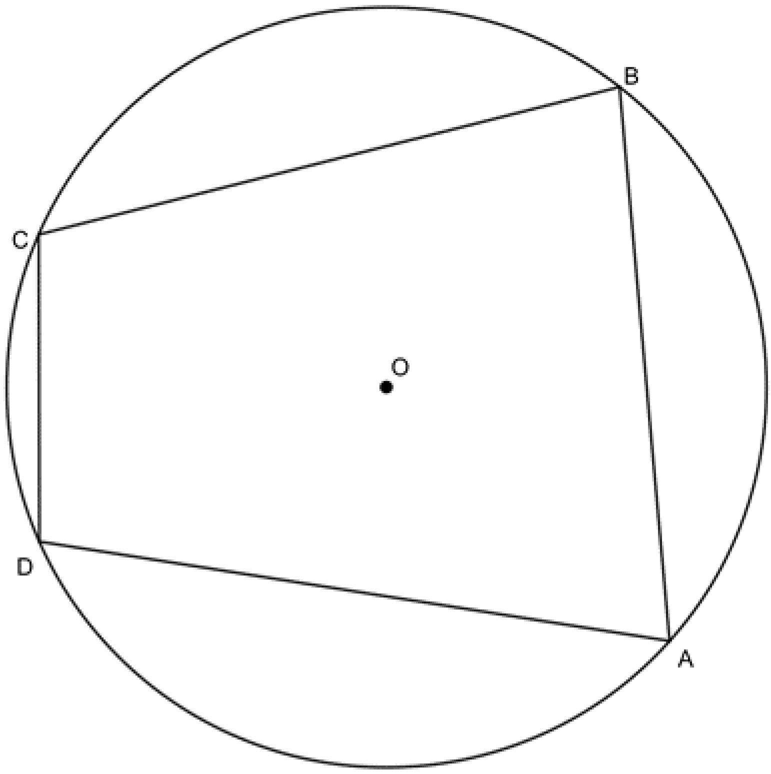
Note that they are numbered only for reference purposes and continue the numbering from unit 2. These theorems do not have official numbers.

Theorem 5

Let's look at the theorem involving opposite angles of a cyclic quadrilateral.

Theorem 5: Opposite interior angles of a cyclic quadrilateral are supplementary

The opposite angles of a cyclic quadrilateral are supplementary (they add up to $180°$).

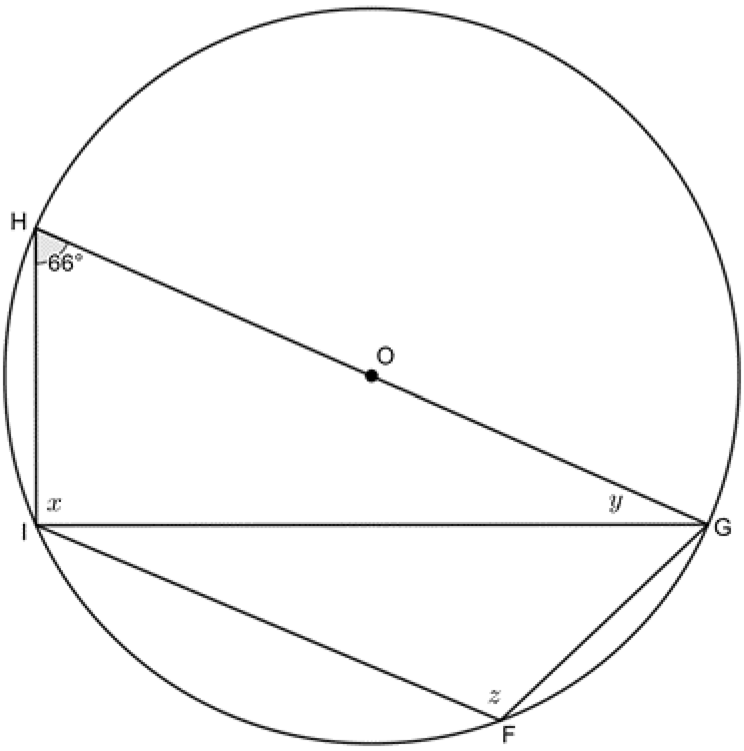


If $ABCD$ is a cyclic quadrilateral, then $\hat{A} + \hat{C} = 180°$ and $\hat{B} + \hat{D} = 180°$.

Reason: opp $\angle$s in cyclic quad

Example 3.1

Given the circle with centre O with diameter $GH$and cyclic quadrilateral $FGHI$. GI is drawn and $G\hat{H}I = 66°$. Determine the values of x, y and z.

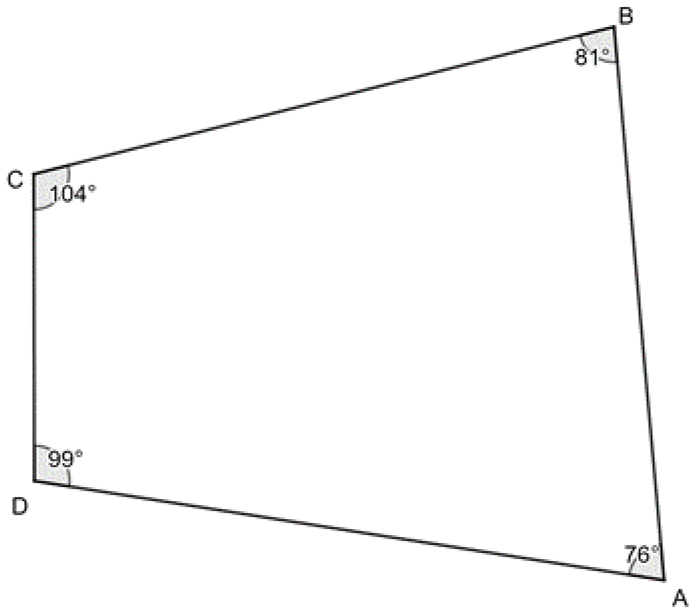


Solution

$x = 90^\circ$ ($\angle$s in semi-circle)
$y = 180^\circ - 66^\circ - 90^\circ = 24^\circ$ ($\angle$s in Δ suppl)
$z = 180^\circ - 66^\circ = 114^\circ$ ($\angle$s in cyclic quad)

Converse to theorem 5: Opposite interior angles are supplementary

If the opposite angles of a quadrilateral are supplementary (they add up to 180°), then the quadrilateral is cyclic.



If $\hat{A} + \hat{C} = 180^\circ$ or $\hat{B} + \hat{D} = 180^\circ$, then $ABCD$ is a cyclic quadrilateral.

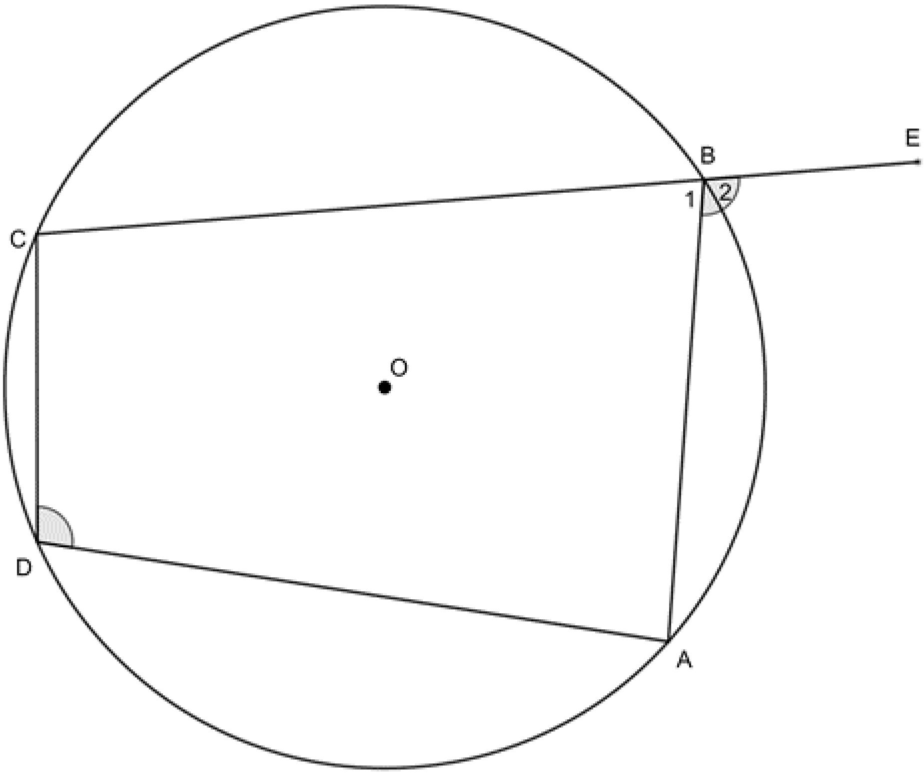
Reason: opp int $\angle$s suppl

Theorem 6

The next theorem looks at exterior angles.

Theorem 6: Exterior angle of a cyclic quadrilateral equal to opposite interior angle

The exterior angle of a cyclic quadrilateral is equal to the opposite interior angle.

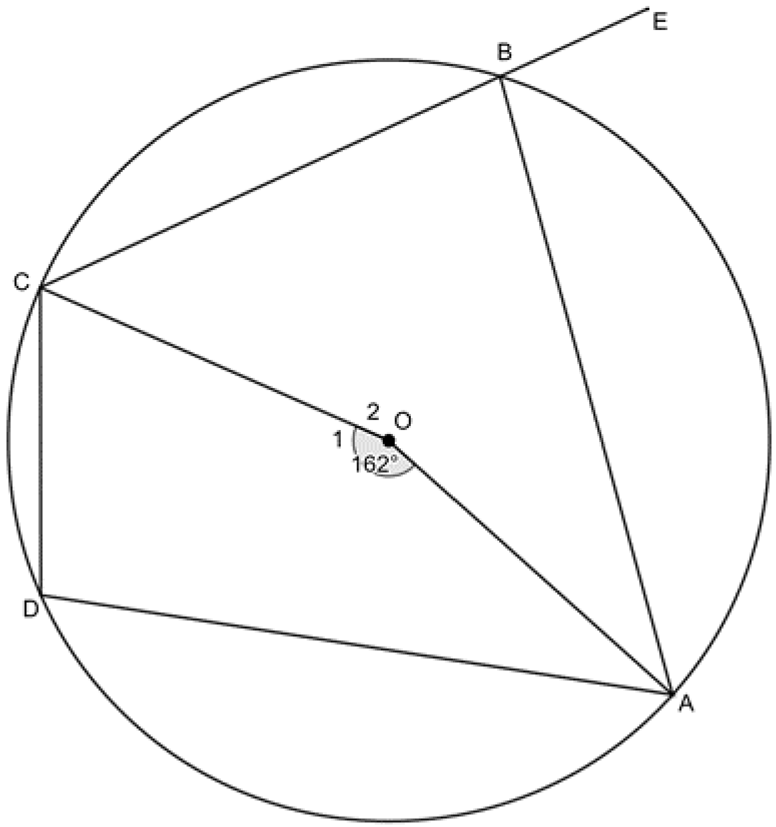


If $ABCD$ is a cyclic quadrilateral, then $\hat{B}_2 + \hat{D}$.

Reason: ext $\angle$ =opp int $\angle$ in cyclic quad

Example 3.2

Given the circle with centre O and cyclic quadrilateral $ABCD$, and $\hat{O}_1 = 162°$. Determine the value the value of $A\hat{B}E$.



Solution

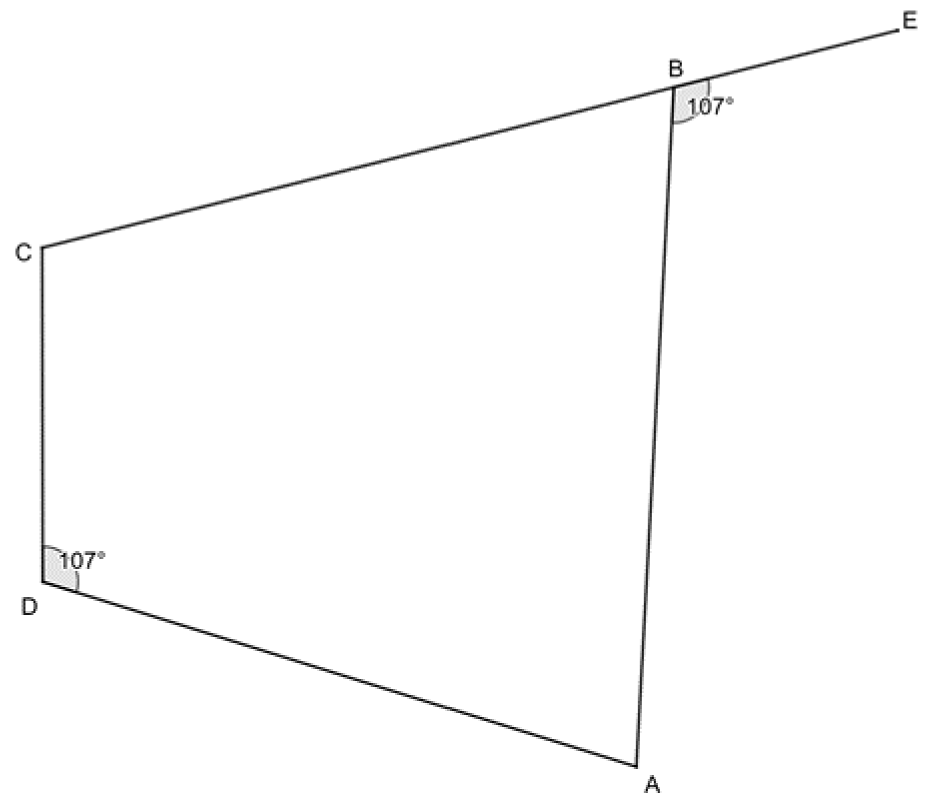
$\hat{O}_2 = 360^\circ - 162^\circ = 198^\circ$ (∠ around a point)

$\therefore \hat{D} = 99^\circ$ (∠ at centre =2∠ at circumference)

$\therefore A\hat{B}E = 99^\circ$ (ext ∠= opp int ∠ in cyclic quad)

Converse to theorem 6: Exterior angle equal to opposite interior angle

If the exterior angle of a quadrilateral is equal to the opposite interior angle, then the quadrilateral is cyclic.



If $A\hat{B}E = \hat{D}$, then $ABCD$ is a cyclic quadrilateral.

Reason: ext $\angle$ =opp int $\angle$

Proving a quadrilateral is a cyclic quadrilateral

So far, we have seen two ways in which we can prove that a quadrilateral is a cyclic quadrilateral.

- If we can prove that the opposite interior angles of the quadrilateral are supplementary, then the quadrilateral is cyclic.
- If we can prove that the exterior angle of the quadrilateral is equal to the opposite interior angle, then the quadrilateral is cyclic.

But there is a third way. Remember theorem 4 from unit 2? It stated that if the angles subtended by a chord of the circle are on the same side of the chord, then the angles are equal.

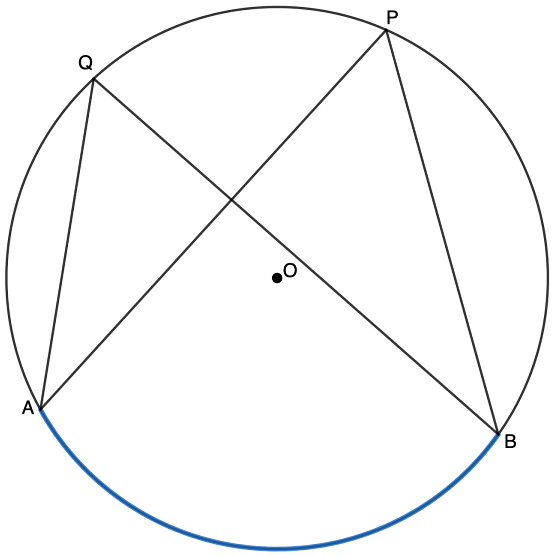


Figure 3: Angles in the same segment of a circle are equal

Because $\hat{Q}$ and $\hat{P}$ are both subtended by arc AB (or chord AB), then we know that $\hat{Q} = \hat{P}$. But now have a look at $ABPQ$. Can you see that all four vertices lie on the circumference of the same circle? Therefore, it is a cyclic quadrilateral.

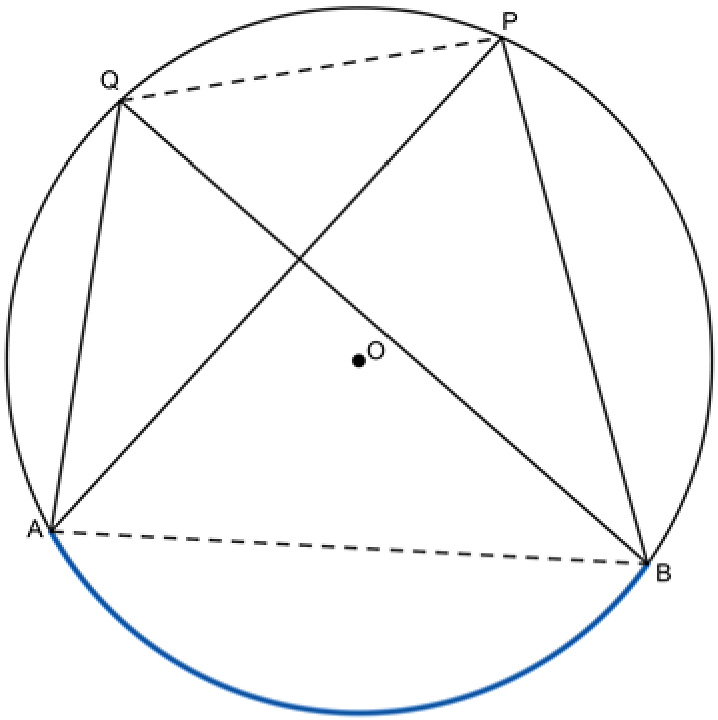


Figure 4: All four vertices lie on the circumference of the same circle

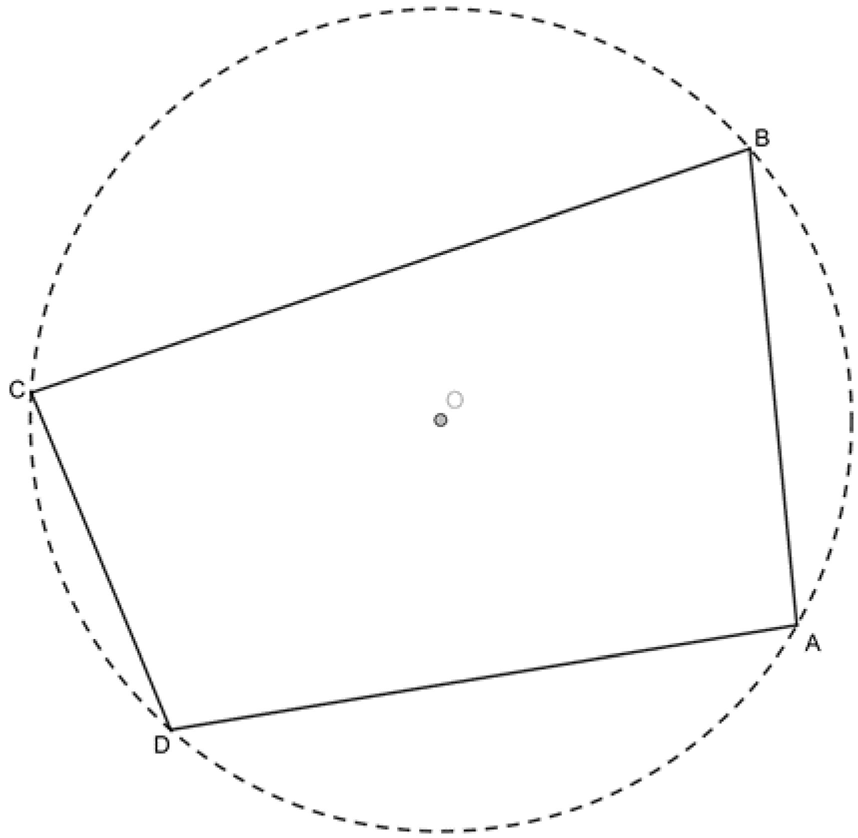
This means that we can use the **converse of theorem 4** to prove that a quadrilateral is cyclic. If the angles in the same segment of a circle are equal, then the quadrilateral made by the chord and the two angles must be a cyclic quadrilateral.

Take note!

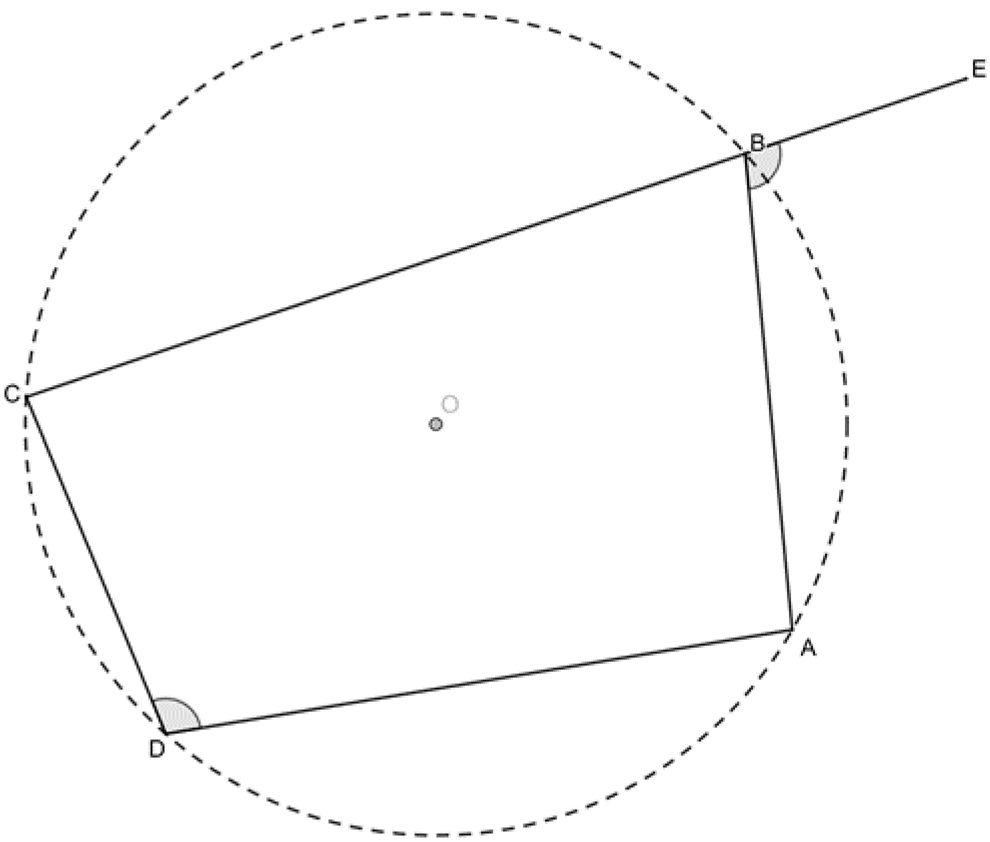
There are three ways to prove that a quadrilateral is a cyclic quadrilateral:

Proof 1: Opp int $\angle$s suppl

If $\hat{A} + \hat{C} = 180^\circ$ or $\hat{B} + \hat{D} = 180^\circ$, then $ABCD$ is a cyclic quadrilateral.

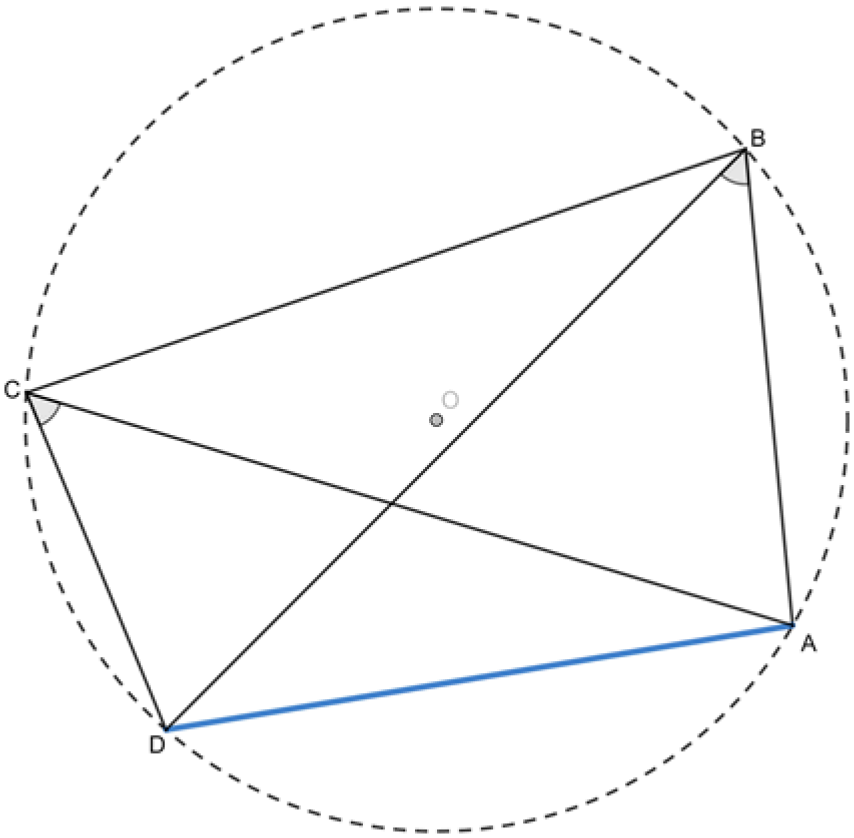


Proof 2: ext $\angle$ =opp int $\angle$



If $A\hat{B}E = \hat{D}$, then $ABCD$ is a cyclic quadrilateral.

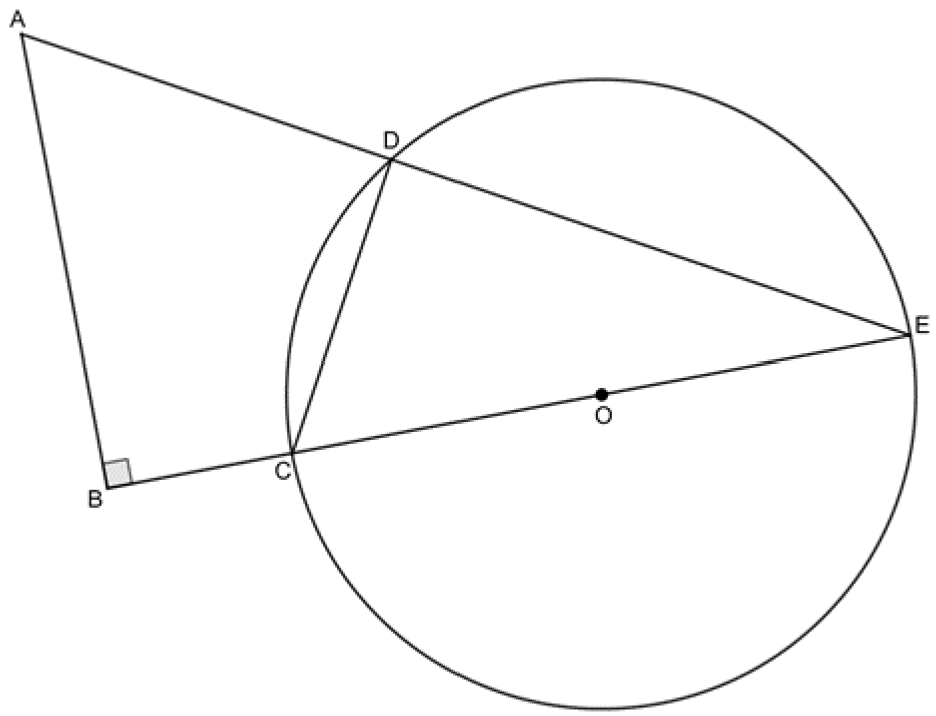
Proof 3: $\angle$s in same segment



If $A\hat{C}D = A\hat{B}D$, then $ABCD$ is a cyclic quadrilateral.

Example 3.3

If COE is a diameter of circle centre O, prove that $ABCD$ is a cyclic quadrilateral.



Solution

$C\hat{D}E = 90^\circ \quad (\angle\text{s in semi-circle})$

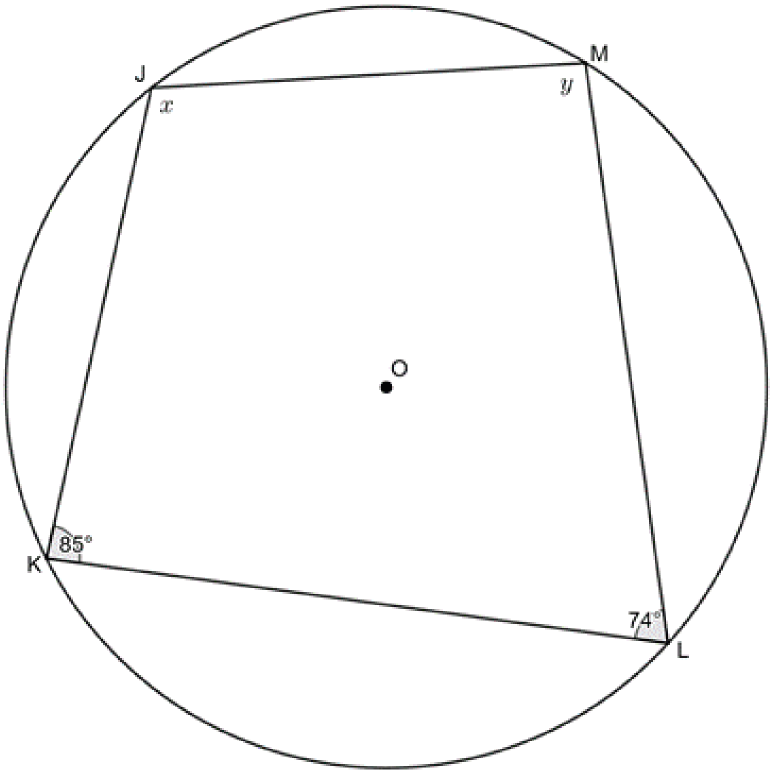
$\therefore C\hat{D}E = A\hat{B}C = 90^\circ$

Therefore $ABCD$ is a cyclic quadrilateral (ext $\angle =$opp int $\angle$).

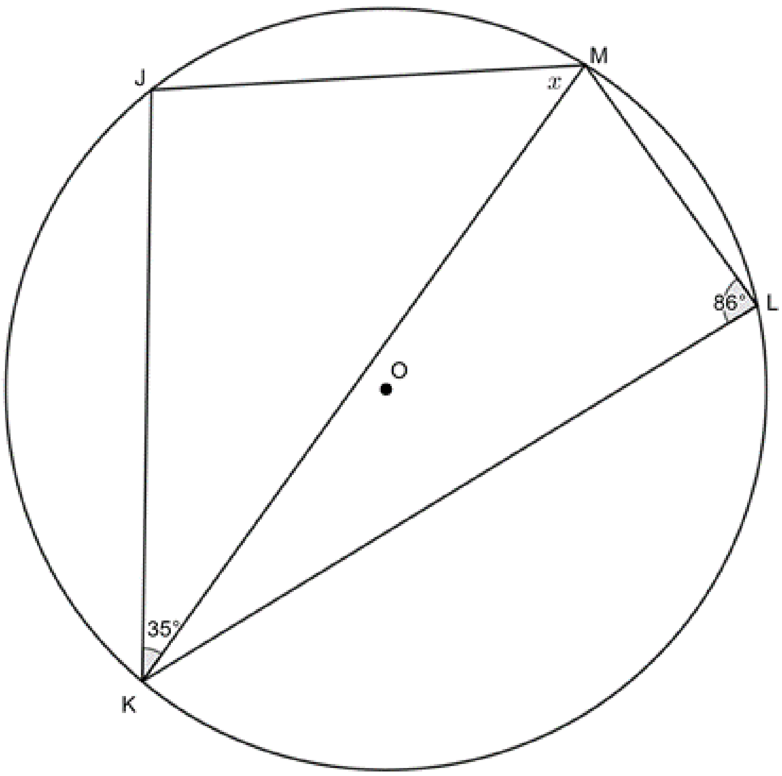
Exercise 3.1

1. Find the value of the unknown angles:

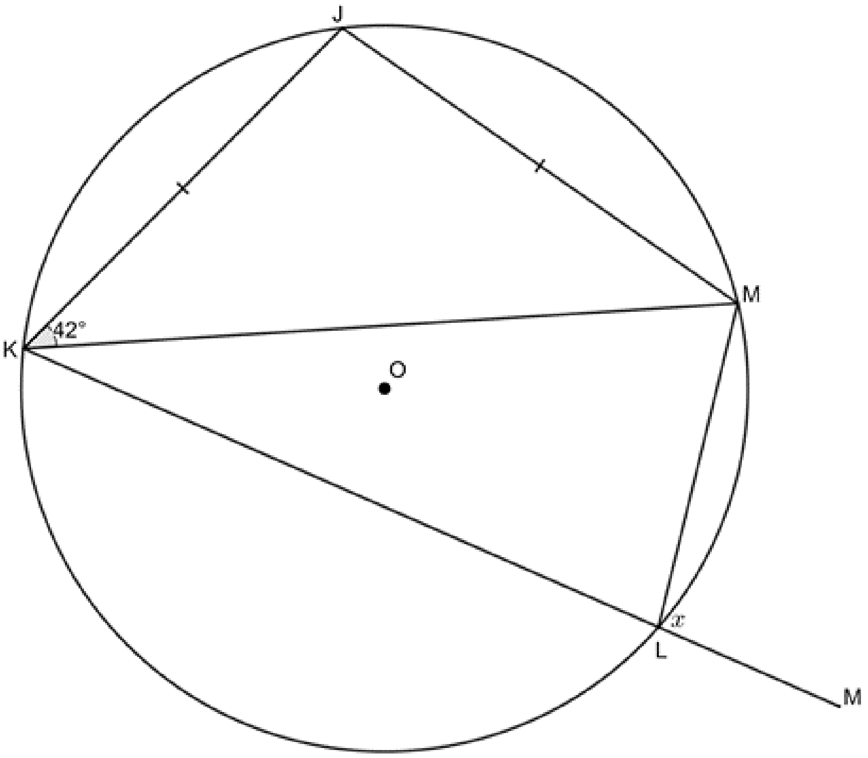
   a.



b.

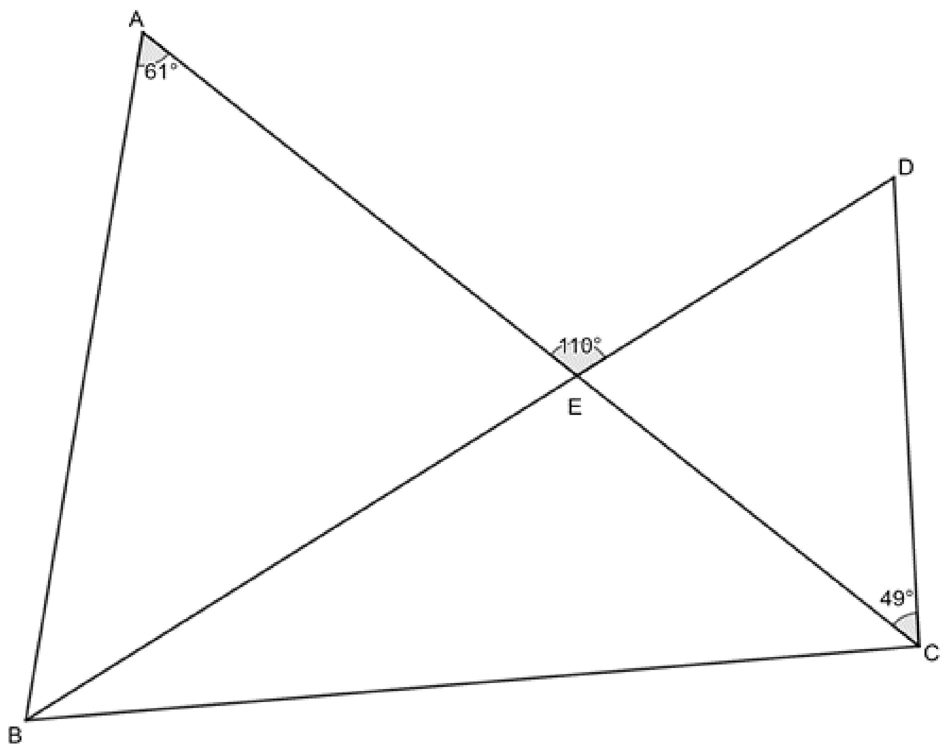


c.

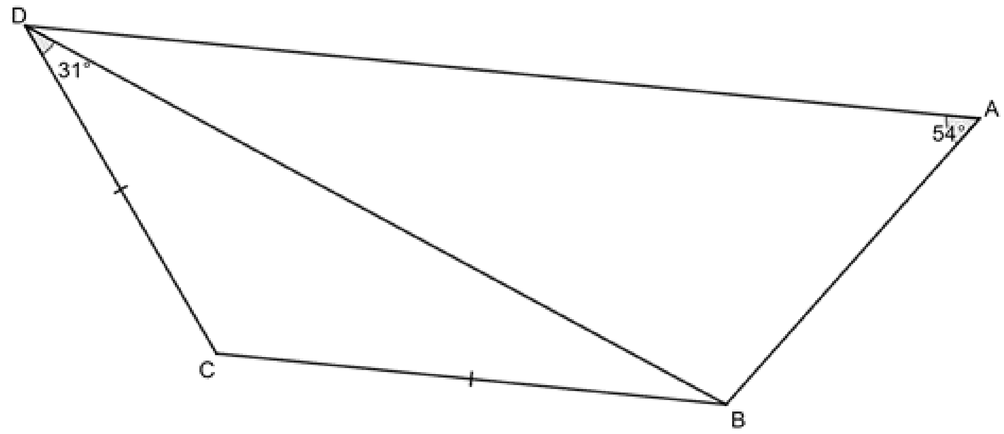


2. In each case, determine if $ABCD$ is a cyclic quadrilateral:

   a.



   b.



The full solutions are at the end of the unit.

Summary

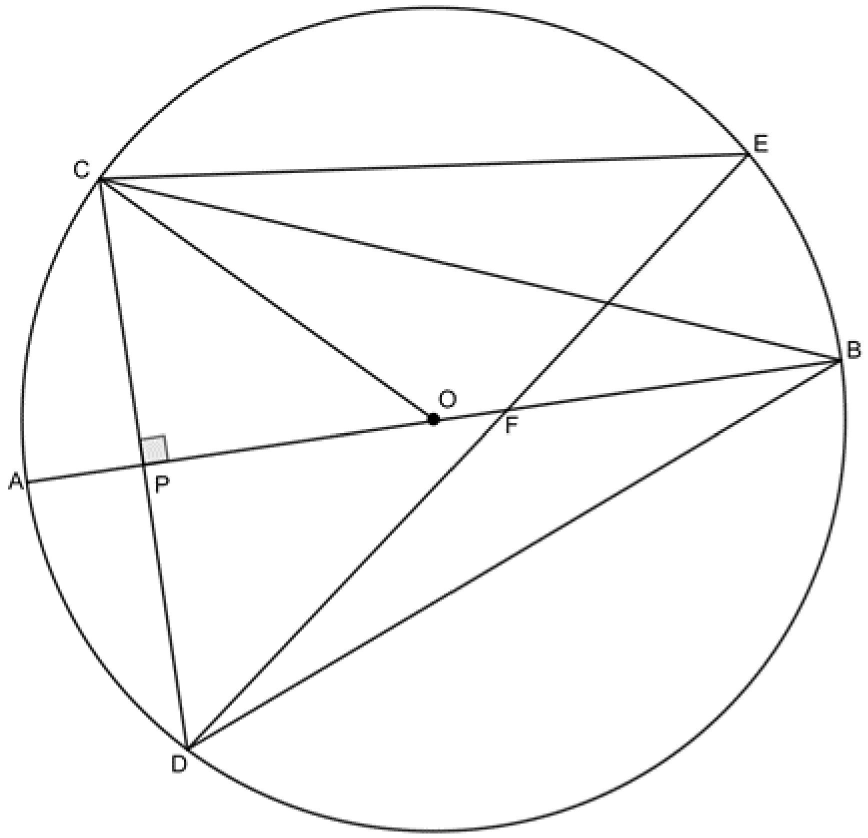
In this unit you have learnt the following:

- That a cyclic quadrilateral is any four-sided shape whose vertices all lie on the circumference of the same circle.
- The opposite angles of a cyclic quadrilateral are supplementary.
- The exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.
- A quadrilateral can be proven to be a cyclic quadrilateral if you can show that:
  - the opposite angles are supplementary
  - the exterior angle is equal to the interior opposite angle
  - the angles subtended by one side of the quadrilateral are equal.

Unit 3: Assessment

Suggested time to complete: 25 minutes

1. O is the centre of the circle with diameter AB. $CD \perp AB$ at P and chord DEcuts AB at F. $CB = DB$.



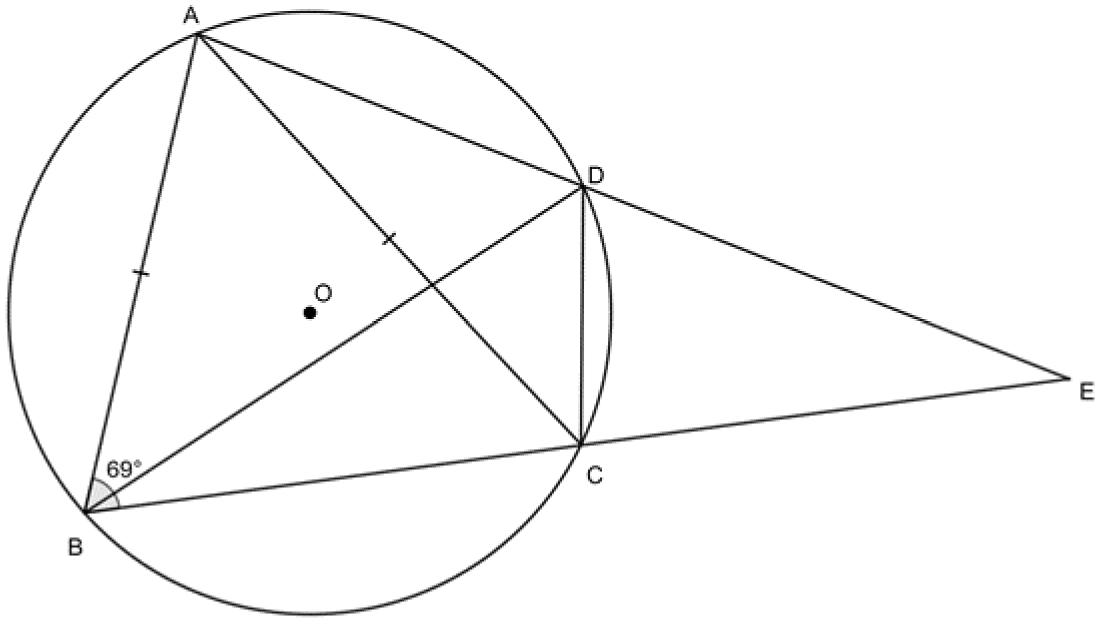
Prove that:

a. $C\hat{B}P = D\hat{B}P$

b. $C\hat{E}D = 2 \times C\hat{B}A$

c. $A\hat{B}D = \frac{1}{2} C\hat{O}A$

2. A, B, C and D are points on circle centre O. $A\hat{B}C = 69^\circ$. AD and BC are extended to meet at E. $AB = AC$.



a. Determine, with reasons, two more angles equal to $A\hat{B}C$.
b. If $B\hat{A}D = 82^\circ$, calculate $B\hat{C}D$, $C\hat{E}D$ and $A\hat{B}D$.

The full solutions are at the end of the unit.

Unit 3: Solutions

Exercise 3.1

1.
   a.
   $$\begin{aligned} x &= 180^\circ - 74^\circ \quad \text{(opp } \angle\text{s in cyclic quad)} \\ &= 106^\circ \\ y &= 180^\circ - 85^\circ \quad \text{(opp } \angle\text{s in cyclic quad)} \\ &= 95^\circ \end{aligned}$$
   b.
   $$\begin{aligned} \hat{J} &= 180^\circ - 86^\circ \quad \text{(opp } \angle\text{s in cyclic quad)} \\ &= 94^\circ \\ x &= 180^\circ - 94^\circ - 35^\circ \quad (\angle\text{s in } \Delta \text{ suppl)} \\ &= 51^\circ \end{aligned}$$
   c.
   In ΔJKM:

$$K\hat{M}J = 42^\circ \quad \text{(isosc } \Delta)$$
$$\therefore K\hat{J}M = 180^\circ - 42^\circ - 42^\circ \quad (\angle\text{s in } \Delta \text{ suppl})$$
$$= 96^\circ$$
$$x = K\hat{J}M = 96^\circ \quad (\text{ext } \angle = \text{opp int } \angle \text{ in cyclic quad})$$

2.

a.

$$C\hat{E}D = 180^\circ - 110^\circ \quad (\angle\text{s on a str line})$$
$$= 70^\circ$$
$$B\hat{D}C = 180^\circ - 70^\circ - 49^\circ \quad (\angle\text{s in } \Delta \text{ suppl})$$
$$= 61^\circ$$
$$\therefore B\hat{A}E = C\hat{D}E$$

Therefore $ABCD$ is a cyclic quadrilateral ($\angle$s in same segment).

b.

$$C\hat{B}D = 31^\circ \quad \text{(isosc } \Delta)$$
$$\therefore B\hat{C}D = 180^\circ - 31^\circ - 31^\circ \quad (\angle\text{s in } \Delta \text{ suppl})$$
$$= 118^\circ$$
$$B\hat{C}D + B\hat{A}D = 118^\circ + 54^\circ = 172^\circ$$

Therefore $ABCD$ is not a cyclic quadrilateral.

Back to Exercise 3.1

Unit 3: Assessment

1.

a.

$$CB = DB \quad \text{(given)}$$
$$\therefore B\hat{C}D = C\hat{D}B \quad \text{(isosc } \Delta)$$
$$D\hat{P}O = 180^\circ - 90^\circ = 90^\circ \quad (\angle\text{s on str line suppl})$$
$$D\hat{P}O = C\hat{P}O$$
In $\Delta\ CBP$:
$$C\hat{B}P = 180^\circ - C\hat{P}O - B\hat{C}D \quad (\angle\text{s in } \Delta \text{ suppl})$$
$$= 90^\circ - B\hat{C}D$$
In $\Delta\ DBP$:
$$D\hat{B}P = 180^\circ - D\hat{P}O - C\hat{D}B \quad (\angle\text{s in } \Delta \text{ suppl})$$
$$= 90^\circ - C\hat{D}B$$
$$\therefore C\hat{B}P = D\hat{B}P$$

b.

$$C\hat{E}D = C\hat{B}D \quad (\angle\text{s in same seg})$$
$$\text{But } C\hat{B}P = D\hat{B}P \quad \text{(proven in a.)}$$
$$\therefore 2 \times C\hat{B}A = C\hat{B}D$$
$$\therefore C\hat{E}D = 2 \times C\hat{B}A$$

c.

$$C\hat{O}A = 2 \times C\hat{B}A \quad (\angle \text{ at centre} = 2\angle \text{ at circumference})$$
$$\text{But } C\hat{B}P = D\hat{B}P \quad \text{(proven in a.)}$$
$$\therefore D\hat{B}P = \frac{1}{2} C\hat{O}A$$

2.

a. $A\hat{C}B = A\hat{B}C = 69^\circ \quad \text{(isosc } \Delta)$

$C\hat{D}E = A\hat{B}C = 69^\circ \quad (\text{ext } \angle = \text{ opp int } \angle \text{ in cyclic quad})$

b. If $B\hat{A}D = 82^\circ$

$$B\hat{C}D = 180^\circ - B\hat{A}D \quad \text{(opp } \angle\text{s in cyclic quad)}$$
$$\therefore B\hat{C}D = 180^\circ - 82^\circ = 98^\circ$$
$$C\hat{D}E = 69^\circ \quad \text{(proven in a.)}$$
$$D\hat{C}E = B\hat{A}D = 82^\circ \quad \text{(ext } \angle = \text{ opp int } \angle \text{ in cyclic quad)}$$
$$C\hat{E}D = 180 - C\hat{D}E - D\hat{C}E \quad (\angle\text{s in } \Delta \text{ suppl)}$$
$$\therefore C\hat{E}D = 180^\circ - 69^\circ - 82^\circ = 29^\circ$$
$$A\hat{C}B = 69^\circ \quad \text{(proven in a.)}$$
$$D\hat{C}E = 82^\circ \quad \text{(proven above)}$$
$$A\hat{C}D = 180^\circ - A\hat{C}B - D\hat{C}E \quad (\angle\text{s on a str line suppl)}$$
$$\therefore A\hat{C}D = 180^\circ - 69^\circ - 82^\circ = 29^\circ$$
$$A\hat{B}D = A\hat{C}D = 29^\circ \quad (\angle\text{s in same seg)}$$

Back to Unit 3: Assessment

Media Attributions

Unit 4: Tangent to circle theorems

DYLAN BUSA

Unit outcomes

By the end of this unit you will be able to:

- Apply the theorem tangent perpendicular to radius at point of contact.
- Apply the converse theorem of tangent perpendicular to radius.
- Apply the theorem two tangents drawn from the same point outside a circle are equal.
- Apply the tangent-chord theorem.

What you should know

Before you start this unit, make sure you can:

- State and use all the circle theorems covered in unit 2:
  - A line drawn perpendicular to a chord from the centre of the circle bisects the chord.
  - A line drawn from the circle centre to the mid-point of a chord is perpendicular to the chord.
  - The angle subtended by an arc or chord at the centre of a circle is twice the size of the angle subtended at the circumference.
  - The diameter of a circle subtends a right angle at the circumference.
  - If an angle subtended by a chord at a point on the circumference is a right angle, then the chord is a diameter.
  - Angles subtended by the same arc or chord in the same segment of a circle (on the same side of the chord) are equal.
- State and use all the cyclic quadrilateral theorems covered in unit 3:
  - The opposite angles of a cyclic quadrilateral are supplementary.
  - The exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.
- Prove that a quadrilateral is a cyclic quadrilateral. Refer to unit 3 if you need help with this.

Introduction

We have come across the concept of a tangent several times so far, most recently in subject outcome 3.1. We therefore know that a tangent is a straight line that touches a curve at one point and one point only.

In subject outcome 3.1, we dealt with finding the equation of a tangent to a circle. We noted that the tangent is perpendicular to the line drawn from the centre of the circle to the circumference at the point of tangency. We will use this fact as the basis for one of the theorems in this unit.

Tangent to circle theorems

We already know that the radius of a circle is perpendicular to a tangent at the point of contact (the point of tangency) (figure 1). This will be our first theorem in this unit.

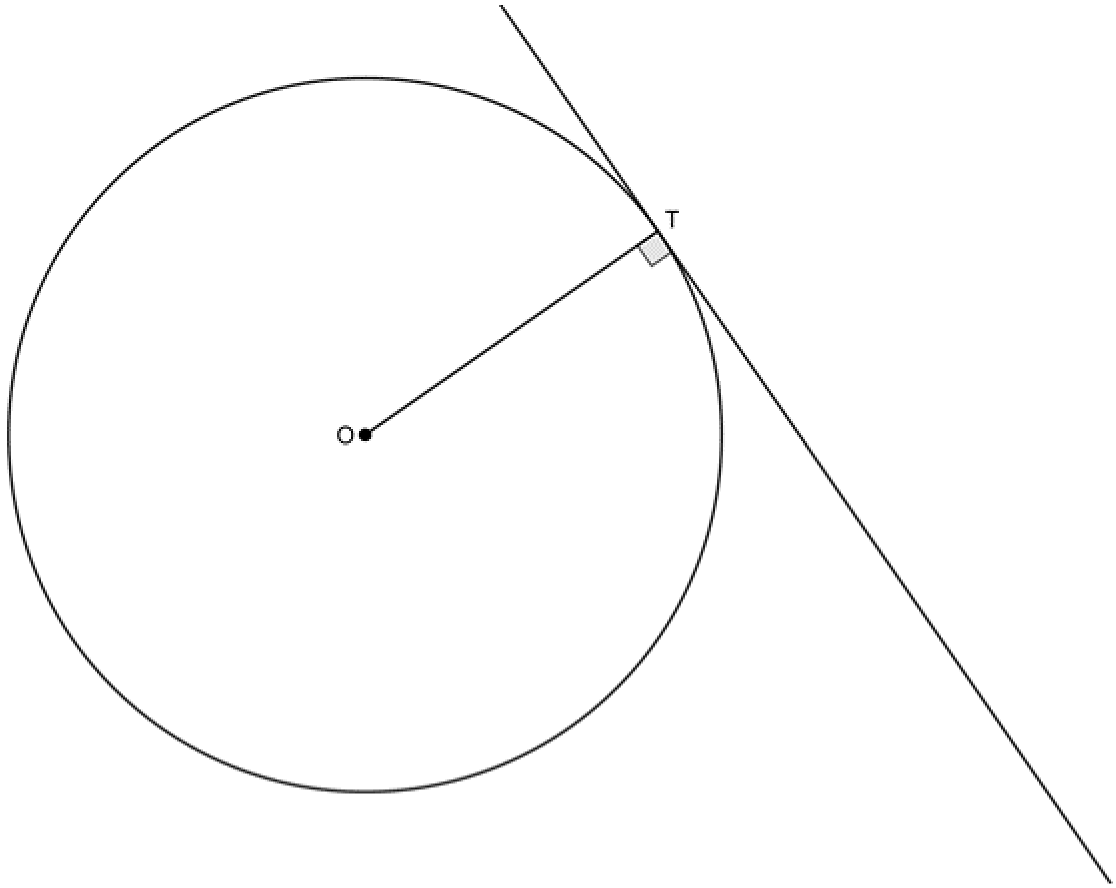


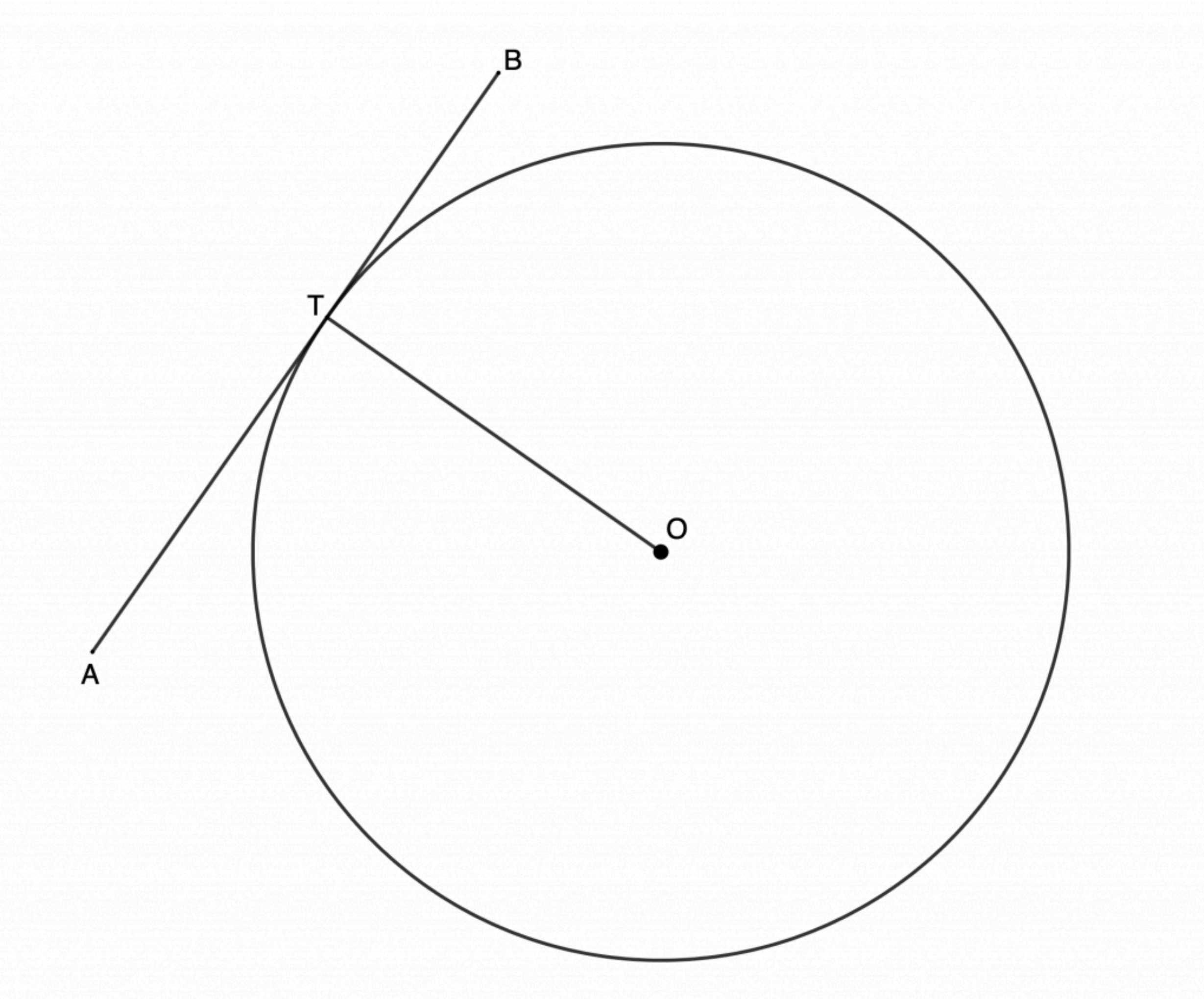
Figure 1: The radius is perpendicular to the tangent at the point of tangency

You do not need to be able to prove any of the tangent to circle theorems yourself. You can simply assume that they are true. The following sections explain the theorems that you need to be able to state and use. Note that they are numbered only for reference purposes and continue the numbering from unit 3. These theorems do not have official numbers.

Theorem 7

This theorem covers what you've already learnt about radii and tangents to circles.

> **Theorem 7: The radius of a circle is perpendicular to the tangent at the point of tangency**
>
> If a tangent is drawn to a circle, then it is perpendicular to the radius at the point of contact.



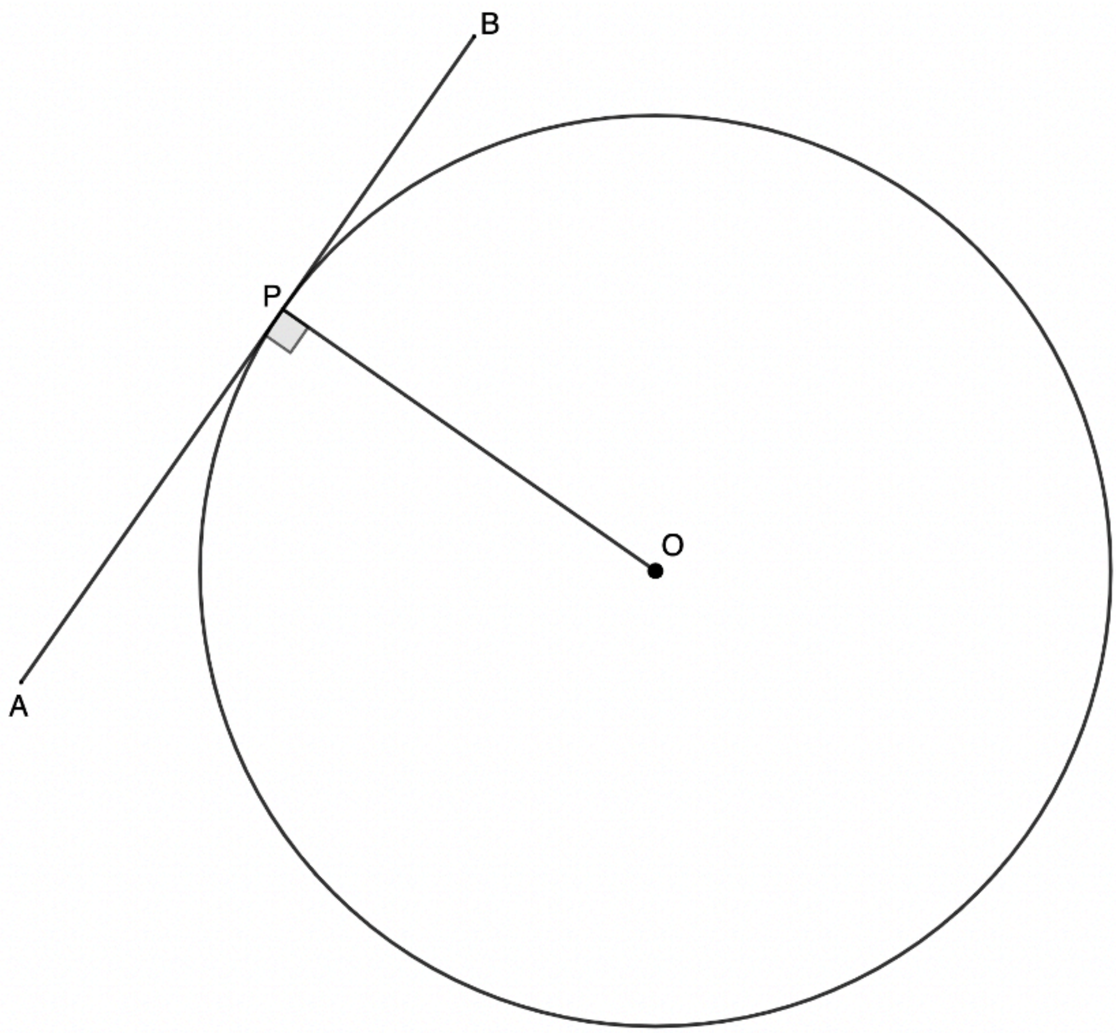
The tangent AT is perpendicular to the radius OT at point T.

Reason: radius $\perp$ tangent

Now let's look at the converse to this theorem.

Converse to theorem 7: If a line is perpendicular to a radius at a point on the circumference of a circle, then it is a tangent to the circle

If a line is drawn perpendicular to a radius at the point where the radius meets the circle, then it is a tangent to the circle at that point.

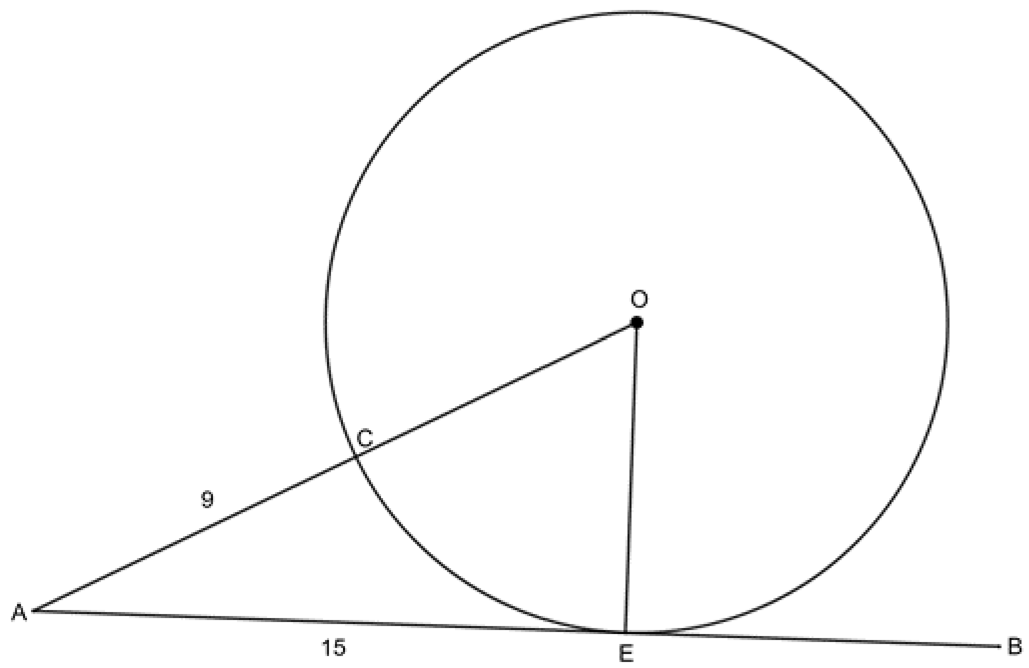


If line AB is perpendicular to radius OP at point P, then AB is a tangent to the circle at P.

Reason: radius $\perp$ line at point of contact

Example 4.1

AB is a tangent to circle centre O at E. C is a point on AO on the circumference of the circle. $AC = 9$ and $AE = 15$. Determine the length of the radius of the circle.



Solution

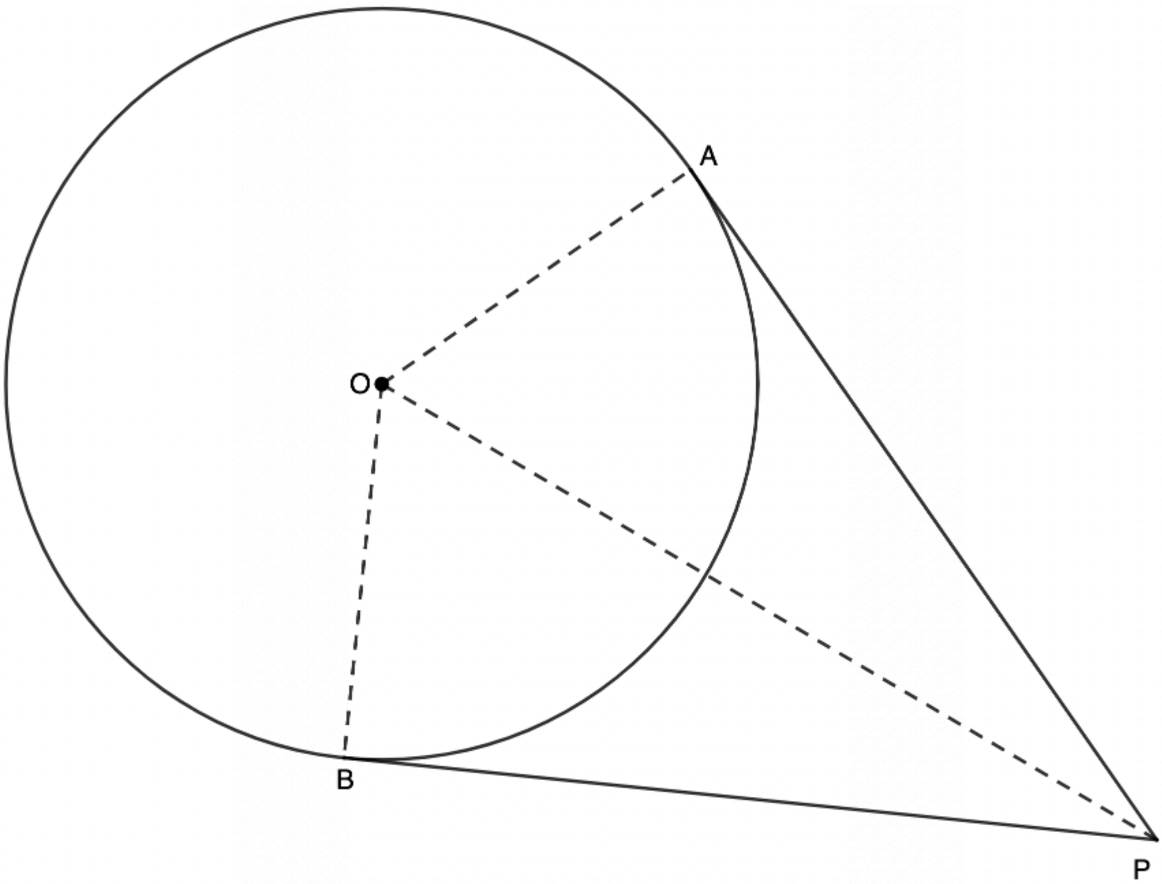
$$\begin{aligned} O\hat{E}A &= 90^\circ \quad \text{(radius} \perp \text{tangent)} \\ \therefore AO^2 &= OE^2 + AE^2 \quad \text{(Pythagoras)} \\ \text{Let } OE &= x \\ \therefore AO &= 9 + x \\ \therefore (9+x)^2 &= x^2 + 15^2 \\ \therefore 81 + 18x + x^2 &= x^2 + 225 \\ \therefore 18x &= 144 \\ \therefore x &= \frac{144}{18} \\ &= 8 \end{aligned}$$

Theorem 8

This next theorem looks at two tangents to a circle from the same point.

Theorem 8: Two tangents from the same point outside a circle

If two tangents are drawn from the same point outside a circle, then they are equal in length.

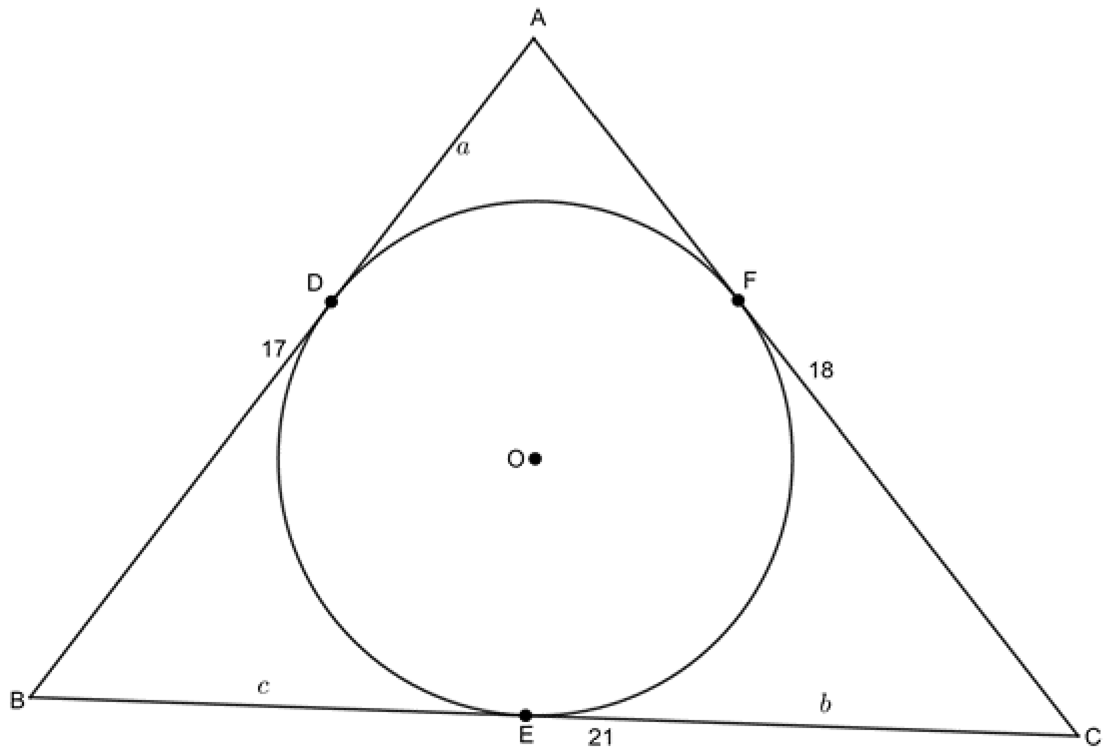


If tangents PA and PB are drawn from the same point P, then $PA = PB$.

Reason: tangents from same point are equal

Example 4.2

$AB = 17$, $AC = 18$ and $BC = 21$. AB is a tangent at D. AC is a tangent at F. BC is a tangent at E. Determine the lengths of a, b and c.



Solution

$AD = AF = a$ (tangents from same point $=$)

$CF = CE = b$ (tangents from same point $=$)

$BE = BD = c$ (tangents from same point $=$)

But

$AD + BD = 17$ (given)

$\therefore a + c = 17$ (1)

$AF + CF = 18$ (given)

$\therefore a + b = 18$ (2)

$CE + BE = 21$ (given)

$\therefore b + c = 21$ (3)

We have three equations that we can solve simultaneously.

Subtract (2) from (1):

$c - b = -1$

$\therefore c = b - 1$ (4)

Substitute (4) into (3):

$b + (b - 1) = 21$

$\therefore 2b = 22$

$\therefore b = 11$

Substitute $b = 11$ into (4):

$c = 11 - 1 = 10$

Substitute $c = 10$ into (1):

$a + 10 = 17$

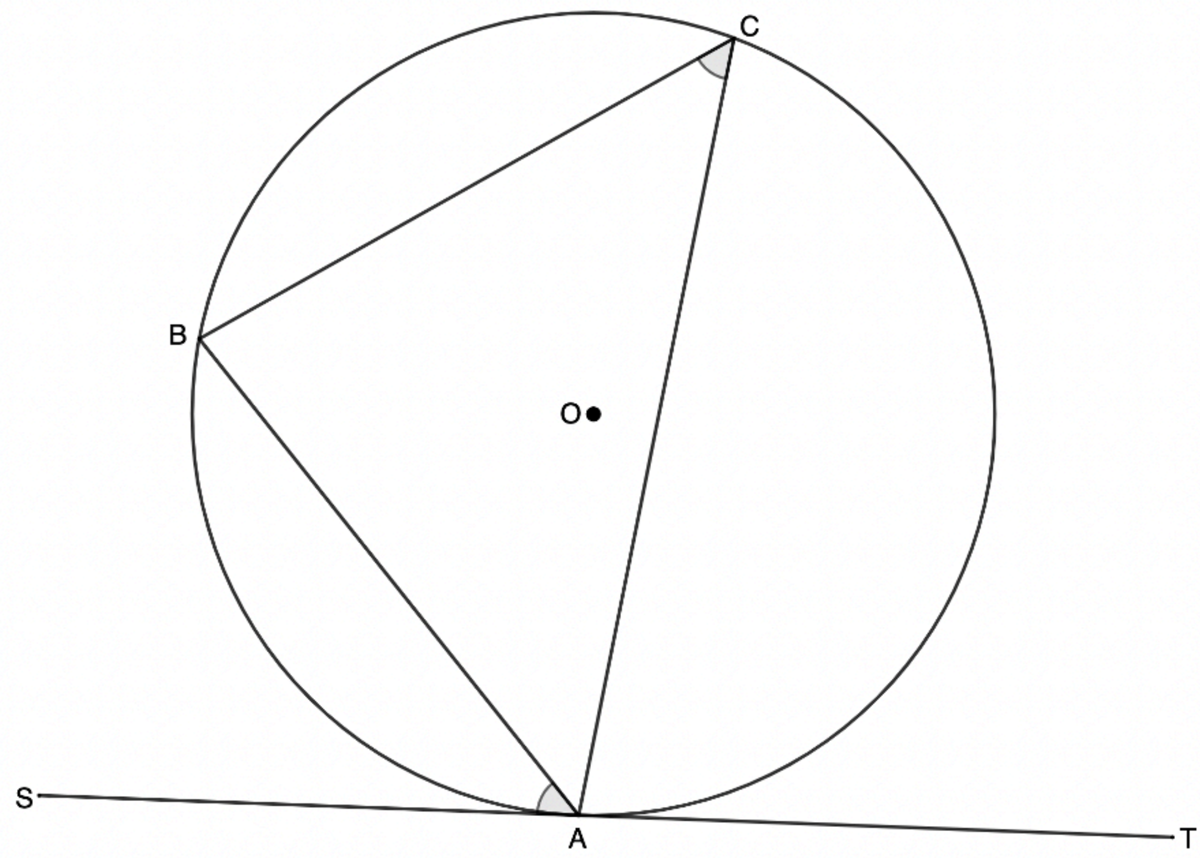
$\therefore a = 7$

Theorem 9

Let's now look at the tan-chord theorem.

Theorem 9: Tan-chord theorem

The angle between a tangent to a circle and a chord drawn at the point of contact, is equal to the angle which the chord subtends in the alternate segment.



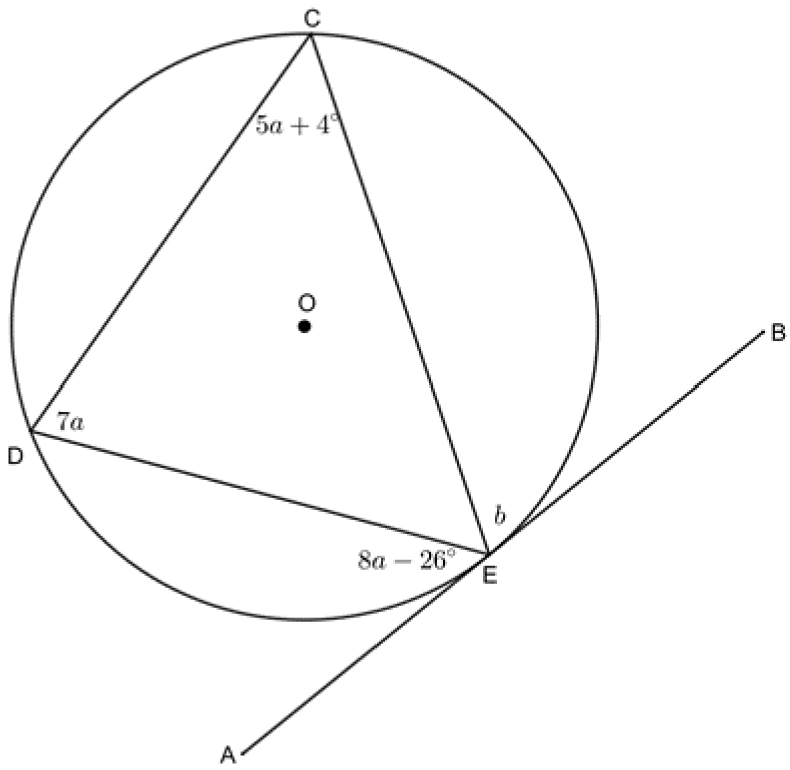
If ST is a tangent to the circle at A, then $S\hat{A}B$ (the angle between the tangent and the chord) is equal to $A\hat{C}B$ (the angle subtended by the chord in the alternate segment).

Reason: tan. chord theorem

Note: The chord must meet the circumference at the point of tangency. The angle subtended by the chord must be in the alternate segment of the circle (i.e. not the segment nearest to the tangent).

Example 4.3

AB is a tangent to the circle at E. Determine the values of a and b.

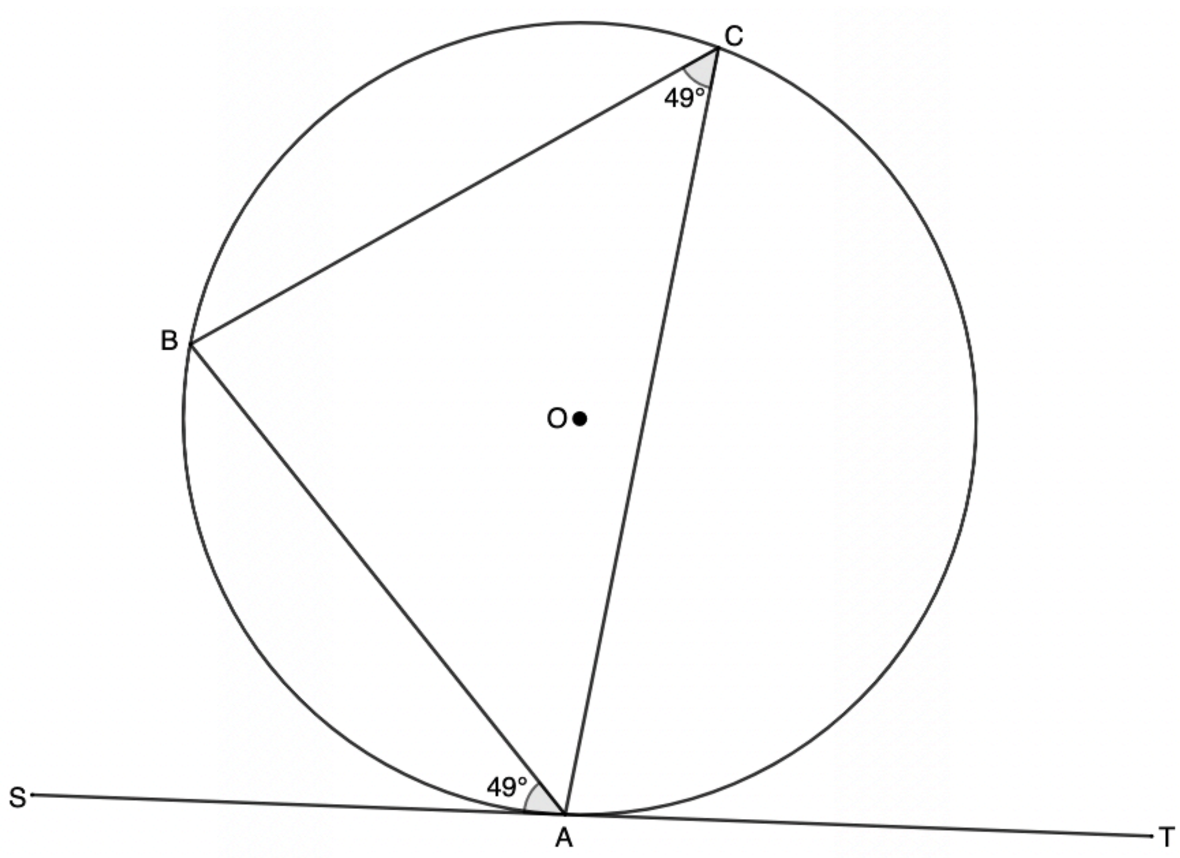


Solution

$$A\hat{E}D = D\hat{C}E \quad \text{(tan. chord theorem)}$$
$$\therefore 8a - 26^{\circ} = 5a + 4^{\circ}$$
$$\therefore 3a = 30^{\circ}$$
$$\therefore a = 10^{\circ}$$
$$B\hat{E}C = E\hat{D}C \quad \text{(tan. chord theorem)}$$
$$\therefore b = 7a = 7 \times 10^{\circ} = 70^{\circ}$$

Converse to theorem 9: Angle in opposite segment equal

We can prove that a line is a tangent to a circle at a point if we can show that the angle between the line and a chord drawn from that point on the line is equal to the angle subtended by the chord in the alternate segment.

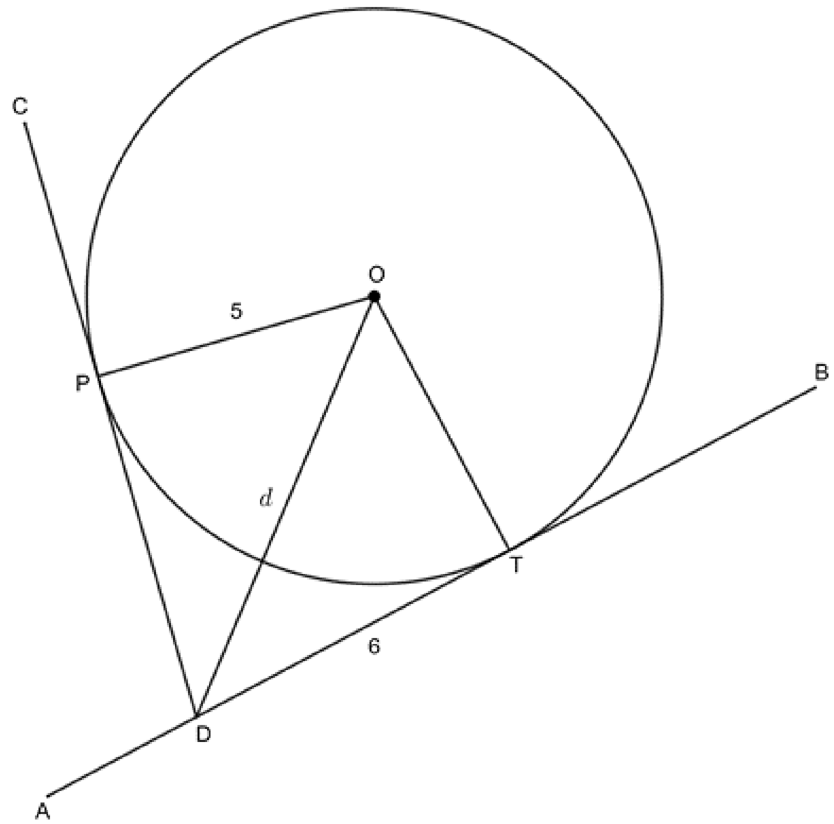


If $S\hat{A}B$ (the angle between the line and the chord) is equal to $A\hat{C}B$ (the angle subtended by the chord in the alternate segment), then ST is a tangent to circle ABC at A.

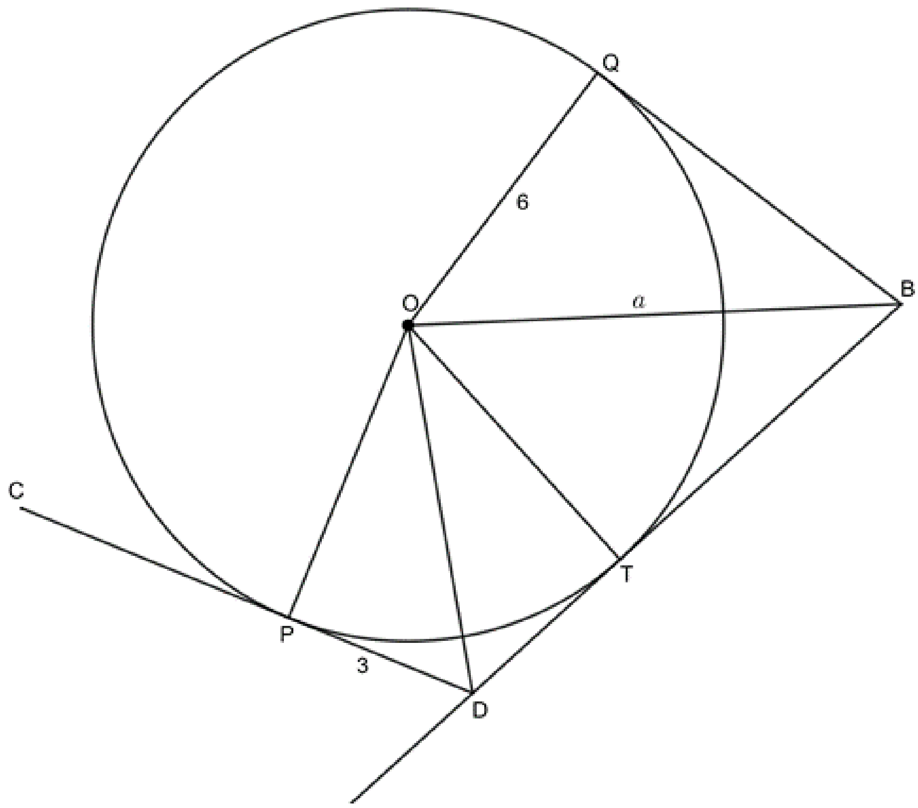
Reason: $\angle$ between tangent and chord $=$ $\angle$ in opp seg

Exercise 4.1

1. Determine the value of the unknown lengths:
   a.

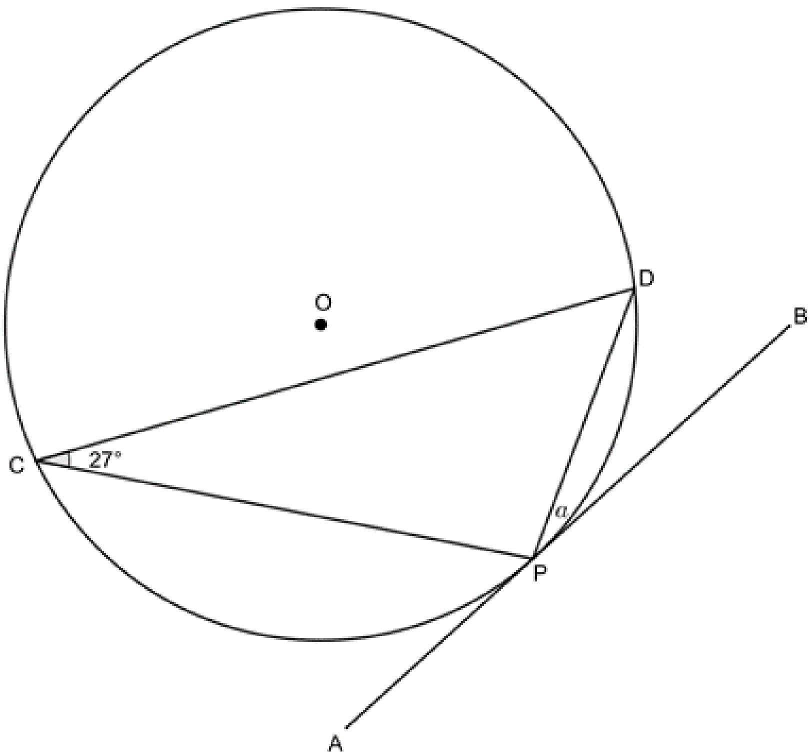


b. $BD = 13$

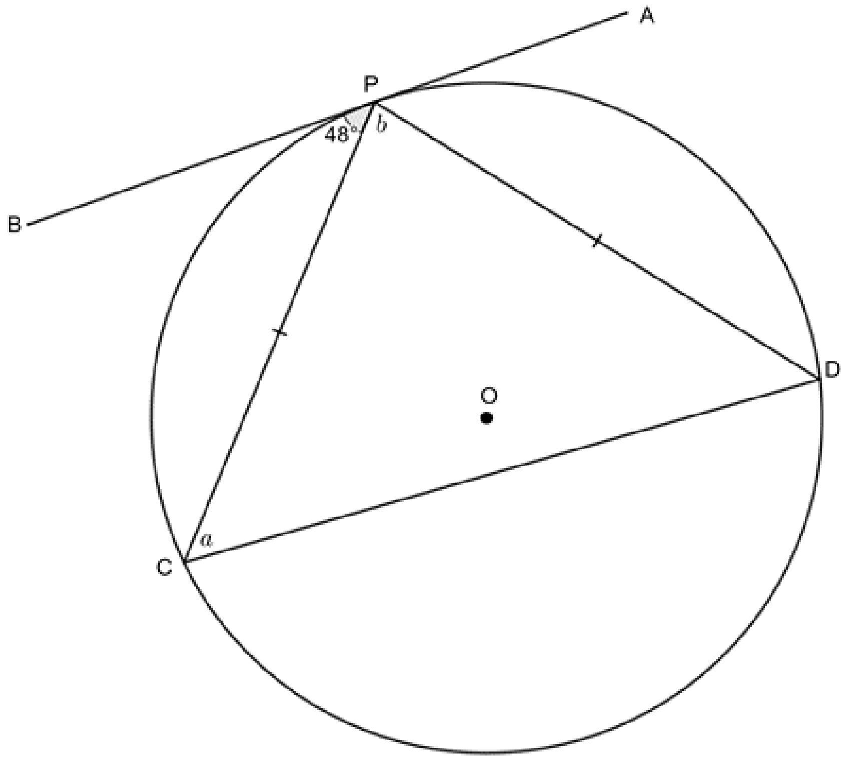


2. Determine the size of the unknown angles:

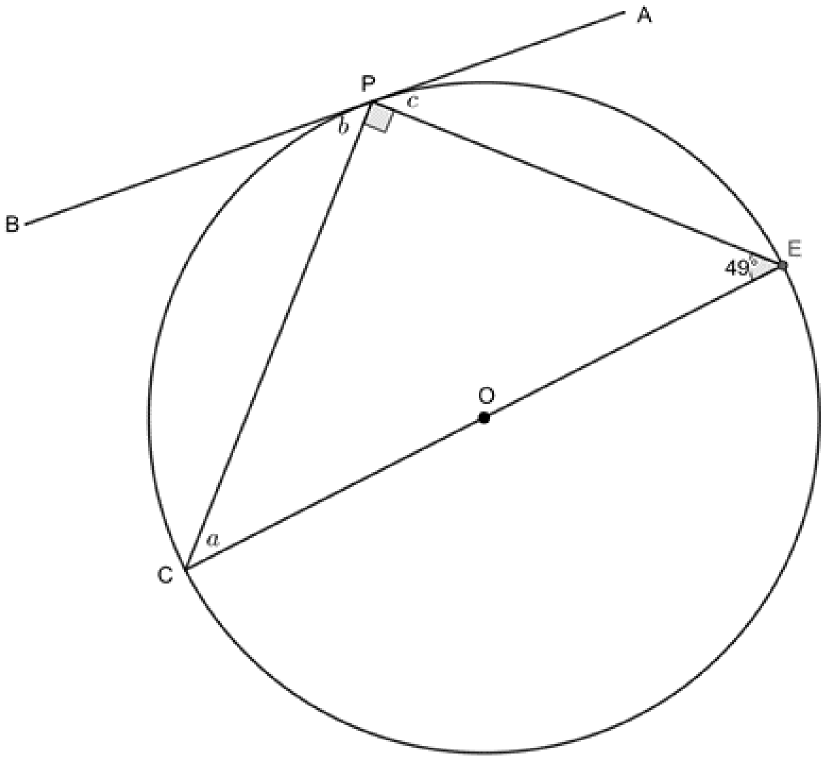
   a.



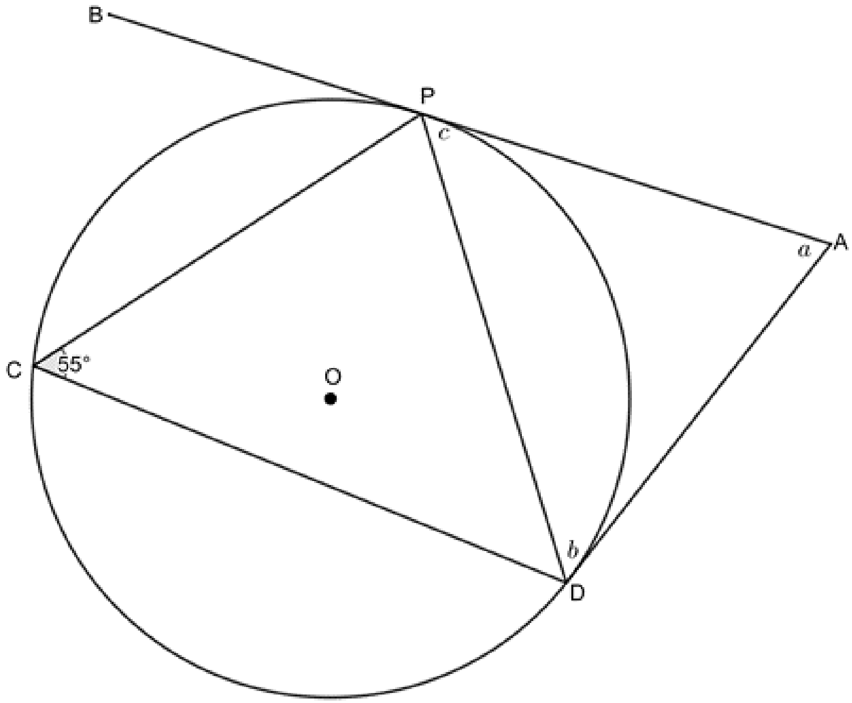
b.



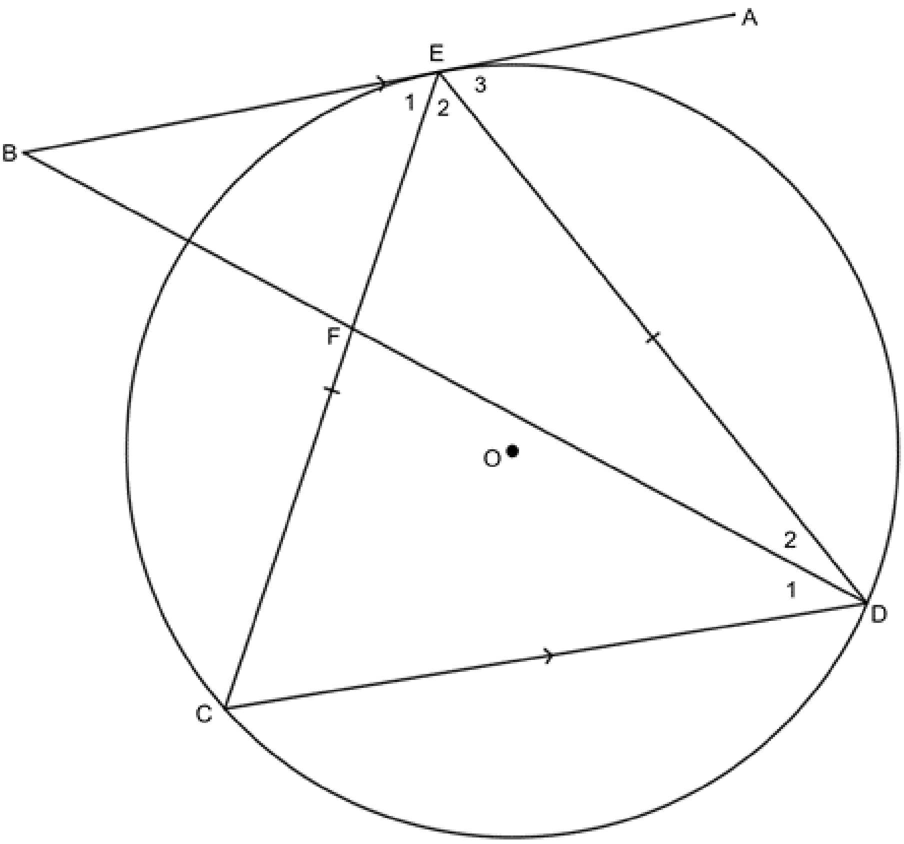
c.



d.



3. Given $EC = ED$, $AB \parallel CD$ and $\hat{E}_2 = \hat{D}_1$. Prove:



a. AEB is a tangent to circle CDE.

b. DE is a tangent to the circle EFB.

The full solutions are at the end of the unit.

Summary

In this unit you have learnt the following:

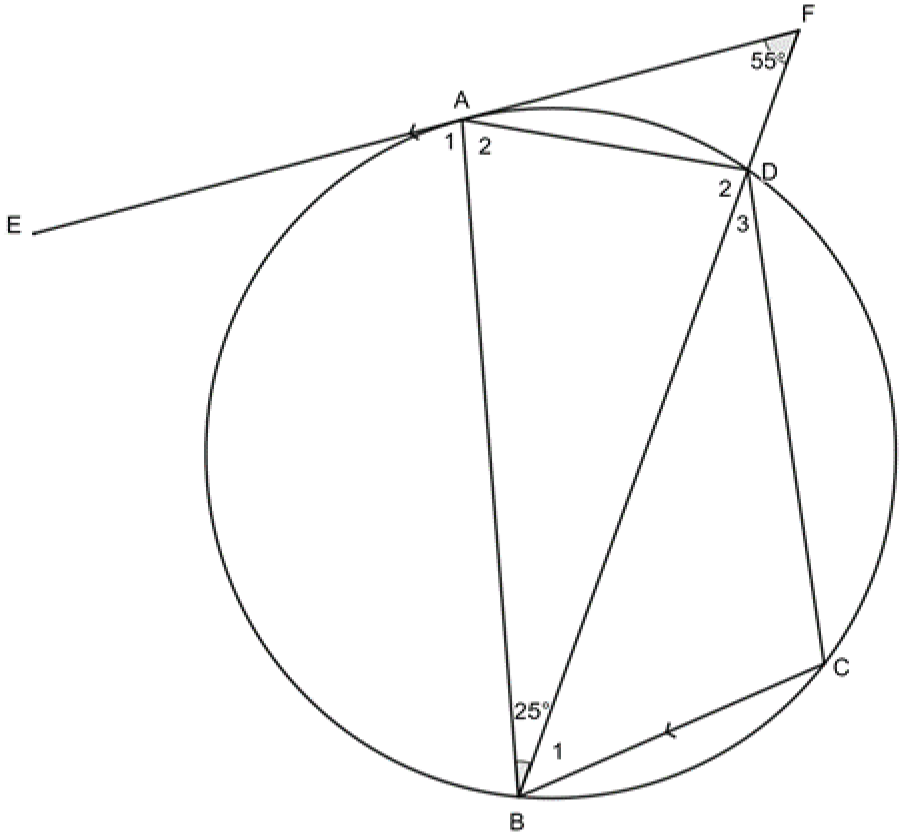
- That the tangent to a circle is perpendicular to the radius at point of contact.
- That if a line is perpendicular to a radius, then the line is a tangent to the circle at that point.
- That two tangents drawn from the same point outside a circle are equal in length.
- That the angle between a tangent to a circle and a chord draw from the point of contact is equal to the angle subtended by the chord in the alternate segment.
- That if the angle between a line and a chord drawn from the line is equal to the angle the chord subtends in the alternate segment, then the line is a tangent to the circle at that point.

Unit 4: Assessment

Suggested time to complete: 35 minutes

Question 1 adapted from NC(V) Mathematics Level 4 November 2011 question 1.3

1. Given below is a circle with a cyclic quadrilateral $ABCD$ and EF a tangent at A. $EF \parallel BC$ and $\hat{F} = 55°$. $A\hat{B}D = 25°$.

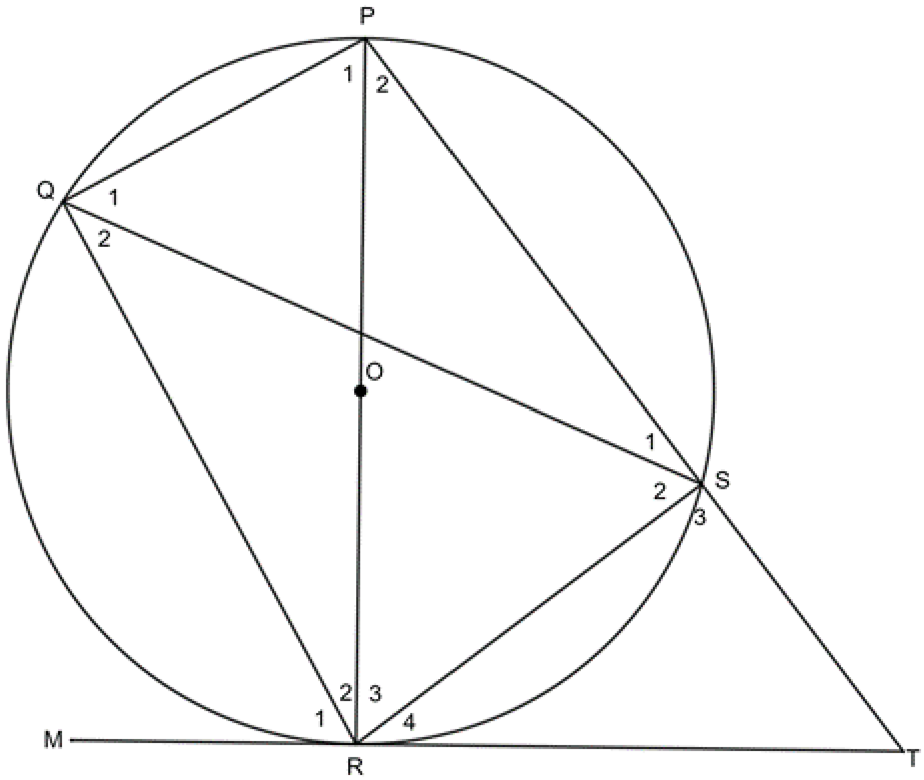


Calculate, with reasons, the magnitudes of the following angles:

a. $D\hat{A}F$
b. $F\hat{D}A$
c. $\hat{D}_2$
d. $\hat{A}_2$
e. $\hat{C}$
f. $\hat{B}_1$

Question 2 adapted from NC(V) Mathematics Level 4 November 2013 question 1.4

2. In the figure, O is the centre of the circle. P, Q, R and S are points on the circumference of the circle. PR is a diameter of the circle and MRT is a tangent at R. $\hat{S}_2 = 61°$ and $\hat{R}_3 = 52°$.



Determine, with reasons, the sizes of:

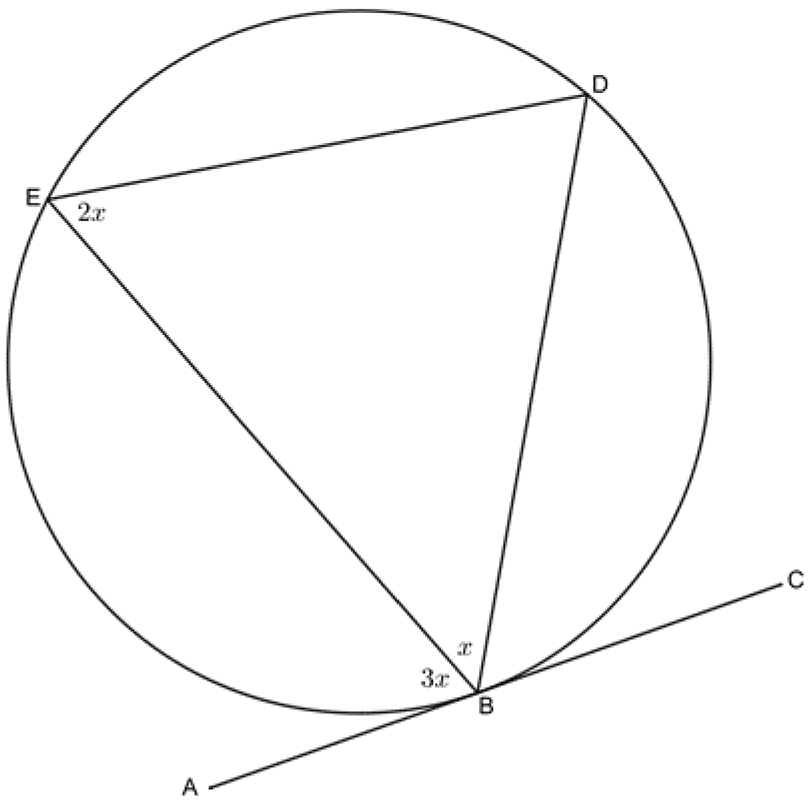
a. $\hat{R}_1$

b. $\hat{R}_4$

c. $\hat{S}_3$

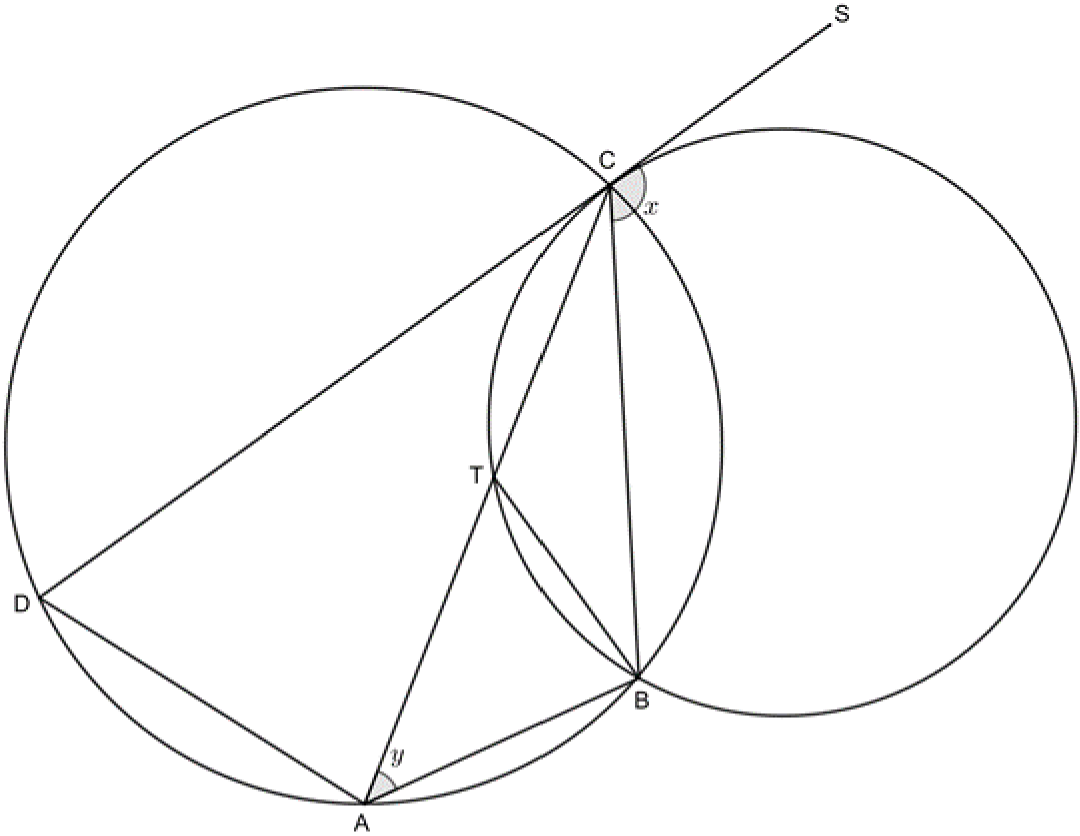
Question 3 adapted from NC(V) Mathematics Level 4 November 2015 question 1.2

3. Determine the value of x in the following diagram:



Question 4 adapted from NC(V) Mathematics Level 4 November 2012 question 1.3

4. In the following figure, $ABCD$ is a cyclic quadrilateral. DC is a tangent to a circle through B, C and T. AC passes through T on the circle. DC is produced to S and TB is joined.
   Let $B\hat{C}S = x$ and $B\hat{A}C = y$.



a. Name, giving reasons, TWO other angles that are equal to x.

b. Determine, giving reasons, the value of $A\hat{B}T$ in terms of x and y.

c. Hence, prove that DA is a tangent to circle ABT.

The full solutions are at the end of the unit.

Unit 4: Solutions

Exercise 4.1

1.

a.

$$O\hat{P}D = 90^\circ \quad \text{(radius} \perp \text{tangent)}$$
$$DT = DP = 6 \quad \text{(tangents from same point equal)}$$
$$d^2 = OP^2 + DP^2 \quad \text{(Pythagoras)}$$
$$\therefore d^2 = 25 + 36 = 61$$
$$\therefore d = \sqrt{61}$$

b. $DB = 13 \quad \text{(given)}$

$$PD = DT = 3 \quad \text{(tangents from same point equal)}$$
$$\therefore TB = 13 - 3 = 10$$
$$TB = QB = 10 \quad \text{(tangents from same point equal)}$$

$O\hat{Q}B = 90^\circ \quad \text{(radius} \perp \text{tangent)}$
$\therefore a^2 = 6^2 + 10^2 = 136$
$\therefore a = 2\sqrt{34}$

2.

a. $a = 27^\circ \quad \text{(tan. chord theorem)}$

b.
$P\hat{D}C = 48^\circ \quad \text{(tan. chord theorem)}$
$P\hat{D}C = P\hat{C}D \quad \text{(isosc } \Delta)$
$\therefore a = 48^\circ$
$A\hat{P}D = 48^\circ \quad \text{(tan. chord theorem)}$
$b = 180^\circ - 48^\circ - 48^\circ \quad (\angle\text{s on str line suppl)}$
$\therefore b = 84^\circ$

c. $a = 180^\circ - 90^\circ - 49^\circ = 41^\circ \quad (\angle\text{s in } \Delta \text{ suppl)}$
$b = 49^\circ \quad \text{(tan. chord theorem)}$
$c = 41^\circ \quad \text{(tan. chord theorem)}$

d. $c = 55^\circ \quad \text{(tan. chord theorem)}$
$b = 55^\circ \quad \text{(tan. chord theorem)}$
$a = 180^\circ - 55^\circ - 55^\circ = 70^\circ \quad (\angle\text{s in } \Delta \text{ suppl)}$

3.

a.
$\hat{E}_1 = D\hat{C}E \quad \text{(alt } \angle\text{s } =;\ AB \parallel CD)$
$D\hat{C}E = \hat{D}_{1+2} = C\hat{D}E \quad \text{(isosc } \Delta)$
$\therefore \hat{E}_1 = C\hat{D}E$
$\therefore AEB$ is a tangent to circle $CDE \quad (\angle$ between tangent and chord $= \angle$ in opp seg$)$

b.
$\hat{E}_2 = \hat{D}_1 \quad \text{(given)}$
$E\hat{B}F = \hat{D}_1 \quad \text{(alt } \angle\text{s } =;\ AB \parallel CD)$
$\therefore E\hat{B}F = \hat{E}_2$
$\therefore DE$ is a tangent to circle EFB $\quad (\angle$ between tangent and chord $= \angle$ in opp seg$)$

Back to Exercise 4.1

Unit 4: Assessment

1.

a. $D\hat{A}F = 25^\circ \quad \text{(tan. chord theorem)}$

b.
$F\hat{D}A = 180^\circ - A\hat{F}D - D\hat{A}F \quad (\angle\text{s in } \Delta \text{ suppl)}$
$= 180^\circ - 55^\circ - 25^\circ$
$= 100^\circ$

c.
$\hat{D}_2 = 180^\circ - F\hat{D}A \quad (\angle\text{s on str lin suppl)}$
$= 180^\circ - 100^\circ$
$= 80^\circ$

d.

$$\hat{A}_2 = 180^\circ - A\hat{B}D - \hat{D}_2 \quad (\angle\text{s in } \Delta \text{ suppl})$$
$$= 180^\circ - 25^\circ - 80^\circ$$
$$= 75^\circ$$

e.

$$\hat{C} = 180^\circ - \hat{A}_2 \quad (\text{opp } \angle\text{s of cyclic quad suppl})$$
$$= 180^\circ - 75^\circ$$
$$= 105^\circ$$

f. $\hat{B}_1 = \hat{F} = 55^\circ \quad (\text{alt } \angle\text{s } =;\ EF \parallel BC)$

2.

a. $\hat{R}_1 = S_2 = 61^\circ \quad (\text{tan. chord theorem})$

b.

$$\hat{R}_3 = \hat{Q}_1 = 52^\circ \quad (\angle\text{s in same seg})$$
$$\hat{Q}_1 + \hat{Q}_2 = 90^\circ \quad (\angle\text{s in semi-circle})$$
$$\therefore \hat{Q}_2 = 38^\circ$$
$$R_4 = \hat{Q}_2 = 38^\circ \quad (\text{tan. chord theorem})$$

c. $\hat{S}_3 = \hat{Q}_{1+2} = 90^\circ \quad (\text{ext } \angle = \text{opp int } \angle \text{ in cyclic quad})$

3.

$$C\hat{B}D = B\hat{E}D = 2x \quad (\text{tan. chord theorem})$$
$$A\hat{B}E + E\hat{B}D + C\hat{B}D = 180^\circ \quad (\angle\text{s on str line suppl})$$
$$\therefore 3x + x + 2x = 180^\circ$$
$$\therefore 6x = 180^\circ$$
$$\therefore x = 30^\circ$$

4.

a. $C\hat{T}B = x \quad (\text{tan. chord theorem})$

$B\hat{A}D = x \quad (\text{ext } \angle = \text{opp int } \angle \text{ in cyclic quad})$

b.

$$A\hat{T}B = 180 - C\hat{T}B \quad (\angle\text{s on str line suppl})$$
$$\therefore A\hat{T}B = 180^\circ - x$$
$$A\hat{B}T = 180^\circ - A\hat{T}B - B\hat{A}T \quad (\angle\text{s in } \Delta \text{ suppl})$$
$$\therefore A\hat{B}T = 180^\circ - (180^\circ - x) - y$$
$$= x - y$$

c.

$$B\hat{A}D = x \quad (\text{proven in a.})$$
$$\therefore C\hat{A}D = x - y$$
$$\therefore C\hat{A}D = A\hat{B}T$$

Therefore, DA is a tangent to circle ABT ($\angle$ between tangent and chord $= \angle$ in opp seg).

Back to Unit 4: Assessment

Media Attributions

SUBJECT OUTCOME X

SPACE, SHAPE AND MEASUREMENT: SOLVE PROBLEMS BY CONSTRUCTING AND INTERPRETING TRIGONOMETRIC MODELS

Subject outcome

Subject outcome 3.3: Solve problems by constructing and interpreting trigonometric models

Learning outcomes

- Use the following compound angle identities:
  $\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$
  $\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$
  to derive and apply the following double angle identities:
  $\sin 2\alpha = 2\sin\alpha\cos\alpha$
  Formula does not parse
- Determine the specific solutions of trigonometric expressions using compound and double angle identities without a calculator (e.g. $\sin 120^\circ$, $\cos 75^\circ$, etc.).
- Use compound angle identities to simplify trigonometric expressions and to prove trigonometric equations.
- Determine the specific solutions of trigonometric equations by using knowledge of compound angles and identities.
  Note:
  - Solutions: $[0^\circ, 360^\circ]$.
  - Identities limited to:
    $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ and $\sin^2\theta + \cos^2\theta = 1$.
  - Double and compound angle identities are included.
  - Radians are excluded.
- Solve problems from a given diagram in two and three dimensions by applying the sine and cosine rule.
  Note: Area formula and compound angle identities are excluded.

Unit 1 outcomes

By the end of this unit you will be able to:

- Expand the compound angles of $\sin(\alpha \pm \beta)$ and $\cos(\alpha \pm \beta)$.
- Use the compound expansions to simplify expressions.
- Use compound angles to prove identities.

Unit 2 outcomes

By the end of this unit you will be able to:

- Solve equations involving double and compound angles.

Unit 3 outcomes

By the end of this unit you will be able to:

- Apply the sine rule correctly to solve 2-D and 3-D problems.
- Apply the cosine rule correctly to solve 2-D and 3-D problems.

Unit 4 outcomes

By the end of this unit you will be able to:

- Define radian measure.
- Convert from degrees to radians.
- Convert from radians to degrees.

Unit 1: Work with compound angles

DYLAN BUSA

Unit outcomes

By the end of this unit you will be able to:

- Expand the compound angles of $\sin(\alpha \pm \beta)$ and $\cos(\alpha \pm \beta)$.
- Use the compound expansions to simplify expressions.
- Use compound angles to prove identities.

What you should know

Before you start this unit, make sure you can:

- Simplify trigonometric expressions without a calculator, by using the special angles of $30°$, $45°$ and $60°$. Refer to level 3 subject outcome 3.3 unit 1 if you need help with this.
- Simplify trigonometric expression using the reduction formulae of $(90° \pm \theta)$, $(180° \pm \theta)$ and $(360° \pm \theta)$. Refer to level 3 subject outcome 3.3 unit 2 if you need help with this.
- State and apply the basic trigonometric identities of:
  - $\sin^2 x + \cos^2 x = 1$
  - $\tan x = \dfrac{\sin x}{\cos x}$.

  Refer to level 3 subject outcome 3.3 unit 3 if you need help with this.
- State and use the distance formula. Refer to level 2 subject outcome 3.3 unit 2 if you need help with this.

Introduction

There is a very important property in algebra that you have been using for many years now. It is called the distributive property of multiplication over addition (or just the distributive property). It says that the product of a number and the sum of two or more other numbers is equal to the sum of the products or that $x(a+b) = xa + xb$. You know it well, right?

Now it might seem reasonable to apply the distributive property to an expression such as $\cos(30° + 20°)$ and say that $\cos(30° + 20°) = \cos 30° + \cos 20°$. But is this true? Use a calculator to work out the values of the left-hand and the right-hand sides of the equation. Are they the same? Is the equation true?

You should have found that $\cos(30° + 20°) = \cos 50° = 0.643$ (rounded to three decimal places) and that $\cos 30° + \cos 20° = 1.806$ (rounded to three decimal places). The answers are not even close. Therefore, we know that:
$\cos(30° + 20°) \neq \cos 30° + \cos 20°$

There is a very good reason why we cannot apply the distributive property in this case. **We are not multiplying** $\cos$ by $(30° + 20°)$. Cosine is a function that we are applying to the sum of two angles. The distributive property applies only if we are finding a product (multiplying two or more numbers).

So, that leaves us with the question, 'what is $\cos(30^\circ + 20^\circ)$ equal to?'. We are going to answer this question in activity 1.1.

The compound angle identities

For this next activity, we are going to use the cosine rule and the distance formula. If you don't remember what these are, look at the references in the 'What you should know' section above before continuing.

Activity 1.1: What is cos(30° + 20°) equal to?

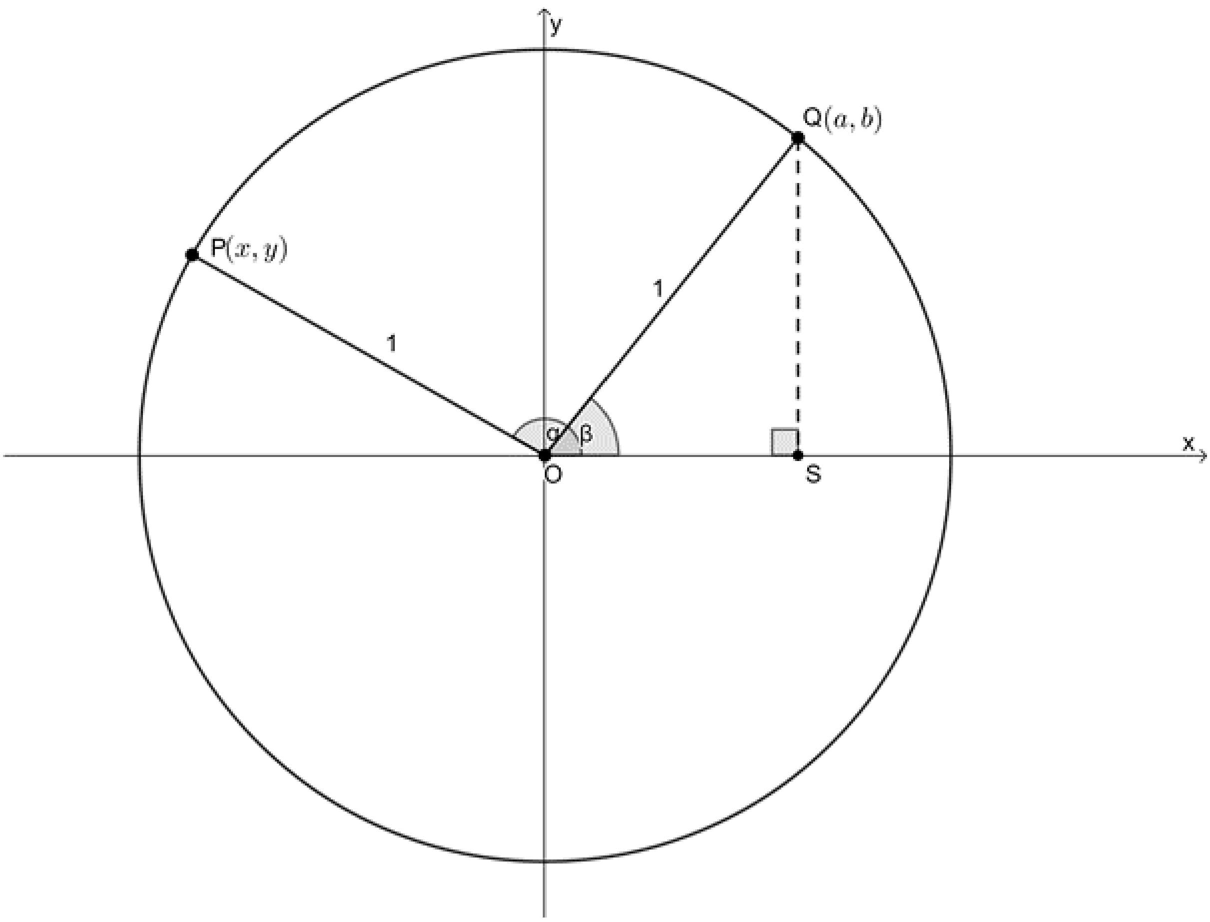
Time required: 10 minutes

What you need:

- a pen or pencil
- a blank piece of paper
- a calculator

What to do:

Have a look at this diagram of a unit circle on the Cartesian plane. Two points, $P(x, y)$ and $Q(a, b)$ have been drawn. P is at an angle of α from the positive x-axis and Q is at an angle of β from the positive x-axis.



1. Express the coordinates of Q as trigonometric ratios.
2. Express the coordinates of P as trigonometric ratios. Hint: drop another perpendicular.
3. Write down an expression for the distance between P and Q in terms of the coordinates for these points that you found in 1. and 2. Hint: Use the distance formula.
4. Use this expression to find PQ^2.
5. What is the angle formed between P and Q i.e. what is the angle $P\hat{O}Q$?
6. Write down another expression for the distance between P and Q but this time in terms of the included angle $P\hat{O}Q$. Hint: Use the cosine rule.
7. If both of these expressions are the distance PQ, write down an expression for $\cos(\alpha - \beta)$.

What did you find?

1. In ΔQOS, $\sin\beta = \frac{opp}{hyp} = \frac{b}{1}$. Therefore, $b = \sin\beta$.
   $\cos\beta = \frac{adj}{hyp} = \frac{a}{1}$. Therefore, $a = \cos\beta$.
   So, the coordinates of Q are $Q\left(\cos\beta, \sin\beta\right)$.
2. We can drop a perpendicular to the x-axis from P, and call this point T.
   Then in ΔPOT, $\sin\alpha = \frac{opp}{hyp} = \frac{y}{1}$. Therefore, $y = \sin\alpha$.
   $\cos\alpha = \frac{adj}{hyp} = \frac{x}{1}$. Therefore, $x = \cos\alpha$.
   So, the coordinates of P are $P\left(\cos\alpha, \sin\alpha\right)$.
3. We know that the distance between two points is given by $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. Therefore, the distance PQ will be $PQ = \sqrt{(x-a)^2 + (y-b)^2}$. But we have trig-based expressions for all these coordinates. Therefore, $PQ = \sqrt{(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2}$
4. We need to square both sides.
   $$\begin{aligned} PQ &= \sqrt{(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2} \\ \therefore PQ^2 &= (\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2 \\ &= \cos^2\alpha - 2\cos\alpha\cos\beta + \cos^2\beta + \sin^2\alpha - 2\sin\alpha\sin\beta + \sin^2\beta \\ &= (\cos^2\alpha + \sin^2\alpha) + (\cos^2\beta + \sin^2\beta) - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta \\ &= 1 + 1 - 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta) \qquad (\text{Remember that } \sin^2\theta + \cos^2\theta = 1) \\ &= 2 - 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta) \end{aligned}$$
5. $P\hat{O}Q = \alpha - \beta$
6. The cosine rule states that $a^2 = b^2 + c^2 - 2bc\cos A$ where a is the side opposite $\hat{A}$. In our case, PQ is the side opposite $(\alpha - \beta)$ in ΔPQO. Therefore: $PQ^2 = PO^2 + QO^2 - 2PO \cdot QO\cos(\alpha - \beta)$. But both PO and QO are equal to 1. Therefore:
   $$\begin{aligned} PQ^2 &= PO^2 + QO^2 - 2PO \cdot QO\cos(\alpha - \beta) \\ &= 1 + 1 - 2(1)(1)\cos(\alpha - \beta) \\ &= 2 - 2\cos(\alpha - \beta) \end{aligned}$$
7. We can now equate our two expressions and then rearrange the equation to get $\cos(\alpha - \beta)$ on the one side.

$$PQ^2 = 2 - 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta) = 2 - 2\cos(\alpha - \beta)$$
$$\therefore 2\cos(\alpha - \beta) = 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta)$$
$$\therefore \cos(\alpha - \beta) = (\cos\alpha\cos\beta + \sin\alpha\sin\beta)$$

We now have an expression for $\cos(\alpha - \beta)$, the cosine of the **compound** angle $\alpha - \beta$, in terms of trig ratios of the **single angles** α and β.

$\cos(\alpha - \beta) = (\cos\alpha\cos\beta + \sin\alpha\sin\beta)$.

Notice that the signs are different. There is a negative on the left-hand side but a positive on the right-hand side.

What do you think $\cos(\alpha + \beta)$ is equal to? Try figure this out on your own by using the fact that $\cos(\alpha + \beta) = \cos(\alpha - (-\beta))$.

Does your reasoning agree with the following?
$\cos(\alpha - (-\beta)) = \cos\alpha\cos(-\beta) + \sin\alpha\sin(-\beta)$. But we know that $\cos(-\beta) = \cos\beta$ and $\sin(-\beta) = -\sin\beta$.

Therefore:
$$\begin{aligned}\cos(\alpha - (-\beta)) &= \cos\alpha\cos(-\beta) + \sin\alpha\sin(-\beta)\\ &= \cos\alpha\cos\beta + \sin\alpha(-\sin\beta)\\ &= \cos\alpha\cos\beta - \sin\alpha\sin\beta\end{aligned}$$

Don't you love the symmetry?
$\cos(\alpha - \beta) = (\cos\alpha\cos\beta + \sin\alpha\sin\beta)$
$\cos(\alpha + \beta) = (\cos\alpha\cos\beta - \sin\alpha\sin\beta)$

We call these the **cosine compound angle identities**.

Now, we can use these identities to work out identities that express $\sin(\alpha - \beta)$ and $\sin(\alpha + \beta)$ in terms of trig ratios of the single angles α and β. On your own, use the fact that $\sin\theta = \cos(90^\circ - \theta)$, combined with the $\cos(\alpha + \beta)$ identity, to work out an identity that expresses $\sin(\alpha - \beta)$ in terms of trig ratios of the single angles α and β.

$$\begin{aligned}\sin(\alpha - \beta) &= \cos(90^\circ - (\alpha - \beta))\\ &= \cos((90^\circ - \alpha) + \beta))\\ &= \cos(90^\circ - \alpha)\cos\beta - \sin(90^\circ - \alpha)\sin\beta\\ &= \sin\alpha\cos\beta - \cos\alpha\sin\beta\end{aligned}$$

Now, use the $\sin(\alpha - \beta)$ identity to work out a similar identity for $\sin(\alpha + \beta)$.

$$\begin{aligned}\sin(\alpha + \beta) &= \sin(\alpha - (-\beta))\\ &= \sin\alpha\cos(-\beta) - \cos\alpha\sin(-\beta)\\ &= \sin\alpha\cos(\beta) - \cos\alpha(-\sin\beta)\\ &= \sin\alpha\cos\beta + \cos\alpha\sin\beta\end{aligned}$$

Compound angle identities:
$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$
$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$
For sine, the signs are the same but the ratios in each term are different.

$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$
For cosine, the signs are different but the ratios in each term are the same.

Example 1.1

Determine $\cos 75^\circ$ without a calculator.

Solution

Whenever you see a trigonometry question state that you must not use a calculator, you know that you need to use the special angles.

We know that $75^\circ = 45^\circ + 30^\circ$. Therefore, we can use a compound angle identity to rewrite $\cos 75^\circ$ with special angles and calculate its value without a calculator.

$$\begin{aligned}\cos 75^\circ &= \cos(45^\circ + 30^\circ)\\ &= \cos 45^\circ \cos 30^\circ - \sin 45^\circ \sin 30^\circ\\ &= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \times \frac{1}{2}\\ &= \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}\\ &= \frac{\sqrt{3}-1}{2\sqrt{2}} \qquad \text{Multiply by } \frac{\sqrt{2}}{\sqrt{2}}\\ &= \frac{\sqrt{2}\left(\sqrt{3}-1\right)}{4}\end{aligned}$$

Example 1.2

Determine the value of the following expression without the use of a calculator.
$\cos 65^\circ \cos 35^\circ + \cos 25^\circ \cos 55^\circ$

Solution

Once again, we are told that a calculator must not be used. Therefore, we need to rely on special angles. Unfortunately, as it stands, the expression does not match any of our compound angle identities. When this is the case, it is often a good idea to change sine into cosine, or vice versa. In this case, let's use the fact that $\cos 65^\circ = \cos(90^\circ - 25^\circ) = \sin 25^\circ$ and $\cos 55^\circ = \cos(90^\circ - 35^\circ) = \sin 35^\circ$

$$\begin{aligned}\cos 65^\circ \cos 35^\circ + \cos 25^\circ \cos 55^\circ &= \cos(90^\circ - 25^\circ)\cos 35^\circ + \cos 25^\circ \cos(90^\circ - 35^\circ)\\ &= \sin 25^\circ \cos 35^\circ + \cos 25^\circ \sin 35^\circ\end{aligned}$$

Now we have an expression that does match one of the compound angle identities:
$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$ where $\alpha = 25^\circ$ and $\beta = 35^\circ$.

$$\begin{aligned} \sin 25^\circ \cos 35^\circ + \cos 25^\circ \sin 35^\circ &= \sin(25^\circ + 35^\circ) \\ &= \sin 60^\circ \\ &= \frac{\sqrt{3}}{2} \end{aligned}$$

Exercise 1.1

1. Calculate the value of the following without using a calculator:
   a. $\cos 105^\circ$
   b. $\sin 15^\circ$
2. Determine the value of the following expressions without using a calculator:
   a. $\sin 10^\circ \cos 20^\circ + \cos 10^\circ \sin 20^\circ$
   b. $\cos 50^\circ \sin 80^\circ - \cos 40^\circ \sin 10^\circ$
   c. $\cos^2 15^\circ - \sin^2 15^\circ$
   d. $\sin x \cos(30^\circ + x) - \cos x \sin(30^\circ + x)$

Question 3 adapted from Everything Maths Grade 12 Exercise 4-2 question 3

3.
   a. Prove that $\sin(60^\circ - x) + \sin(60^\circ + x) = \sqrt{3}\cos x$.
   b. Hence, evaluate $\sin 15^\circ + \sin 105^\circ$ without a calculator.
4. Simplify the following expression without a calculator:
$$\frac{\sin x \cos(30^\circ + x) - \cos x \sin(30^\circ + x)}{\cos x \cos(60^\circ + x) + \sin x \sin(60^\circ + x)}$$

The full solutions are at the end of the unit.

Double angle identities

The **double angle identities** $\sin 2\theta$ and $\cos 2\theta$are special cases of the compound angle identities $\sin(\alpha + \beta)$ and $\cos(\alpha + \beta)$. You should be able to derive these identities yourself. Try doing this now before reading on.

Here are the derivations for you to check your own work.

$$\begin{aligned} \sin 2\theta &= \sin(\theta + \theta) \\ &= \sin\theta\cos\theta + \cos\theta\sin\theta \\ &= 2\sin\theta\cos\theta \end{aligned}$$

$$\begin{aligned} \cos 2\theta &= \cos(\theta + \theta) \\ &= \cos\theta\cos\theta - \sin\theta\sin\theta \\ &= \cos^2\theta - \sin^2\theta \end{aligned}$$

Remember that $\cos^2\theta + \sin^2\theta = 1$. Therefore, we can write two more alternative versions of the cosine double angle identity.

$$\cos^2\theta - \sin^2\theta = (1 - \sin^2\theta) - \sin^2\theta$$
$$= 1 - 2\sin^2\theta$$

Or

$$\cos^2\theta - \sin^2\theta = \cos^2\theta - (1 - \cos^2\theta)$$
$$= 2\cos^2\theta - 1$$

Double angle identities:

$\sin 2\theta = 2\sin\theta\cos\theta$

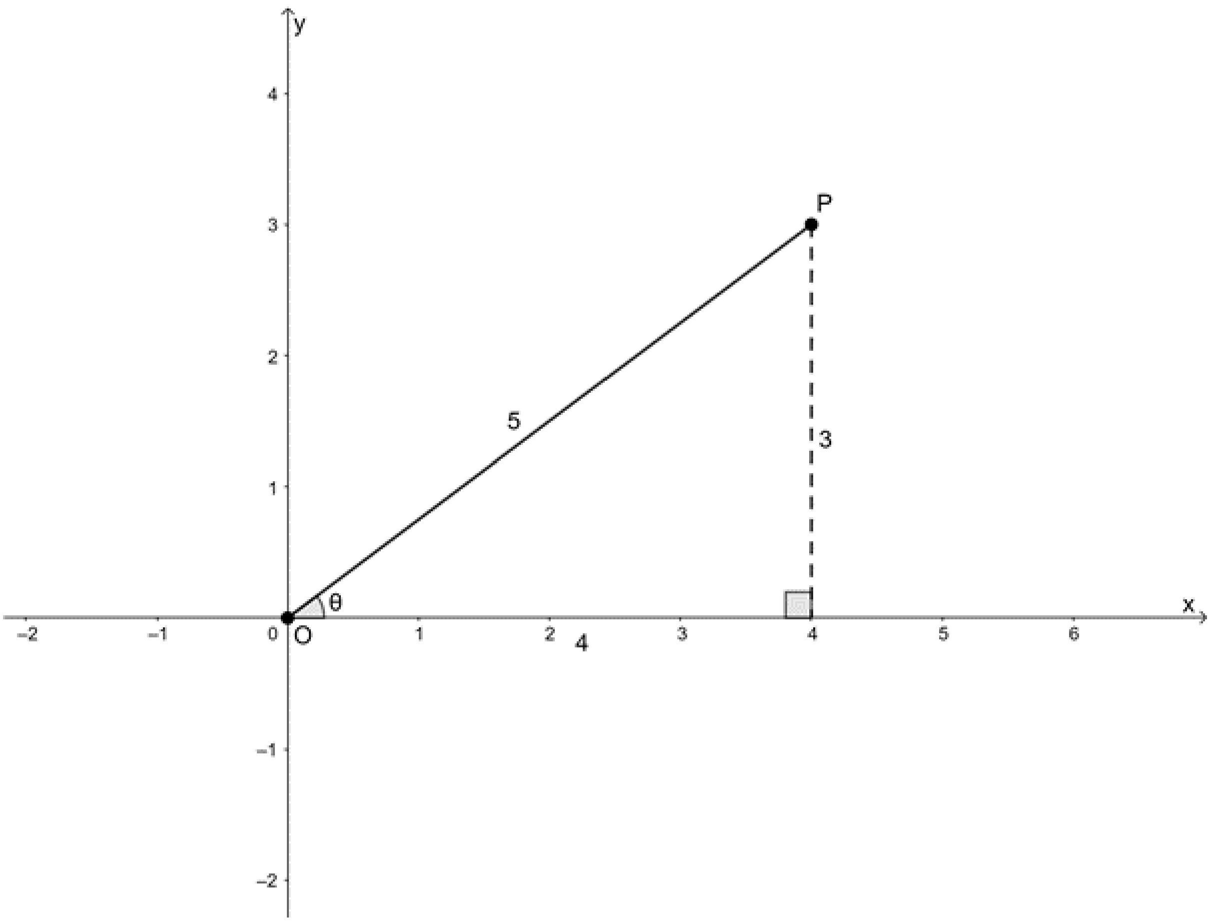
$$\cos 2\theta = \left\{ \begin{array}{c} \cos^2\theta - \sin^2\theta \\ 2\cos^2\theta - 1 \\ 1 - 2\sin^2\theta \end{array} \right\}$$

Example 1.3

If $0^\circ \leq \theta \leq 90^\circ$ and $\tan\theta = 0.75$, determine the value of $\sin 2\theta$ without using a calculator.

Solution

We know that θ is an acute angle in the first quadrant. We also know that $\tan\theta = 0.75 = \frac{3}{4}$. We can use Pythagoras to find the length of the hypotenuse.



Now we need to determine $\sin 2\theta$:

$$\begin{aligned}\sin 2\theta &= 2\sin\theta\cos\theta \\ &= 2\times\frac{3}{5}\times\frac{4}{5} \\ &= \frac{24}{25}\end{aligned}$$

Often, we use the double angle identities to simplify more complicated expressions as shown in example 1.4.

Example 1.4

Prove that $\frac{1+\cos 2\theta+\cos\theta}{\sin\theta+\sin 2\theta}=\frac{1}{\tan\theta}$.

Solution

It is usually a good idea to start by trying to simplify the more complicated side of the expression to see if you can get to the less complicated expression. In this case, the LHS is the more complicated side. Often, you will need to play around with various options to find the one that works. Don't give up if your first idea does not work.

$$\begin{aligned}\text{LHS} &= \frac{1+\cos 2\theta+\cos\theta}{\sin\theta+\sin 2\theta} \\ &= \frac{1+\sin^2\theta-\cos^2\theta+\cos\theta}{\sin\theta+2\sin\theta\cos\theta} && \text{Try another cosine double angle identity} \\ &= \frac{1+(2\cos^2\theta-1)+\cos\theta}{\sin\theta(1+2\cos\theta)} \\ &= \frac{2\cos^2\theta+\cos\theta}{\sin\theta(1+2\cos\theta)} \\ &= \frac{\cos\theta(2\cos\theta+1)}{\sin\theta(1+2\cos\theta)} \\ &= \frac{\cos\theta(1+2\cos\theta)}{\sin\theta(1+2\cos\theta)} \\ &= \frac{\cos\theta}{\sin\theta} \\ &= \frac{1}{\tan\theta} = \text{RHS}\end{aligned}$$

Example 1.5

Given that $\cos 75^\circ = a$, determine the following in terms of a without using a calculator.

1. $\cos 150^\circ$
2. $\sin 165^\circ$

3. $\sin 75°$

Solutions

1. This first question is straight forward. We can simply apply the cosine double angle identity.
$$\begin{aligned} \cos 150° &= \cos(75° + 75°) \\ &= 2\cos^2 75° - 1 \\ &= 2a^2 - 1 \end{aligned}$$
2. This question does not need us to apply any double angle identity.
$$\begin{aligned} \sin 165° &= \sin(90° + 75°) \\ &= \cos 75° \\ &= a \end{aligned}$$
3. This is a tricky question. We need to express $\sin 150°$ in terms of $a = \cos 75°$ so we cannot use either the fact that $\sin 150° = \sin(180° - 30°)$ or $\sin 150° = \sin(90° + 60°)$ directly. We need to apply the double angle identity in reverse. We know that:
$$\begin{aligned} \sin 150° &= \sin(75° + 75°) \\ &= 2\sin 75° \cos 75° \end{aligned}$$
We already know that $\cos 75° = a$ but in order to find an expression for $\sin 75°$ we need to determine the value of $\sin 150°$. Thankfully, we can easily do this without a calculator.
$$\begin{aligned} \sin 150° &= \sin(180° - 30°) \\ &= \sin 30° \\ &= \frac{1}{2} \end{aligned}$$
Now we can say that:
$$\begin{aligned} 2\sin 75° \cos 75° &= \frac{1}{2} \\ \therefore \sin 75° \cos 75° &= \frac{1}{4} \\ \text{But } \cos 75° &= a \\ \therefore \sin 75° &= \frac{1}{4a} \end{aligned}$$

Exercise 1.2

1. Given $\sin 50° = a$ and $\sin 40° = b$, determine the value of the following in terms of a and b:
   a. $\sin 100°$
   b. $\cos 40°$
   c. $\cos 140°$
   d. $\cos 25°$
   e. $\cos 80°$
   f. $\cos(-800°)$

Question 2 adapted from Everything Maths Grade 12 Exercise 4-3 question 3

2. Prove the following identity:
$$\frac{1}{\sin 2A} - \frac{1}{\tan 2A} = \tan A$$
3. Given $13\cos\theta = -5$ and $180^\circ \leq \theta \leq 360^\circ$, determine the value of the following expressions without a calculator:
   a. $\cos 2\theta$
   b. $\sin 2\theta$
   c. $\tan 4\theta$
   d. $\sin(180^\circ - 2\theta)$

The full solutions are at the end of the unit.

Summary

In this unit you have learnt the following:

- $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$
- $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$
- $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
- $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$
- $\sin 2\theta = 2\sin\theta\cos\theta$
- $\cos 2\theta = \left\{\begin{matrix} \cos^2\theta - \sin^2\theta \\ 2\cos^2\theta - 1 \\ 1 - 2\sin^2\theta \end{matrix}\right\}$

Unit 1: Assessment

Suggested time to complete: 60 minutes

1. Prove that $\sin 75^\circ + \sin 15^\circ = \frac{\sqrt{6}}{2}$.
2. Prove that $\frac{\sin 2\theta}{\sin\theta} - \frac{\cos 2\theta}{\cos\theta} = \frac{1}{\cos\theta}$.
3. Given that $\cos\theta = a$, determine $\cos 3\theta$ in terms of a.
4.
   a. Prove that $\sin(S+T) - \sin(S-T) = 2\cos S\sin T$.
   b. Hence, calculate the value of $\cos 15^\circ \sin 75^\circ$ without a calculator.
5.
   a. Prove that $\tan 2x + \frac{1}{\cos 2x} = \frac{\sin x + \cos x}{\cos x - \sin x}$.
   b. Explain why the identity is undefined for $x = 45^\circ$.
6. Determine the value of $\cos 67.5^\circ$ without a calculator.

Question 7 adapted from NC(V) Level 4 November 2011 Paper 2 question 3.3

7. Prove that $\sin(A+B) \times \sin(A-B) = \sin^2 A - \sin^2 B$.

Question 8 and 9 adapted from NC(V) Level 4 November 2012 Paper 2 question 3.3 and 3.4

8. If $90^\circ < A < 360^\circ$ and $\tan A = \frac{3}{2}$ show, without the use of calculator that $\cos 2A - \sin 2A = -\frac{17}{13}$.
9. Prove that $\cos(A+B) - \cos(A-B) = -2\sin A \sin B$.

The full solutions are at the end of the unit

Unit 1: Solutions

Exercise 1.1

1.
   a.
$$\begin{aligned}\cos 105^\circ &= \cos(60^\circ + 45^\circ)\\ &= \cos 60^\circ \cos 45^\circ - \sin 60^\circ \sin 45^\circ\\ &= \frac{1}{2}\cdot\frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{2}}\\ &= \frac{1}{2\sqrt{2}} - \frac{\sqrt{3}}{2\sqrt{2}}\\ &= \frac{1-\sqrt{3}}{2\sqrt{2}}\\ &= \frac{\sqrt{2}\left(1-\sqrt{3}\right)}{4}\end{aligned}$$
   b.
$$\begin{aligned}\sin 15^\circ &= \sin(45^\circ - 30^\circ)\\ &= \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ\\ &= \frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}\cdot\frac{1}{2}\\ &= \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}\\ &= \frac{\sqrt{3}-1}{2\sqrt{2}}\\ &= \frac{\sqrt{2}\left(\sqrt{3}-1\right)}{4}\end{aligned}$$
   c.

$$
\begin{aligned}
\tan 15^\circ &= \frac{\sin 15^\circ}{\cos 15^\circ} \\
&= \frac{\sin(45^\circ - 30^\circ)}{\cos(45^\circ - 30^\circ)} \\
&= \frac{\frac{\sqrt{2}\left(\sqrt{3}-1\right)}{4}}{\cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ} && \text{See part b. for numerator calculations} \\
&= \frac{\frac{\sqrt{2}\left(\sqrt{3}-1\right)}{4}}{\frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2}} \\
&= \frac{\frac{\sqrt{2}\left(\sqrt{3}-1\right)}{4}}{\frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}} \\
&= \frac{\frac{\sqrt{2}\left(\sqrt{3}-1\right)}{4}}{\frac{\sqrt{3}+1}{2\sqrt{2}}} \\
&= \frac{\frac{\sqrt{2}\left(\sqrt{3}-1\right)}{4}}{\frac{\sqrt{2}\left(\sqrt{3}+1\right)}{4}} \\
&= \frac{\sqrt{2}\left(\sqrt{3}-1\right)}{4} \times \frac{4}{\sqrt{2}\left(\sqrt{3}+1\right)} \\
&= \frac{\left(\sqrt{3}-1\right)}{\left(\sqrt{3}+1\right)}
\end{aligned}
$$

2.

a.

$$
\begin{aligned}
\sin 10^\circ \cos 20^\circ + \cos 10^\circ \sin 20^\circ &= \sin(10^\circ + 20^\circ) \\
&= \sin 30^\circ \\
&= \frac{1}{2}
\end{aligned}
$$

b.

$$
\begin{aligned}
\cos 50^\circ \sin 80^\circ - \cos 40^\circ \sin 10^\circ &= \cos 50^\circ \sin(90^\circ - 10^\circ) - \cos(90^\circ - 50^\circ) \sin 10^\circ \\
&= \cos 50^\circ \cos 10^\circ - \sin 50^\circ \sin 10^\circ \\
&= \cos(50^\circ + 10^\circ) \\
&= \cos 60^\circ \\
&= \frac{1}{2}
\end{aligned}
$$

c.

$$
\begin{aligned}
\cos^2 15^\circ - \sin^2 15^\circ &= \cos 15^\circ \cos 15^\circ - \sin 15^\circ \sin 15^\circ \\
&= \cos(15^\circ + 15^\circ) \\
&= \cos 30^\circ \\
&= \frac{\sqrt{3}}{2}
\end{aligned}
$$

d.

$$\begin{aligned}\sin x\cos(30^\circ+x)-\cos x\sin(30^\circ+x)&=\sin[x-(30^\circ+x)]\\&=\sin(x-30^\circ-x)\\&=\sin(-30^\circ)\\&=-\sin 30^\circ\\&=-\frac{1}{2}\end{aligned}$$

3.

a.

$$\begin{aligned}\text{LHS}&=\sin(60^\circ-x)+\sin(60^\circ+x)\\&=(\sin 60^\circ\cos x-\cos 60^\circ\sin x)+(\sin 60^\circ\cos x+\cos 60^\circ\sin x)\\&=\sin 60^\circ\cos x+\sin 60^\circ\cos x\\&=2\times\sin 60^\circ\cos x\\&=2\times\frac{\sqrt{3}}{2}\cos x\\&=\sqrt{3}\cos x=\text{RHS}\end{aligned}$$

b.

$$\begin{aligned}\sin 15^\circ+\sin 105^\circ&=\sin(60^\circ-45^\circ)+\sin(60^\circ+45^\circ)\\\text{But }\ \sin(60^\circ-x)+\sin(60^\circ+x)&=\sqrt{3}\cos x\\\therefore\text{Let }\ x&=45^\circ\\\sin(60^\circ-45^\circ)+\sin(60^\circ+45^\circ)&=\sqrt{3}\cos 45^\circ\\&=\sqrt{3}\times\frac{1}{\sqrt{2}}\\&=\frac{\sqrt{3}}{\sqrt{2}}\end{aligned}$$

4.

$$\begin{aligned}\frac{\sin x\cos(30^\circ+x)-\cos x\sin(30^\circ+x)}{\cos x\cos(60^\circ+x)+\sin x\sin(60^\circ+x)}&=\frac{\sin[x-(30^\circ+x)]}{\cos[x-(60^\circ+x)]}\\&=\frac{\sin(x-30^\circ-x)}{\cos(x-60^\circ-x)}\\&=\frac{\sin(-30^\circ)}{\cos(-60^\circ)}\\&=\frac{-\sin 30^\circ}{\cos 60^\circ}\\&=\frac{-\frac{1}{2}}{\frac{1}{2}}\\&=-1\end{aligned}$$

Back to Exercise 1.1

Exercise 1.2

1.

a.

$$\begin{aligned}\sin 100^\circ&=\sin(50^\circ+50^\circ)\\&=2\sin 50^\circ\cos 50^\circ\\&=2a\cos(90^\circ-40^\circ)\\&=2a\sin 40^\circ\\&=2ab\end{aligned}$$

b.

$$\begin{aligned}\cos 40^\circ &= \cos(90^\circ - 50^\circ)\\ &= \sin 50^\circ\\ &= a\end{aligned}$$

c.

$$\begin{aligned}\cos 140^\circ &= \cos(180^\circ - 40^\circ)\\ &= -\cos 40^\circ\\ &= -[\cos(90^\circ - 50^\circ)]\\ &= -\sin 50^\circ\\ &= -a\end{aligned}$$

d.

$$\begin{aligned}\cos 50^\circ &= \cos(25^\circ + 25^\circ)\\ = 2\cos^2 25^\circ - 1\\ \therefore 2\cos^2 25^\circ &= \cos 50^\circ + 1\\ \therefore \cos^2 25^\circ &= \frac{\cos 50^\circ + 1}{2}\\ \therefore \cos 25^\circ &= \sqrt{\frac{\cos 50^\circ + 1}{2}} \qquad \cos 25^\circ > 0 \therefore \cos^2 25^\circ > 0\\ &= \sqrt{\frac{\cos(90^\circ - 40^\circ) + 1}{2}}\\ &= \sqrt{\frac{\sin 40^\circ + 1}{2}}\\ &= \sqrt{\frac{b+1}{2}}\end{aligned}$$

e.

$$\begin{aligned}\cos 80^\circ &= \cos(40^\circ + 40^\circ)\\ &= 1 - 2\sin^2 40^\circ\\ &= 1 - 2b^2\end{aligned}$$

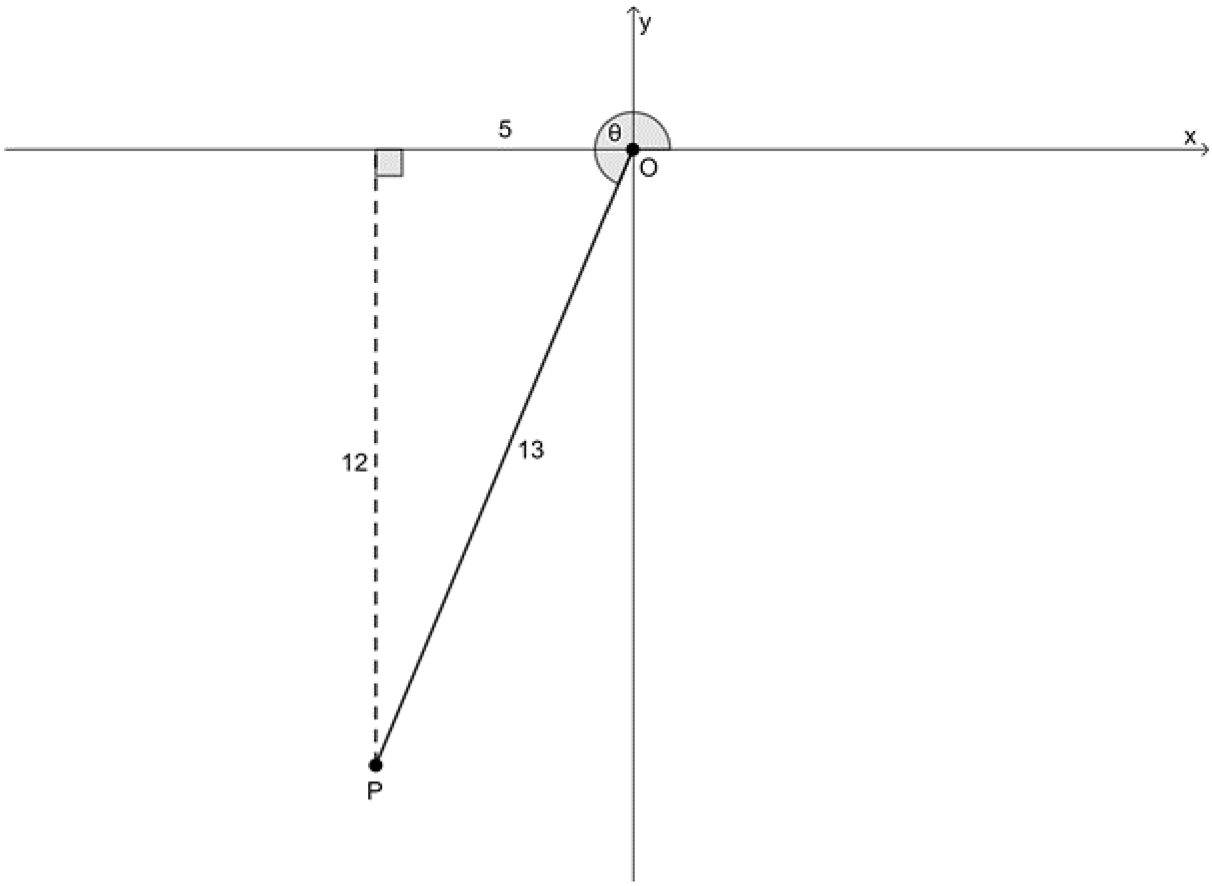
f.

$$\begin{aligned}\cos(-800^\circ) &= \cos(-80^\circ - 2 \cdot 360^\circ)\\ &= \cos(-80^\circ)\\ &= \cos 80^\circ\\ &= 1 - 2b^2 \qquad \text{From e.}\end{aligned}$$

2.

$$\begin{aligned}\text{LHS} &= \frac{1}{\sin 2A} - \frac{1}{\tan 2A}\\ &= \frac{1}{\sin 2A} - \frac{\cos 2A}{\sin 2A}\\ &= \frac{1 - \cos 2A}{\sin 2A}\\ &= \frac{1 - (1 - 2\sin^2 A)}{2 \sin A \cos A}\\ &= \frac{2\sin^2 A}{2 \sin A \cos A}\\ &= \frac{\sin A}{\cos A}\\ &= \tan A = \text{RHS}\end{aligned}$$

3. $13 \cos\theta = -5$ and $180^\circ \leq \theta \leq 360^\circ$



a.

$$\begin{aligned}\cos\theta &= -\frac{5}{13}\\ \cos 2\theta &= 2\cos^2\theta - 1\\ &= 2\left(-\frac{5}{13}\right)^2 - 1\\ &= 2\left(\frac{25}{169}\right) - 1\\ &= \frac{50}{169} - 1\\ &= \frac{50-169}{169}\\ &= -\frac{119}{169}\end{aligned}$$

b.

$$\begin{aligned}\sin 2\theta &= 2\sin\theta\cos\theta\\ &= 2\times\left(-\frac{12}{13}\times-\frac{5}{13}\right)\\ &= 2\times\left(\frac{60}{169}\right)\\ &= \frac{120}{169}\end{aligned}$$

c.

$$\begin{aligned}\tan 2\theta &= \frac{\sin 2\theta}{\cos 2\theta} \\ &= \frac{\frac{120}{169}}{-\frac{119}{169}} \\ &= -\frac{120}{169} \times \frac{169}{119} \\ &= -\frac{120}{119}\end{aligned}$$

d.

$$\begin{aligned}\sin(180^\circ - 2\theta) &= \sin 2\theta \\ &= \frac{120}{169}\end{aligned}$$

Back to Exercise 1.2

Unit 1: Assessment

1.

$$\begin{aligned}\text{LHS} &= \sin 75^\circ + \sin 15^\circ \\ &= \sin(45^\circ + 30^\circ) + \sin(45^\circ - 30^\circ) \\ &= \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ + \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ \\ &= 2 \sin 45^\circ \cos 30^\circ \\ &= 2 \times \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} \\ &= \frac{\sqrt{3}}{\sqrt{2}} \\ &= \frac{\sqrt{6}}{2} = \text{RHS}\end{aligned}$$

2.

$$\begin{aligned}\text{LHS} &= \frac{\sin 2\theta}{\sin \theta} - \frac{\cos 2\theta}{\cos \theta} \\ &= \frac{2 \sin \theta \cos \theta}{\sin \theta} - \frac{2\cos^2\theta - 1}{\cos \theta} \\ &= \frac{2 \sin \theta \cos^2\theta - 2 \sin \theta \cos^2\theta + \sin \theta}{\sin \theta \cos \theta}. \\ &= \frac{\sin \theta}{\sin \theta \cos \theta} \\ &= \frac{1}{\cos \theta} = \text{RHS}\end{aligned}$$

3. $\cos \theta = a$

$$\begin{aligned}\cos 3\theta &= \cos(\theta + 2\theta) \\ &= \cos \theta \cos 2\theta - \sin \theta \sin 2\theta \\ &= \cos \theta \left(2\cos^2\theta - 1\right) - \sin \theta \left(2 \sin \theta \cos \theta\right) \\ &= 2\cos^3\theta - \cos \theta - 2\sin^2\theta \cos \theta \\ &= \cos \theta \left(2\cos^2\theta - 1 - 2\sin^2\theta\right) \\ &= \cos \theta \left(2\cos^2\theta - 1 - 2(1 - \cos^2\theta)\right) \\ &= \cos \theta \left(2\cos^2\theta - 1 - 2 + 2\cos^2\theta\right) \\ &= \cos \theta \left(4\cos^2\theta - 3\right) \\ &= a(4a^2 - 3)\end{aligned}$$

4.

a.

$$\begin{aligned}\text{LHS} &= \sin(S+T) - \sin(S-T)\\ &= \sin S\cos T + \cos S\sin T - (\sin S\cos T - \cos S\sin T)\\ &= \sin S\cos T + \cos S\sin T - \sin S\cos T + \cos S\sin T\\ &= 2\cos S\sin T = \text{RHS}\end{aligned}$$

b.

$$\begin{aligned}2\cos S\sin T &= \sin(S+T) - \sin(S-T)\\ \therefore \cos S\sin T &= \frac{1}{2}(\sin(S+T) - \sin(S-T))\\ \text{Let } S = 15^\circ &\text{ and } T = 75^\circ\\ \therefore \cos 15^\circ \sin 75^\circ &= \frac{1}{2}(\sin(15^\circ + 75^\circ) - \sin(15^\circ - 75^\circ))\\ &= \frac{1}{2}(\sin 90^\circ - \sin(-60^\circ))\\ &= \frac{1}{2}(1 + \sin 60^\circ)\\ &= \frac{1}{2}\left(1 + \frac{\sqrt{3}}{2}\right)\\ &= \frac{1}{2}\left(\frac{2+\sqrt{3}}{2}\right)\\ &= \frac{2+\sqrt{3}}{4}\end{aligned}$$

5.

a.

$$\begin{aligned}\text{LHS} &= \tan 2x + \frac{1}{\cos 2x}\\ &= \frac{\sin 2x}{\cos 2x} + \frac{1}{\cos 2x}\\ &= \frac{\sin 2x + 1}{\cos 2x}\\ &= \frac{2\sin x\cos x + 1}{\cos^2 x - \sin^2 x} \qquad \text{but } \sin^2 x + \cos^2 x = 1 \text{ so}\\ &= \frac{2\sin x\cos x + \sin^2 x + \cos^2 x}{(\cos x + \sin x)(\cos x - \sin x)}\\ &= \frac{\cos^2 x + 2\sin x\cos x + \sin^2 x}{(\cos x + \sin x)(\cos x - \sin x)}\\ &= \frac{(\cos x + \sin x)(\cos x + \sin x)}{(\cos x + \sin x)(\cos x - \sin x)}\\ &= \frac{\sin x + \cos x}{\cos x - \sin x} = \text{RHS}\end{aligned}$$

b. When $x = 45^\circ$, $\cos x = \sin x$. Therefore, $\cos x - \sin x = 0$.

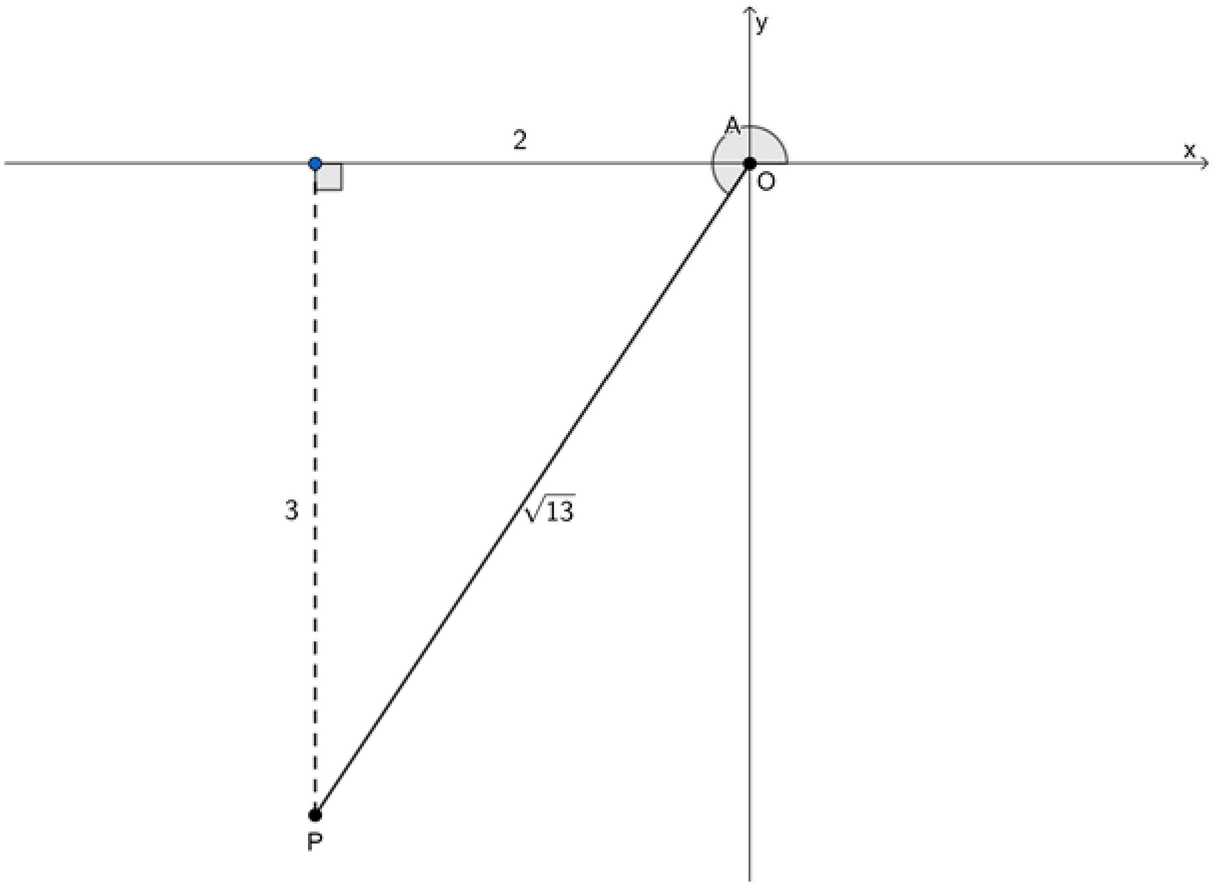
6.

$$\cos 135^\circ = \cos(67.5^\circ + 67.5^\circ)$$
$$= 2\cos^2 67.5^\circ - 1$$
$$\text{But } \cos 135^\circ = \cos(180^\circ - 45^\circ)$$
$$= -\cos 45^\circ$$
$$= -\frac{1}{\sqrt{2}}$$
$$\therefore 2\cos^2 67.5^\circ - 1 = -\frac{1}{\sqrt{2}}$$
$$\therefore 2\cos^2 67.5^\circ = 1 - \frac{1}{\sqrt{2}} = \frac{\sqrt{2}-1}{\sqrt{2}}$$
$$\therefore \cos^2 67.5^\circ = \frac{\sqrt{2}-1}{2\sqrt{2}}$$
$$\therefore \cos 67.5^\circ = \sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}}$$
$$= \sqrt{\frac{2-\sqrt{2}}{4}}$$
$$= \frac{\sqrt{2-\sqrt{2}}}{2}$$

7.

$$\text{LHS} = \sin(A+B) \times \sin(A-B)$$
$$= (\sin A \cos B + \cos A \sin B) \times (\sin A \cos B - \cos A \sin B) \qquad \text{Difference of two squares}$$
$$= (\sin A \cos B)^2 - (\cos A \sin B)^2$$
$$= \sin^2 A \cos^2 B - \cos^2 A \sin^2 B$$
$$= \sin^2 A(1 - \sin^2 B) - (1 - \sin^2 A)\sin^2 B$$
$$= \sin^2 A - \sin^2 A \sin^2 B - \sin^2 B + \sin^2 A \sin^2 B$$
$$= \sin^2 A - \sin^2 B = \text{RHS}$$

8. $90^\circ < A < 360^\circ$ and $\tan A = \frac{3}{2}$



$$\begin{aligned}
\text{LHS} &= \cos 2A - \sin 2A \\
&= 2\cos^2 A - 1 - (2\sin A\cos A) \\
&= 2\left(-\frac{2}{\sqrt{13}}\right)^2 - 1 - 2\left(-\frac{3}{\sqrt{13}}\right)\left(-\frac{2}{\sqrt{13}}\right) \\
&= 2\left(\frac{4}{13}\right) - 1 - 2\left(\frac{6}{13}\right) \\
&= \frac{8}{13} - 1 - \frac{12}{13} \\
&= \frac{8-13-12}{13} \\
&= -\frac{17}{13} = \text{RHS}
\end{aligned}$$

9.

$$\begin{aligned}
\text{LHS} &= \cos(A+B) - \cos(A-B) \\
&= \cos A\cos B - \sin A\sin B - (\cos A\cos B + \sin A\sin B) \\
&= \cos A\cos B - \sin A\sin B - \cos A\cos B - \sin A\sin B \\
&= -2\sin A\sin B = \text{RHS}
\end{aligned}$$

Back to Unit 1: Assessment

Media Attributions

Unit 2: Solve trigonometric equations

DYLAN BUSA

Unit outcomes

By the end of this unit you will be able to:

- Solve equations involving double and compound angles.

What you should know

Before you start this unit, make sure you can:

- State and apply the reduction formulae for $180^\circ \pm \theta$, $90^\circ \pm \theta$ and $360^\circ \pm \theta$. Refer to level 3 subject outcome 3.3 unit 2 if you need help with this.
- State and apply the basic trigonometric identities of $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ and $\sin^2\theta + \cos^2\theta = 1$. Refer to level 3 subject outcome 3.3 unit 3 if you need help with this.
- State and apply the compound and double angle identities. Refer to unit 1 of this subject outcome if you need help with this.
- Find the general solutions of trigonometric equations. Refer to level 3 subject outcome 3.3 unit 4 if you need help with this.

Introduction

In level 3 subject outcome 3.3 unit 4, we learnt how to find the **general solution** of an equation that involved a trigonometric ratio. If you need to, you should review this unit before continuing.

Remember that the trigonometric functions are repetitive. Sine and cosine repeat every 360° (once a full revolution) and tangent repeats every 180° (twice every revolution). We say that the trigonometric functions are **periodic**. Sine and cosine functions have periods of 360° and the tangent function has a period of 180°.

The general solution revision

Because the trigonometric functions are periodic (they repeat themselves), when we solve an equation such as $\sin\theta = \frac{1}{2}$, there is not only one solution. We may know that $\sin 30^\circ = \frac{1}{2}$ but, for example, so does $\sin(30^\circ + 360^\circ) = \sin 390^\circ$, $\sin(30^\circ + 2 \times 360^\circ) = \sin 750^\circ$ and $\sin(30^\circ - 360^\circ) = \sin -330^\circ$.

We also know that $\sin 150^\circ = \frac{1}{2}$ but so do $\sin(150^\circ + 360^\circ) = \sin 510^\circ$, $\sin(150^\circ + 2 \times 360^\circ) = \sin 870^\circ$ and $\sin(150^\circ - 360^\circ) = \sin -210^\circ$, for example.

The general solution to the equation $\sin\theta = \frac{1}{2}$ is $\theta = 30^\circ + k \cdot 360^\circ$ or $\theta = 150^\circ + k \cdot 360^\circ, k \in \mathbb{Z}$. In other words, θ is equal to $30°$ plus or minus any integer multiple of $360°$ or θ is equal to $150°$ plus or minus any integer multiple of $360°$.

When we learnt how to find the general solution of trigonometric equations, we called the answer we get for θ from the calculator (e.g. $\theta = 30°$) the **reference angle**. It is the basis upon which we build the general solution.

Work through the next two examples and the exercise that follows to remind yourself how to find the general solution of simple trigonometric equations.

Example 2.1

Determine the general solution for $\sin\theta = 0.6$.

Solution

Step 1: Use a calculator to determine the reference angle

$\sin\theta = 0.6$

$\therefore \theta = 36.9°$

Note: Unless told otherwise, we usually round the reference angle to one decimal place.

Step 2: Use the CAST diagram to determine any other possible solutions

$\sin\theta = 0.6$. In other words, sine is positive. Sine is positive in the first and second quadrants. We already have the first quadrant solution (the reference angle of $\theta = 36.9°$). We need to find the second quadrant solution. We know that $\sin(180° - \theta) = \sin\theta$. Therefore, the second quadrant solution is $180° - 36.9° = 143.1°$.

Step 3: Generate the general solution

$\theta = 36.9° + k.360° \text{ or } \theta = 143.1° + k.360°, k \in \mathbb{Z}$

Step 4: Check your general solution

It is always a good idea to check that your final solutions satisfy the original equation. Choose a random value for k.

$k = -2$:

$\theta = 36.9° - 2 \times 360° \text{ or } \theta = 143.1° - 2 \times 360°$

$\therefore \theta = -683.1° \text{ or } \theta = -576.9°$

$\sin(-683.1°) = 0.6$

$\sin(-576.9°) = 0.6$

Our general solution is correct.

Example 2.2

Determine the general solution for $7\cos 2\theta + 4 = 0$.

Solution

$7\cos 2\theta + 4 = 0$

$\therefore \cos 2\theta = -\frac{4}{7}$

Step 1: Use a calculator to determine the reference angle

$\cos 2\theta = -\frac{4}{7}$

$\therefore 2\theta = 124.8^\circ$

Note: We keep working with the reference angle of 2θ until we generate the general solution.

Step 2: Use the CAST diagram to determine any other possible solutions

Our equation is $\cos 2\theta = -\frac{4}{7}$. $\cos 2\theta < 0$. Cosine is negative in the second and third quadrants. Our reference angle is in the second quadrant.

Second quadrant: $2\theta = 124.8^\circ$

Third quadrant: $\cos(360^\circ - \theta) = \cos\theta$

$2\theta = 360^\circ - 124.8^\circ = 235.2^\circ$.

Step 3: Generate the general solution

$2\theta = 124.8^\circ + k.360^\circ$ or $2\theta = 235.2^\circ + k.360^\circ, k \in \mathbb{Z}$

$\therefore \theta = 62.4^\circ + k.180^\circ$ or $\theta = 117.6^\circ + k.180^\circ, k \in \mathbb{Z}$

Step 4: Check your general solution

$k = 2$:

$\therefore \theta = 62.4^\circ + 2 \times 180^\circ = 422.4^\circ$ or $\theta = 117.6^\circ + 2 \times 180^\circ = 477.6^\circ$

$\cos(2 \times 422.4^\circ) = -\frac{4}{7}$

$\cos(2 \times 477.6^\circ) = -\frac{4}{7}$

Take note!

The general solutions for equations involving the three basic trigonometric ratios can be written as follows:

If $\sin\theta = x$ then:

$\theta = \left(\sin^{-1}x + k.360^\circ\right)$ or $\theta = \left(\left(180^\circ - \sin^{-1}x\right) + k.360^\circ\right), k \in \mathbb{Z}$

If $\cos\theta = x$ then:

$\theta = \left(\cos^{-1}x + k.360^\circ\right)$ or $\theta = \left(\left(360^\circ - \cos^{-1}x\right) + k.360^\circ\right), k \in \mathbb{Z}$

If $\tan\theta = x$ then:

$\theta = \left(\tan^{-1}x + k.180^\circ\right), k \in \mathbb{Z}$

Exercise 2.1

1. Find the general solution for θ in the following equations:
   a. $\cos\theta = 0.45$
   b. $2\tan\theta = -5$

c. $8\sin\theta + 3 = 0$

d. $-6\cos 2\theta - 3 = 0$

2. Determine θ for the given interval:

a. $3\cos\theta - 2 = -3$ for the interval $[0^\circ, 360^\circ]$

b. $\tan(3\theta - 42^\circ) = 3.4$ where $\theta \in [0^\circ, 180^\circ]$

The full solutions are at the end of the unit.

Trigonometric equations with double and compound angles

We can use the compound and double angle identities we learnt about in unit 1 to help us solve some more complicated trigonometric equations. They may look complicated to begin with, but once we have applied the compound and double angle identities, they become much simpler.

Example 2.3

Determine the general solution for θ in $\dfrac{1 - \sin\theta - \cos 2\theta}{\sin 2\theta - \cos\theta} = -1$.

Solution

Before we start solving any trigonometric equation, we need to try and simplify it as far as possible. In this case, there are double angles involved. Therefore, we can use the double angle identities. So we start with the LHS of the equation and simplify it.

$$\begin{aligned}\frac{1 - \sin\theta - \cos 2\theta}{\sin 2\theta - \cos\theta} &= \frac{1 - \sin\theta - \left(1 - 2\sin^2\theta\right)}{2\sin\theta\cos\theta - \cos\theta} \\ &= \frac{1 - \sin\theta - 1 + 2\sin^2\theta}{2\sin\theta\cos\theta - \cos\theta} \\ &= \frac{2\sin^2\theta - \sin\theta}{2\sin\theta\cos\theta - \cos\theta} \\ &= \frac{\sin\theta(2\sin\theta - 1)}{\cos\theta(2\sin\theta - 1)} \\ &= \frac{\sin\theta}{\cos\theta} \\ &= \tan\theta\end{aligned}$$

Now our equation becomes easy to solve: $\tan\theta = -1$.

Ref angle: $\theta = -45^\circ$

General solution: $\theta = -45^\circ + k.180^\circ, k \in \mathbb{Z}$

Note: We can also write the general solution with a positive angle as $\theta = 135^\circ + k.180^\circ, k \in \mathbb{Z}$.

Example 2.4

Prove that $4\sin\theta\cos^3\theta - 4\sin^3\theta\cos\theta = \sin 4\theta$ and hence determine the general solution for θ in $4\sin\theta\cos^3\theta - 4\sin^3\theta\cos\theta = 0.8$.

Solution

In this example, we are first asked to prove that $4\sin\theta\cos^3\theta - 4\sin^3\theta\cos\theta = \sin 4\theta$.

$$\begin{aligned}
\text{LHS} &= 4\sin\theta\cos^3\theta - 4\sin^3\theta\cos\theta \\
&= 4\sin\theta\cos\theta(\cos^2\theta - \sin^2\theta) \\
&= 2\times 2\sin\theta\cos\theta\times\cos 2\theta \\
&= 2\times\sin 2\theta\times\cos 2\theta \\
&= \sin 4\theta
\end{aligned}$$

Now we can solve the equation:

$$\begin{aligned}
4\sin\theta\cos^3\theta - 4\sin^3\theta\cos\theta &= 0.8 \\
\therefore \sin 4\theta &= 0.8
\end{aligned}$$

Ref angle: $4\theta = 53.1^\circ$

Sine is positive in the first and second quadrants.

$4\theta = 180^\circ - 53.1^\circ = 126.9^\circ$

General solution:

$$\begin{aligned}
&4\theta = 53.1^\circ + k.360^\circ \text{ or } 4\theta = 126.9^\circ + k.360^\circ, k\in\mathbb{Z} \\
&\therefore \theta = 13.275^\circ + k.90^\circ \text{ or } \theta = 31.725^\circ + k.90^\circ, k\in\mathbb{Z}
\end{aligned}$$

Example 2.5

Find the general solution for $\sin^2\theta\cos\theta = \cos^3\theta$.

Solution

We need to be careful here. We cannot divide through by $\sin\theta$ or $\sin^2\theta$. If we do so, we will lose some of the solution. Instead, we need to proceed as if we were solving a quadratic equation.

$$\begin{aligned}
\sin^2\theta\cos\theta &= \cos^3\theta \\
\therefore \sin^2\theta\cos\theta - \cos^3\theta &= 0 \\
\therefore \cos\theta(\sin^2\theta - \cos^2\theta) &= 0 \quad \text{Take a factor of } -1 \text{ out of the bracket} \\
\therefore -\cos\theta(\cos^2\theta - \sin^2\theta) &= 0 \\
\therefore \cos\theta(\cos^2\theta - \sin^2\theta) &= 0 \\
\therefore \cos\theta\times\cos 2\theta &= 0 \\
\therefore \cos\theta = 0 &\text{ or } \cos 2\theta = 0
\end{aligned}$$

We can deal with each part of the solution separately.

$\cos\theta = 0$:

Ref angle: $\theta = 90^\circ$

General solution: $\theta = 90^\circ + k.360^\circ$ or $\theta = 270^\circ + k.360^\circ, k\in\mathbb{Z}$.

We can simplify this general solution to $\theta = 90^\circ + k.180^\circ, k\in\mathbb{Z}$.

$\cos 2\theta = 0$:

Ref angle: $2\theta = 90^\circ$

General solution:

$$\begin{aligned} 2\theta &= 90^\circ + k.360^\circ \text{ or } 2\theta = 270^\circ + k.360^\circ, k \in \mathbb{Z} \\ \therefore \theta &= 45^\circ + k.180^\circ \text{ or } \theta = 135^\circ + k.180^\circ, k \in \mathbb{Z} \end{aligned}.$$

Again, we can simplify this general solution to $\theta = 45^\circ + k.90^\circ, k \in \mathbb{Z}$.

Take note!

There are two basic strategies for solving these more complicated trigonometric equations:

1. Simplify one side of the equation down to a single trigonometric ratio using the various trigonometric identities at your disposal.
2. Make the one side of the equation equal to zero and then simplify the other side in order to make use of the zero product rule – if $a \times b = 0$ then either $a = 0$ or $b = 0$. This is the technique we use to solve quadratic equations.

Exercise 2.2

1. Determine the general solution in each case:
   a. $\cos 2x = \sin 32^\circ$
   b. $\cos 2x = \sin x$
   c. $\sin\theta \cdot \sin 2\theta + \cos 2\theta = 1$
   d. $5\tan 2x - 1 = \tan^2 2x + 5$

Question 2 adapted from Everything Maths Grade 12 Exercise 4-4 question 5a

2. Given $\sin x \cos 3x - \cos x \sin 3x = \tan 140^\circ$:
   a. Find the general solution.
   b. Determine the solutions for the interval $[0^\circ, 90^\circ]$.

The full solutions are at the end of the unit.

Summary

In this unit you have learnt the following:

- How to apply the trigonometric identities, especially the compound and double angle identities, to solve more complicated trigonometric equations.

Unit 2: Assessment

Suggested time to complete: 45 minutes

1. Solve each equation for each given interval. If no interval is given, find the general solution.
   a. $2\cos\theta - 1 = 0$
   b. $\sin^2\theta + \sin\theta - 1 = 0$
   c. $4\sin x + 3\tan x = \dfrac{3}{\cos x} + 4$ for $[0^\circ, 180^\circ]$
   d. $6\cos\theta - 5 = \dfrac{4}{\cos\theta}$
   e. $\cos 2x - \cos x + 1 = 0$ for $[0^\circ, 360^\circ]$

Question 2 adapted from NC(V) Mathematics Level 4 Paper 2November 2015 question 2.5

2. Given that $\cos 2\theta = 2\cos^2\theta - 1$:
   a. Show that $\cos 2\theta + 3\cos\theta - 1 = 2\cos^2\theta + 3\cos\theta - 2$.
   b. Hence, determine the value(s) of θ if $\cos 2\theta + 3\cos\theta - 1 = 0$ and $0^\circ \leq \theta \leq 360^\circ$.

Question 3 adapted from NC(V) Mathematics Level 4 Paper 2 November 2016 question 2.5

3. If $3\cos 2x - \sin 2x - 1 = 0$ find the value(s) of x in the interval $[0^\circ, 360^\circ]$.

The full solutions are at the end of the unit.

Unit 2: Solutions

Exercise 2.1

1.
   a. $\cos\theta = 0.45$
   Ref angle: $\theta = 63.3^\circ$
   $\cos\theta$ is positive in the first and fourth quadrants.
   $\theta = 360^\circ - 63.3^\circ = 296.7^\circ$
   General solution: $\theta = 63.3^\circ + k.360^\circ \text{ or } \theta = 296.7^\circ + k.360^\circ, k \in \mathbb{Z}$
   b.
   $2\tan\theta = -5$
   $\therefore \tan\theta = -\dfrac{5}{2}$
   Ref angle: $\theta = -68.2^\circ$
   Note: You can also express your reference angle as a positive angle – $360^\circ - 68.2^\circ = 291.8^\circ$
   General solution: $\theta = -68.2^\circ + k.180^\circ, k \in \mathbb{Z}$
   c.
   $8\sin\theta + 3 = 0$
   $\therefore \sin\theta = -\dfrac{3}{8}$
   Ref angle: $\theta = -22.0^\circ$
   Note: You can also express your reference angle as a positive angle – $360^\circ - 22.0^\circ = 338^\circ$

$\sin\theta$ is negative in the third and fourth quadrants.
$\theta = 180^\circ + 22.0^\circ = 202.0^\circ$
General solution: $\theta = -22.0^\circ + k.360^\circ$ or $\theta = 202.0^\circ + k.360^\circ, k \in \mathbb{Z}$

d.

$$-6\cos 2\theta - 3 = 0$$
$$\therefore \cos 2\theta = -0.5$$

Ref angle: $\theta = 120^\circ$
$\cos\theta$ is negative in the second and third quadrants.
$\theta = 360^\circ - 120.0^\circ = 240^\circ$
General solution:

$$2\theta = 120^\circ + k.360^\circ \text{ or } \theta = 240^\circ + k.360^\circ, k \in \mathbb{Z}$$
$$\therefore \theta = 60^\circ + k.180^\circ \text{ or } \theta = 120^\circ + k.180^\circ, k \in \mathbb{Z}$$

2.

a.

$$3\cos\theta - 2 = -3$$
$$\therefore \cos\theta = -\frac{1}{3}$$

Ref angle: $\theta = 109.5^\circ$
$\cos\theta$ is negative in the second and third quadrants.
$\theta = 360^\circ - 109.5^\circ = 250.5^\circ$
General solution: $\theta = 109.5^\circ + k.360^\circ$ or $\theta = 250.5^\circ + k.360^\circ, k \in \mathbb{Z}$
Specific solution: $\theta = 109.5^\circ$ or $\theta = 250.5^\circ$

b. $\tan(3\theta - 42^\circ) = 3.4$
Ref angle: $3\theta - 42^\circ = 73.6^\circ$
General solution:

$$3\theta - 42^\circ = 73.6^\circ + k.180^\circ, k \in \mathbb{Z}$$
$$\therefore 3\theta = 115.6^\circ + k.180^\circ, k \in \mathbb{Z}$$
$$\therefore \theta = 38.5^\circ + k.60^\circ, k \in \mathbb{Z}$$

Specific solution: $\theta = 38.5^\circ$ or $\theta = 98.5^\circ$ or $\theta = 158.5^\circ$

Back to Exercise 2.1

Exercise 2.2

1.

a.

$$\cos 2x = \sin 32^\circ$$
$$\therefore \cos 2x = 0.530$$

Or

$$\cos 2x = \sin 32^\circ$$
$$\therefore \cos 2x = \cos(90^\circ - 32^\circ)$$
$$= \cos 58^\circ$$

Ref angle: $\theta = 58^\circ$
Cosine is positive in the first and fourth quadrants.
$\theta = 360^\circ - 58^\circ = 302^\circ$
General solution: $\theta = 58^\circ + k.360^\circ$ or $\theta = 302^\circ + k.360^\circ, k \in \mathbb{Z}$
So:

$$2x = 58^\circ + k.360^\circ \text{ or } 2x = 302^\circ + k.360^\circ, k \in \mathbb{Z}$$
$$x = 29^\circ + k.180^\circ \text{ or } x = 151^\circ + k.180^\circ,\ k \in \mathbb{Z}$$

b.

$$\cos 2x = \sin x$$
$$\therefore 1 - 2\sin^2 x = \sin x$$
$$\therefore 2\sin^2 x + \sin x - 1 = 0$$
$$\therefore (2\sin x - 1)(\sin x + 1) = 0$$
$$\therefore \sin x = \frac{1}{2} \text{ or } \sin x = -1$$

$\sin x = \frac{1}{2}$:
Ref angle: $x = 30^\circ$
Sine is positive in the first and second quadrants.
$x = 180^\circ - 30^\circ = 150^\circ$
General solution: $x = 30^\circ + k.360^\circ \text{ or } x = 150^\circ + k.360^\circ, k \in \mathbb{Z}$
$\sin x = -1$:
Ref angle: $x = -90^\circ$
General solution: $x = 270^\circ + k.360^\circ, k \in \mathbb{Z}$

c.

$$\sin\theta \cdot \sin 2\theta + \cos 2\theta = 1$$
$$\therefore \sin\theta \cdot 2\sin\theta\cos\theta + \cos^2\theta - \sin^2\theta - 1 = 0$$
$$\therefore 2\sin^2\theta\cos\theta + \cos^2\theta - 1 - \sin^2\theta = 0$$
$$\therefore 2\sin^2\theta\cos\theta - \sin^2\theta - \sin^2\theta = 0$$
$$\therefore 2\sin^2\theta\cos\theta - 2\sin^2\theta = 0$$
$$\therefore 2\sin^2\theta(\cos\theta - 1) = 0$$
$$\therefore 2\sin^2\theta = 0 \quad \text{or } \cos\theta = 1$$
$$\therefore \sin\theta = 0 \quad \text{or } \cos\theta = 1$$

$\sin\theta = 0$:
Ref angle: $\theta = 0^\circ$
General solution: $\theta = 0^\circ + k.180^\circ, k \in \mathbb{Z}$
$\cos\theta = 1$:
Ref angle: $\theta = 0^\circ$
General solution: $\theta = 0^\circ + k.360^\circ, k \in \mathbb{Z}$
Overall general solution: $\theta = 0^\circ + k.180^\circ, k \in \mathbb{Z}$

d.

$$5\tan 2x - 1 = \tan^2 2x + 5$$
$$\therefore \tan^2 2x - 5\tan 2x + 6 = 0$$
$$\therefore (\tan 2x - 3)(\tan 2x - 2) = 0$$
$$\therefore \tan 2x = 3 \text{ or } \tan 2x = 2$$

$\tan 2x = 3$:
Ref angle: $2x = 71.6^\circ$
General solution:
$2x = 71.57^\circ + k.180^\circ, k \in \mathbb{Z}$
$\therefore x = 35.8^\circ + k.90^\circ, k \in \mathbb{Z}$
$\tan 2x = 2$:
Ref angle: $2x = 63.4^\circ$
General solution:
$2x = 63.4^\circ + k.180^\circ, k \in \mathbb{Z}$
$\therefore x = 31.71^\circ + k.90^\circ, k \in \mathbb{Z}$

2.

a.

$$\begin{aligned}\text{LHS} &= \sin x \cos 3x - \cos x \sin 3x \\ &= \sin x \cos(2x+x) - \cos x \sin(2x+x) \\ &= \sin x \left(\cos 2x \cos x - \sin 2x \sin x\right) - \cos x \left(\sin 2x \cos x + \cos 2x \sin x\right) \\ &= \sin x \left(\left(2\cos^2 x - 1\right)\cos x - 2\sin x \cos x \sin x\right) - \cos x \left(2 \sin x \cos x \cos x + (2\cos^2 x - 1)\sin x\right) \\ &= \sin x \left(2\cos^3 x - \cos x - 2\sin^2 x \cos x\right) - \cos x \left(2\sin x\cos^2 x + 2\sin x\cos^2 x - \sin x\right) \\ &= 2\sin x\cos^3 x - \sin x \cos x - 2\sin^3 x \cos x - 2\sin x\cos^3 x - 2\sin x\cos^3 x + \sin x \cos x \\ &= -2\sin x\cos^3 x - 2\sin^3 x \cos x \\ &= -2\sin x \cos x(\cos^2 x + \sin^2 x) \\ &= -\sin 2x\end{aligned}$$

Therefore:

$$\begin{aligned}-\sin 2x &= \tan 140^\circ \\ \therefore \sin 2x &= -\tan 140^\circ \\ &= 0.839\end{aligned}$$

Ref angle: $2x = 57.05^\circ$

Sine is positive in the first and second quadrants.

$2x = 180^\circ - 57.05^\circ = 122.95^\circ$

General solution:

$$\begin{aligned}2x &= 57.05 + k.360^\circ \text{ or } 2x = 122.95^\circ + k.360^\circ, k \in \mathbb{Z} \\ \therefore x &= 28.525^\circ + k.180^\circ \text{ or } x = 61.475^\circ + k.180^\circ, k \in \mathbb{Z}\end{aligned}$$

b. Solutions for the interval $[0^\circ, 90^\circ]$: $x = 28.525^\circ$ or $x = 61.475^\circ$

Back to Exercise 2.2

Unit 2: Assessment

1.

a.

$$\begin{aligned}2\cos\theta - 1 &= 0 \\ \therefore \cos\theta &= \frac{1}{2}\end{aligned}$$

Ref angle: $\theta = 60^\circ$

Cosine is positive in the first and fourth quadrants.

$\theta = 360^\circ - 60^\circ = 300^\circ$

General solution: $\theta = 60^\circ + k.360^\circ$ or $\theta = 300^\circ + k.360^\circ, k \in \mathbb{Z}$

b.

$$\begin{aligned}2\sin^2\theta + \sin\theta - 1 &= 0 \\ \therefore (2\sin\theta - 1)(\sin\theta + 1) &= 0 \\ \therefore \sin\theta = \frac{1}{2} &\text{ or } \sin\theta = -1\end{aligned}$$

$\sin\theta = \frac{1}{2}$:

Ref angle: $\theta = 30^\circ$

Sine is positive in the first and second quadrants.

$\theta = 180^\circ - 30^\circ = 150^\circ$

General solution: $\theta = 30^\circ + k.360^\circ$ or $\theta = 150^\circ + k.360^\circ, k \in \mathbb{Z}$

$\sin\theta = 1$:

Ref angle: $\theta = 90^\circ$

General solution: $\theta = 90^\circ + k.180^\circ, k \in \mathbb{Z}$

c.

$$4\sin x+3\tan x=\frac{3}{\cos x}+4 \qquad \cos x\neq 0 \therefore x\neq 90^\circ+k.180^\circ, k\in\mathbb{Z}$$

$$\therefore 4\sin x+3\frac{\sin x}{\cos x}-\frac{3}{\cos x}-4=0$$

$$\therefore 4\sin x\cos x+3\sin x-3-4\cos x=0$$

$$\therefore 4\cos x(\sin x-1)+3(\sin x-1)=0$$

$$\therefore (\sin x-1)(4\cos x+3)=0$$

$$\therefore \sin x=1 \text{ or } \cos x=-\frac{3}{4}$$

$\sin x=1$:

Ref angle: $x=90^\circ$

General solution: $x=90^\circ+k.360^\circ, k\in\mathbb{Z}$

$\cos x=-\frac{3}{4}$:

Ref angle: $x=138.59^\circ$

Cosine is negative in the second and third quadrants.

$x=360^\circ-138.59^\circ=221.41^\circ$

General solution: $x=138.59^\circ+k.360^\circ$ or $x=221.49^\circ+k.360^\circ, k\in\mathbb{Z}$

For the interval $[0^\circ,180^\circ]$: $x=90^\circ$ or $x=138.59^\circ$

d.

$$6\cos\theta-5=\frac{4}{\cos\theta} \qquad \cos\theta\neq 0 \therefore \theta\neq 90^\circ+k.180^\circ, k\in\mathbb{Z}$$

$$\therefore 6\cos^2\theta-5\cos\theta-4=0$$

$$\therefore (3\cos\theta-4)(2\cos\theta+1)=0$$

$$\therefore \cos\theta=\frac{4}{3} \text{ or } \cos\theta=-\frac{1}{2}$$

$\cos\theta=\frac{4}{3}$ – No solution

$\cos\theta=-\frac{1}{2}$:

Ref angle: $\theta=120^\circ$

Cosine is negative in the second and third quadrants.

$\theta=360^\circ-(120^\circ)=240^\circ$

General solution: $\theta=120^\circ+k.360^\circ$ or $\theta=240^\circ+k.360^\circ, k\in\mathbb{Z}$

e.

$$\cos 2x-\cos x+1=0$$

$$\therefore 2\cos^2 x-1-\cos x+1=0$$

$$\therefore 2\cos^2 x-\cos x=0$$

$$\therefore \cos x(2\cos x-1)=0$$

$$\therefore \cos x=0 \text{ or } \cos x=\frac{1}{2}$$

$\cos x=0$:

Ref angle: $x=90^\circ$

General solution: $x=90^\circ+k.180^\circ, k\in\mathbb{Z}$

$\cos x=\frac{1}{2}$

Ref angle: $\theta=60^\circ$

Cosine is positive in the first and fourth quadrants.

$\theta=360^\circ-60^\circ=300^\circ$

General solution: $\theta=60^\circ+k.360^\circ$ or $\theta=300^\circ+k.360^\circ, k\in\mathbb{Z}$

For the interval $[0^\circ,360^\circ]$: $x=60^\circ$ or $x=90^\circ$ or $x=180^\circ$ or $x=300^\circ$ or $x=360^\circ$

2. $\cos 2\theta=2\cos^2\theta-1$

a.

$$\begin{aligned}\text{LHS} &= \cos 2\theta + 3\cos\theta - 1\\ &= 2\cos^2\theta - 1 + 3\cos\theta - 1\,.\\ &= 2\cos^2\theta + 3\cos\theta - 2 = \text{RHS}\end{aligned}$$

b.

$$\begin{aligned}\cos 2x - 3\cos x - 1 &= 0\\ \therefore 2\cos^2 x - 3\cos x - 2 &= 0\\ \therefore (2\cos x + 1)(\cos x - 2) &= 0\\ \therefore \cos x = -\frac{1}{2} \text{ or } \cos x &= 2\end{aligned}$$

$\cos\theta = -\dfrac{1}{2}$:

Ref angle: $\theta = 120^\circ$

Cosine is negative in the second and third quadrants.

$\theta = 360^\circ - (120^\circ) = 240^\circ$

General solution: $\theta = 120^\circ + k.360^\circ$ or $\theta = 240^\circ + k.360^\circ, k \in \mathbb{Z}$

$\cos x = 2$ – No solution

For the interval $[0^\circ, 360^\circ]$: $\theta = 120^\circ$ or $\theta = 240^\circ$

3.

$$\begin{aligned}3\cos 2x - \sin 2x - 1 &= 0\\ \therefore 3(\cos^2 x - \sin^2 x) - 2\sin x\cos x - (\sin^2 x + \cos^2) &= 0\\ \therefore 3\cos^2 - 3\sin^2 x - 2\sin x\cos x - \sin^2 x - \cos^2 x &= 0\\ \therefore 2\cos^2 x - 2\cos x\sin x - 4\sin^2 x &=0\\ \therefore \cos^2 x - \cos x\sin x - 2\sin^2 x &= 0\\ \therefore (\cos x + \sin x)(\cos x - 2\sin x) &= 0\\ \therefore \cos x = -\sin x \text{ or } \cos x &= 2\sin x\end{aligned}$$

$$\begin{aligned}\cos x &= -\sin x\\ \therefore 1 &= -\frac{\sin x}{\cos x}\\ \therefore \tan x &= -1\end{aligned}$$

Ref angle: $x = -45^\circ$

General solution: $x = 135^\circ + k.180^\circ, k \in \mathbb{Z}$

$$\begin{aligned}\cos x &= 2\sin x\\ \therefore 1 &= 2\cdot\frac{\sin x}{\cos x}\\ \therefore 2\tan x &= 1\\ \therefore \tan x &= \frac{1}{2}\end{aligned}$$

Ref angle: $x = 26.6^\circ$

General solution: $x = 26.6^\circ + k.180^\circ, k \in \mathbb{Z}$

For the interval $[0^\circ, 360^\circ]$: $x = 26.6^\circ$ or $x = 135^\circ$ or $x = 206.6^\circ$ or $x = 315^\circ$

Back to Unit 2: Assessment

Unit 3: Solve 2-D and 3-D trigonometry problems using sine and cosine rules

DYLAN BUSA

Unit outcomes

By the end of this unit you will be able to:

- Apply the sine rule correctly to solve 2-D and 3-D problems.
- Apply the cosine rule correctly to solve 2-D and 3-D problems.

What you should know

Before you start this unit, make sure you can:

- Determine the sides and angles in a right-angled triangle using the three basic trigonometric ratios. Refer to level 2 subject outcome 3.6 units 1 and 2 if you need help with this.
- Use the reduction formulae to simplify trigonometric ratios. Refer to level 3 subject outcome 3.3 units 1 and 2 if you need help with this.
- State and use the sine rule to solve 2-D problems. Refer to level 3 subject outcome 3.3 units 6 and 8 if you need help with this.
- State and use the cosine rule to solve 2-D problems. Refer to level 3 subject outcome 3.3 units 7 and 8 if you need help with this.
- State and use the compound and double angle formulae. Refer to unit 1 of this subject outcome if you need help with this.

Introduction

We learnt about the sine rule and the cosine rule in level 3 subject outcome 3.3 units 6 and 7. Remember that these rules give us a way to find the lengths of unknown sides and the sizes of unknown angles in non-right-angled triangles.

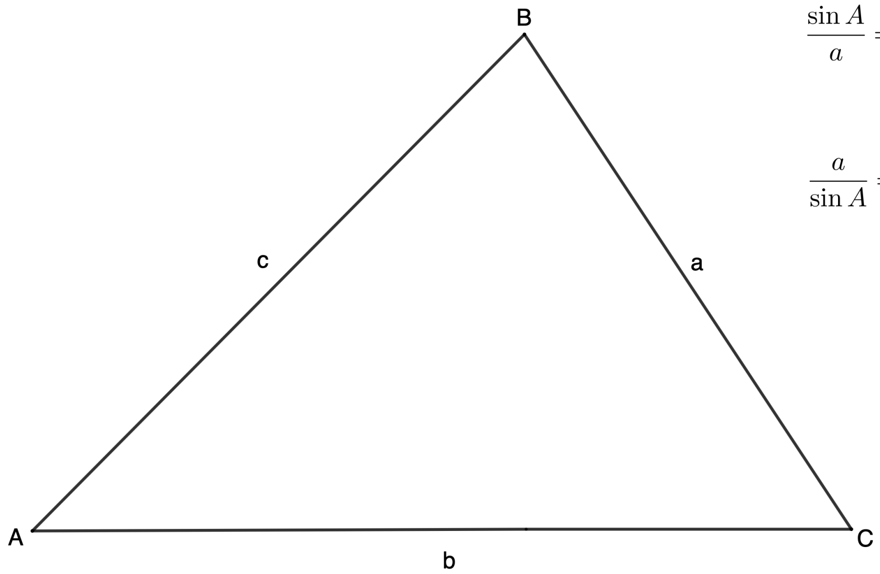
You should revise level 3 subject outcome 3.3 units 6, 7 and 8 before continuing.

Revise the sine and cosine rules

Before we learn anything new, let us revise what we already know.

The sine rule:

In any triangle ΔABC:

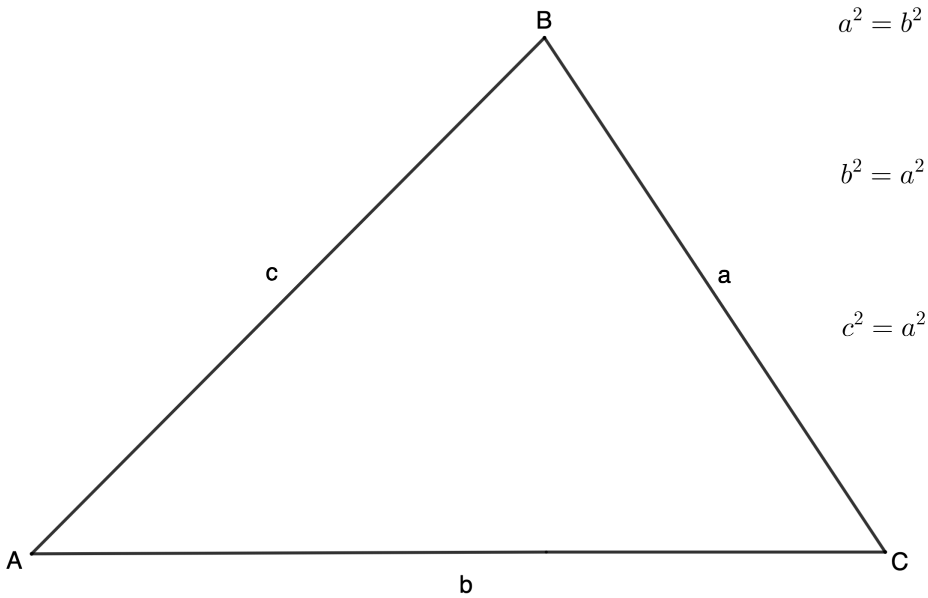


$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

The cosine rule:

In any ΔABC:



$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$b^2 = a^2 + c^2 - 2ac\cos B$$

$$c^2 = a^2 + b^2 - 2ac\cos C$$

- The cosine rule works when we know the lengths of any two sides and the size of the **included** angle.
- The side on the LHS of the formula is always the side **opposite** the angle whose cosine we take.

Take note!

Use the sine rule if:

- no right angle is given
- two sides and an angle are given (not the included angle)
- two angles and a side are given.

Use the cosine rule if:

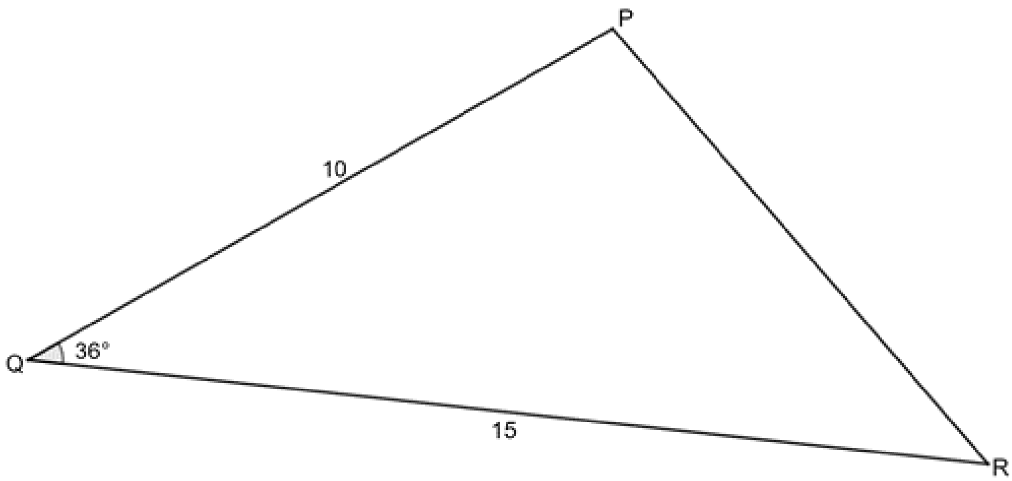
- no right angle is given
- two sides and the included angle are given
- three sides are given.

The difference between level 3 and level 4 is that we will use these rules to solve problems in two and three dimensions.

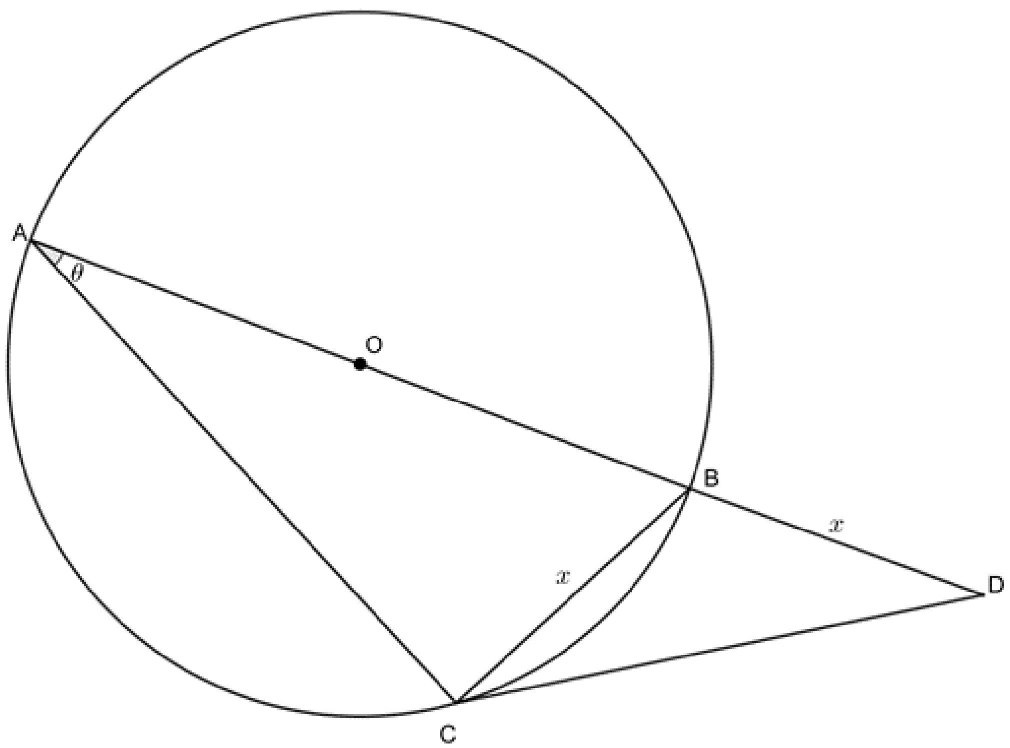
Revise using the sine and cosine rules to solve 2-D problems by completing the following exercise.

Exercise 3.1

1. In ΔPQR, $\hat{Q} = 36^{\circ}$, $p = 15$ and $r = 10$. Solve the triangle:

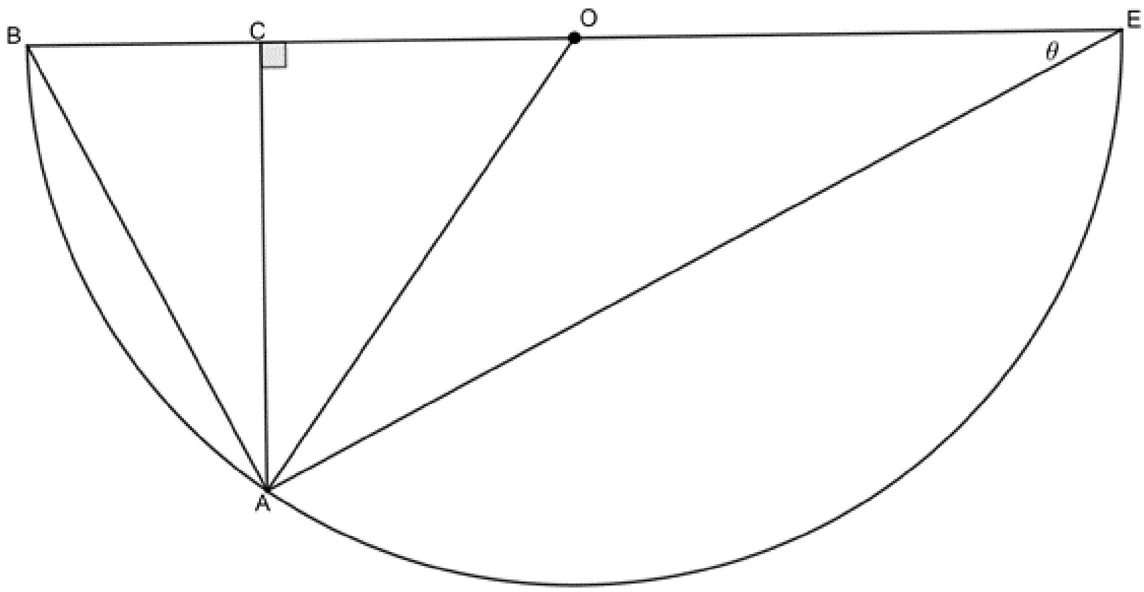


2. In the diagram below, ABD is a straight line through the centre of the circle. $BD = BC = x$ and $B\hat{A}C = \theta$. Show that $CD = 2x^2(1 + \sin\theta)$.



Question 3 adapted from Everything Maths Grade 12 Exercise 4-5 question 1

3. In the diagram below, O is the centre of the semi-circle BAE:



a. Find $A\hat{O}C$ in terms of θ.

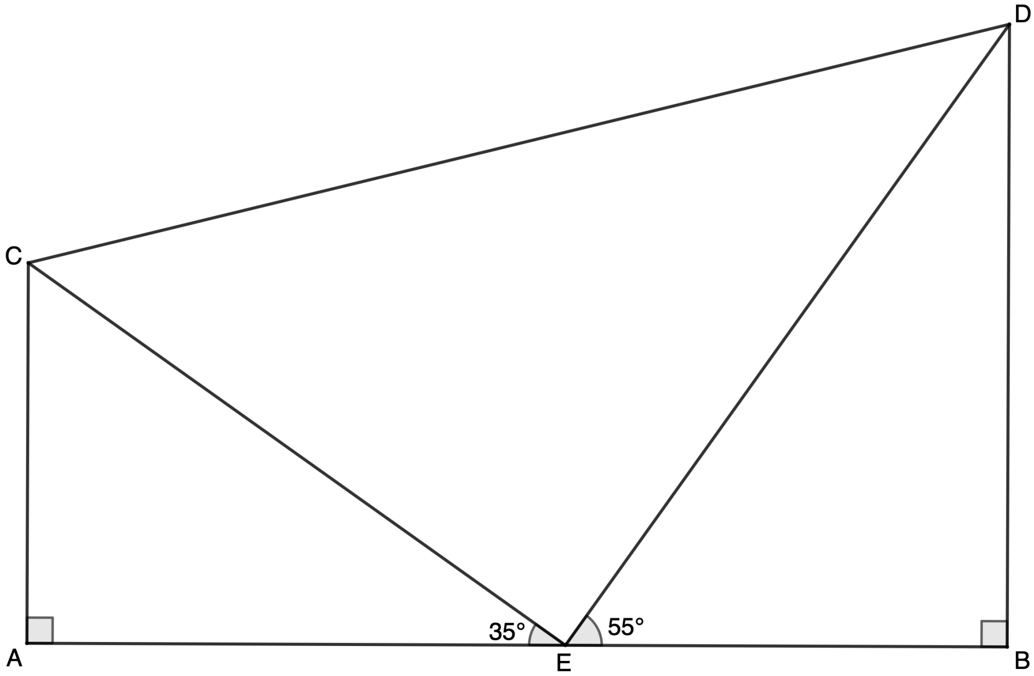
b. In ΔABE, determine an expression for $\cos\theta$.

c. In ΔACE, determine an expression for $\sin\theta$.

d. In ΔACO, determine an expression for $\sin 2\theta$.

e. Use the results from the previous questions to show that $\sin 2\theta = 2\sin\theta\cos\theta$.

4. Two vertical towers AC and BD are $7\ \text{m}$ and $10\ \text{m}$ high, respectively. Point E lies between the two towers. The angle of elevation from E to C is $35°$ and E to D is $55°$. A cable is needed to connect C and D.



a. Determine the minimum length of cable needed to connect C and D (to the nearest metre).

b. How far apart are the bases of the two towers (to the nearest metre)?

The full solutions are at the end of the unit.

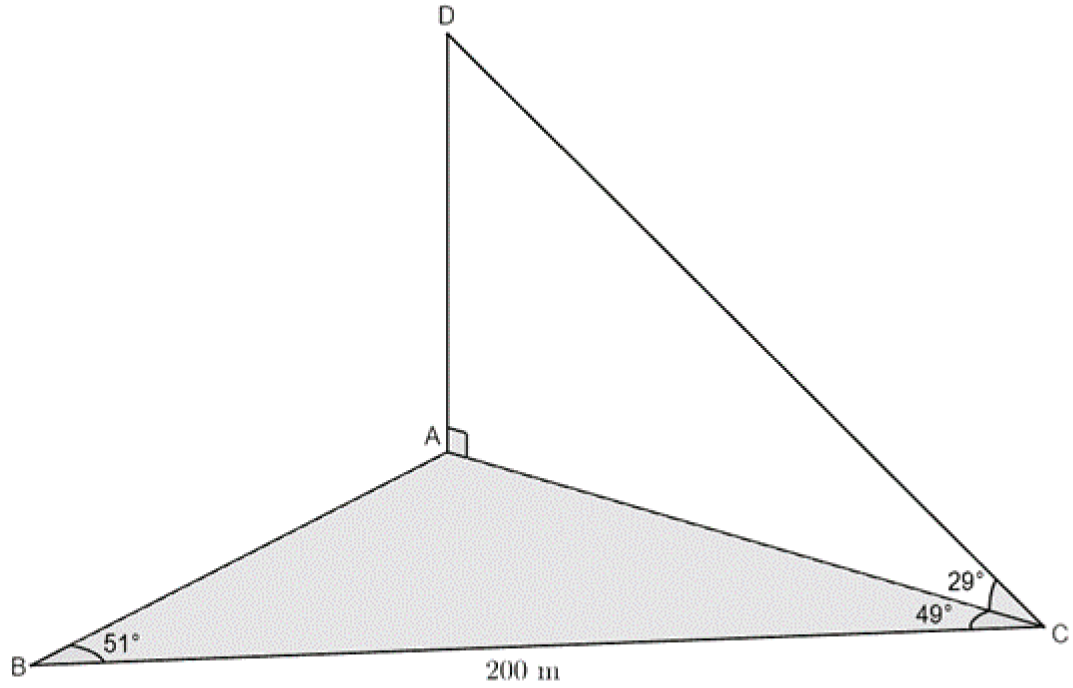
Problems in three dimensions

As we have seen, we can use the sine and cosine rules to find the lengths sides and sizes of angles in non-right-angled triangles in two dimensions. We can apply these same techniques to work in three dimensions. This means that we can start to solve more real-world problems.

But working in three dimensions (especially visualising three dimensions on a two-dimensional screen or piece of paper) takes time and practise for most of us. If at first, you find it hard, don't give up. Working in three dimensions can be learnt. It just needs practise.

Example 3.1

AD is a vertical flagpole and its base, A, is in the same horizontal plane as the points B and C. The angle of elevation from point C to the top of the flagpole is $29°$. The distance BC is 200 m, $A\hat{B}C = 51°$ and $A\hat{C}B = 49°$. Determine the height of the flagpole.



Solution

In this example, a sketch of the situation has been given. Sometimes, you will need to create your own sketch. Notice how ΔABC has been shaded. This helps us see that this triangle lies on the ground and is at right angles to the triangle formed by points A, C and D. Remember that the flagpole is vertical (meaning it is at right-angles to the ground). The given lengths and angles have been filled in.

We need to calculate the length of AD. We have one angle inside this triangle, and it is a right-angled triangle. Therefore, if we can find the length of AC, we can use the tangent ratio to find AD. We choose AC because it is the side shared by both triangles.

We can use the sine rule to calculate AC but, to do so, we first need to calculate the size of $B\hat{A}C$.

$$B\hat{A}C = 180° - A\hat{B}C - A\hat{C}B \quad (\angle\text{s in } \Delta \text{ suppl})$$
$$\therefore B\hat{A}C = 180° - 51° - 49° = 80°$$
$$\frac{AC}{\sin 51°} = \frac{BC}{\sin 80°}$$
$$\therefore AC = \frac{200 \times \sin 51°}{\sin 80°}$$
$$\therefore AC = 157.83$$

Remember to keep the full answer for AC in your calculator's memory. Don't round off as this will affect the accuracy of your final answer.

Now that we know AC, we can find AD:

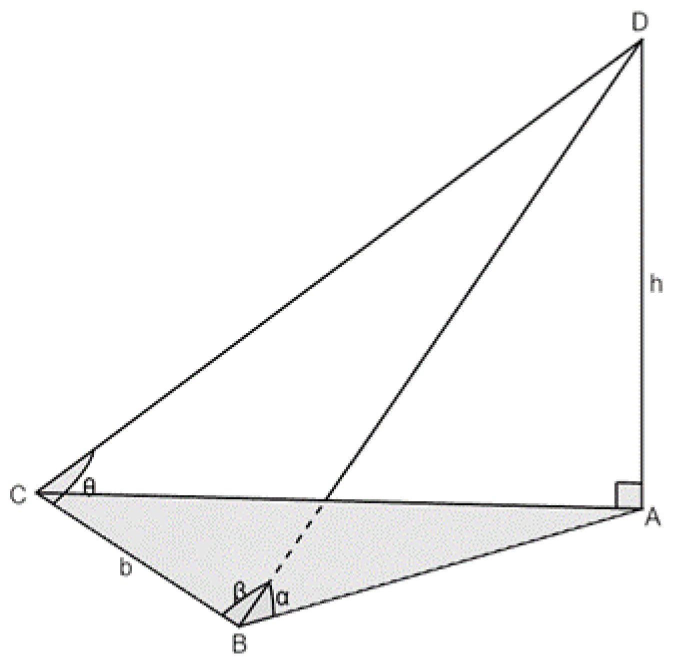
$$\tan 29^\circ = \frac{AD}{BC}$$
$$\therefore AD = 157.83 \times \tan 29^\circ$$
$$= 87.48 \text{ m}$$

If you are still having difficulty picturing the above situation in your mind, visit this "interactive version of the diagram". You can click and drag to view the situation from different angles.

Example 3.2

Example adapted from Everything Maths Grade 12 Worked example 16

D is the top of a building of height h. The base of the building is at A and ΔABC lies on the ground (a horizontal plane). $BC = b$, $D\hat{B}A = \alpha$, $D\hat{B}C = \beta$ and $D\hat{C}B = \theta$.



Show that $h = \dfrac{b \sin\alpha \sin\theta}{\sin(\beta + \theta)}$.

Solution

We are asked to find an expression for h. Because we have been given an angle in ΔABD, this is the triangle we will focus on. To find h, we need to find either AB or BD. DB is a shared side between ΔABD and ΔBCD, and ΔBCD is a good triangle to work in because we have been give information about it. Let's start by writing an expression for h in ΔABD.

$$\sin\alpha = \frac{h}{BD}$$
$$\therefore h = BD \times \sin\alpha$$

Now, let's find an expression for BD in ΔBCD.

$$\frac{BD}{\sin\theta} = \frac{b}{\sin B\hat{D}C}$$
$$\therefore BD = \frac{b \times \sin\theta}{\sin B\hat{D}C}$$

But we can work out what $B\hat{D}C$ is:

$$B\hat{D}C = 180^\circ - B\hat{C}D - C\hat{B}D \quad (\angle\text{s in } \Delta \text{ suppl})$$
$$\therefore B\hat{D}C = 180^\circ - \beta - \theta$$

Therefore:

$$\begin{aligned}\sin BDC &= \sin(180^\circ - \beta - \theta)\\ &= \sin(180^\circ - (\beta + \theta))\\ &= \sin(\beta + \theta)\end{aligned}$$

Therefore:

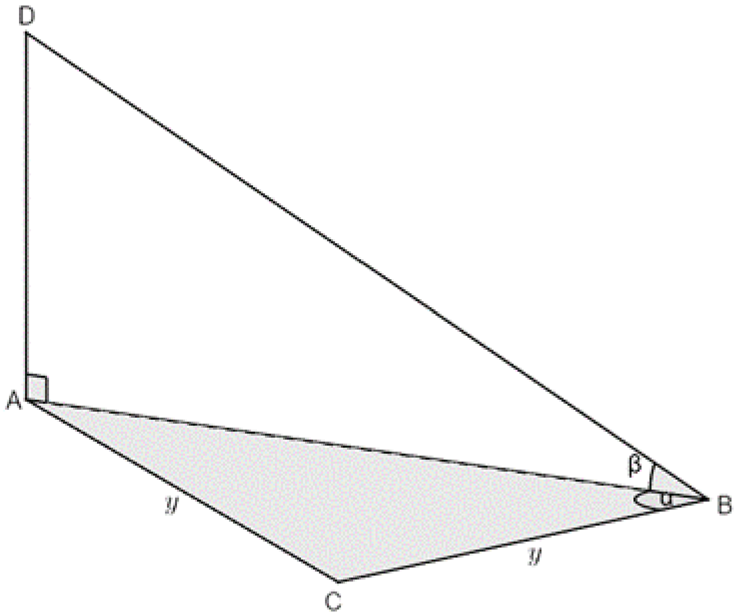
$$BD = \frac{b \times \sin\theta}{\sin(\beta + \theta)}$$

Now we can replace BD in our original expression with this new expression for BD.

$$\begin{aligned}h &= BD \times \sin\alpha\\ &= \frac{b \times \sin\theta}{\sin(\beta+\theta)} \times \sin\alpha\\ &= \frac{b\sin\alpha\sin\theta}{\sin(\beta+\alpha)}\end{aligned}$$

Exercise 3.2

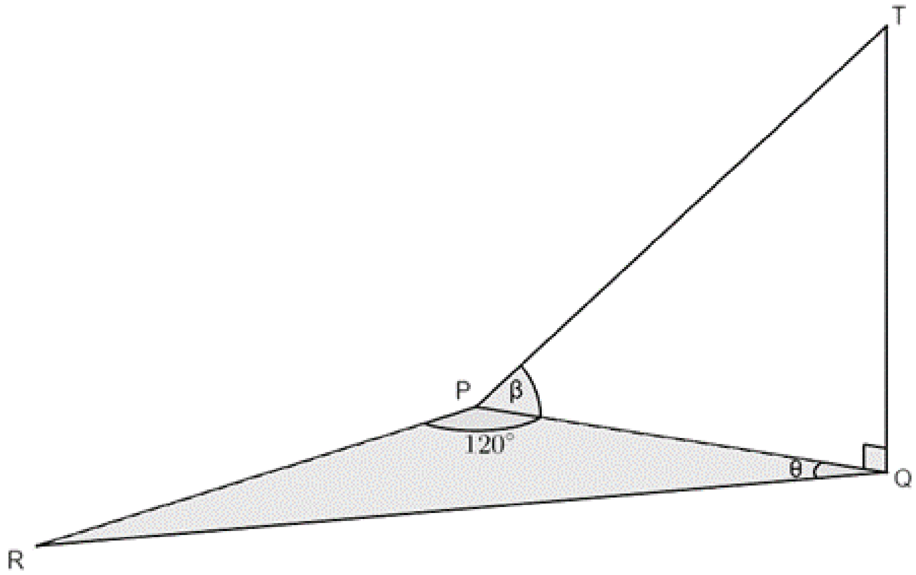
1. A cell tower, AD, has its base at A. ΔABC is a horizontal plane on the ground. The angle of elevation to the top of the tower from B is β. A surveyor is standing at C such that he is the same distance, y, from A and B. $A\hat{B}C = \alpha$.



a. Find an expression for the height of the tower in terms of y, α and β.

b. If it is given that $y = 80\ \text{m}$, $\alpha = 39^\circ$ and $\beta = 23^\circ$, calculate the height of the tower.

2. A building is represented below as QT. ΔPQR is on the horizontal plane with $Q\hat{P}R = 120^\circ$. $Q\hat{P}T = \beta$ and $P\hat{Q}R = \theta$.

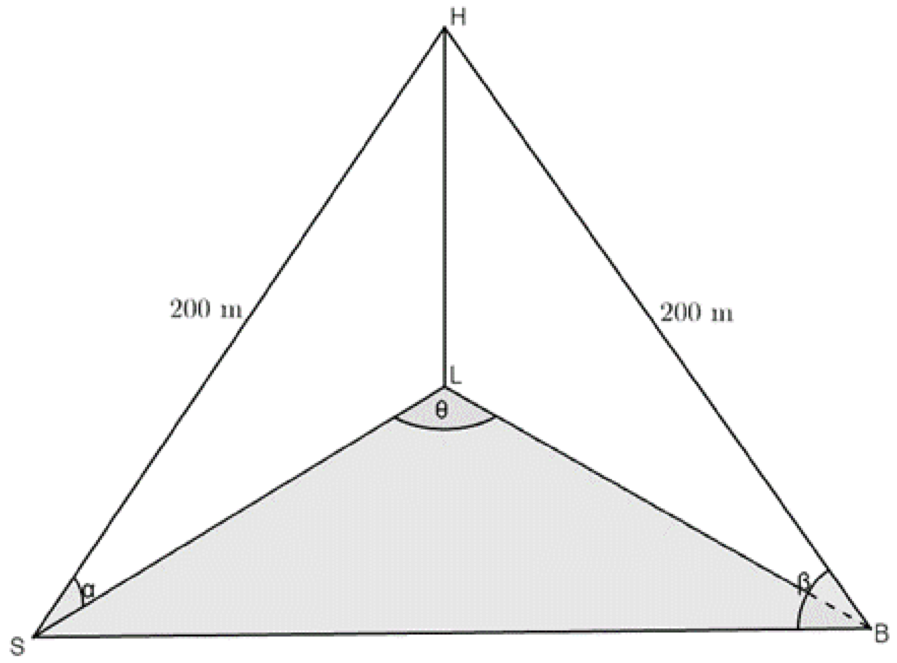


a. If $RQ = p$, show that the height of the building, QT, is given by $p \tan\beta \left(\cos\theta - \frac{\sqrt{3}\sin\theta}{3}\right)$.

b. If it is further given that $QR = 42\ \text{m}$, $\theta = 18^\circ$ and $\beta = 30^\circ$, calculate the height of the building to one decimal place.

Question 3 adapted from Everything Maths Grade 12 Exercise 4-6 question 4

3. Two ships at sea can see a lighthouse on the shore. The distance from the top of the lighthouse (H) to ship S and to ship B is $200\ \text{m}$. The angle of elevation from S to H is α, $H\hat{B}S = \beta$ and $S\hat{L}B = \theta$.



Show that the distance between the two ships is given by $SB = 400\cos\beta$.

The full solutions are at the end of the unit.

Summary

In this unit you have learnt the following:

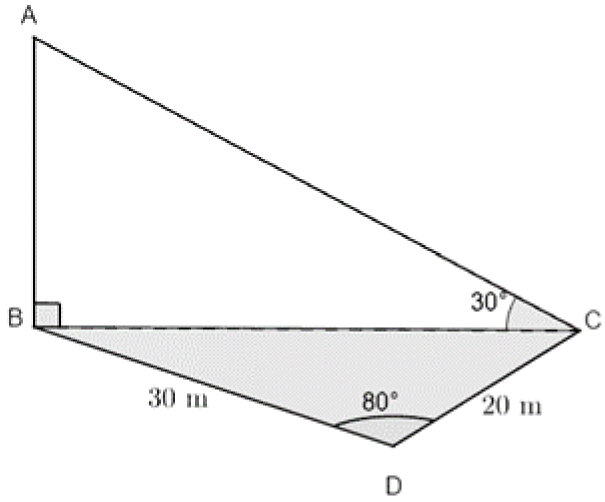
- How to use the trigonometric ratios, reduction formulae, compound and double angle formulae, the sine rule, and the cosine rule to solve problems in three dimensions.

Unit 3: Assessment

Suggested time to complete: 35 minutes

Question 1 adapted from NC(V) Level 4 Paper 2 November 2016 question 2.6

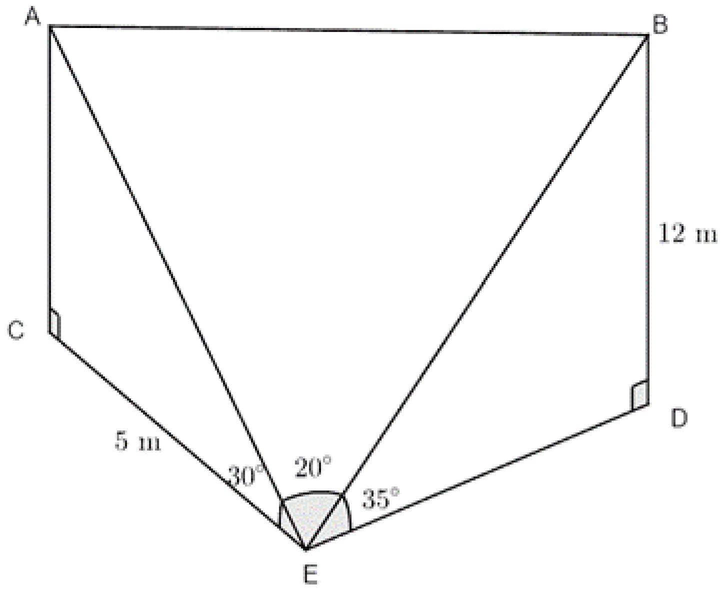
1. Given below is the side view of a vertical cliff. Points B, C and D are on the same horizontal plane. A is a point on the top of the cliff such that $AB \perp BC$. $BD = 30 \text{ m}$, $DC = 20 \text{ m}$, $A\hat{B}C = 90^\circ$, $B\hat{D}C = 80^\circ$ and $A\hat{C}B = 30^\circ$.



Determine the height of the cliff, AB, to one decimal place.

Question 2 adapted from NC(V) Level 4 Paper 2 November 2015 question 2.6

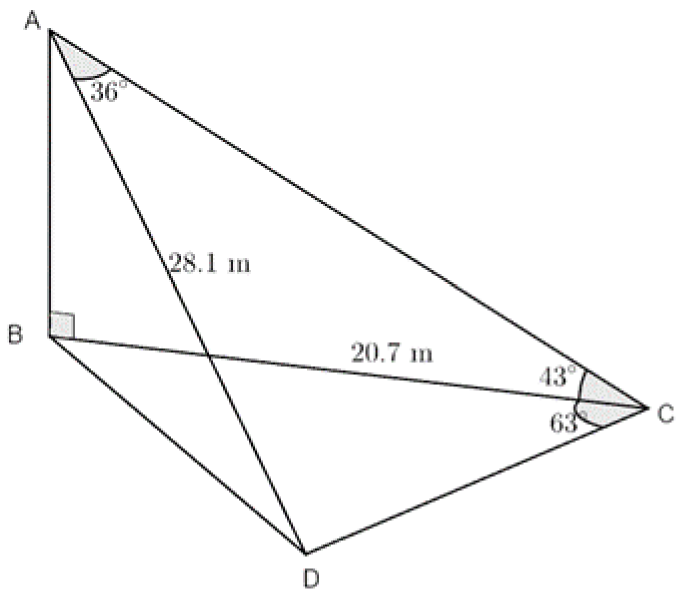
2. In the diagram, AC and BD represent the two vertical towers. The points C, D and E are on the same horizontal plane. From E the angles of elevation of A and B are 30° and 35° respectively. $A\hat{E}B = 20^\circ$, $CE = 5\text{ m}$ and $BD = 12\text{ m}$.



   a. Calculate the length of BE.
   b. Calculate the length of AE.
   c. Determine the distance, AB, between the top of the two towers.

Question 3 adapted from NC(V) Level 4 Paper 2 November 2014 question 2.6

3. An eagle, on top of a vertical cliff AB, spots two rabbits a distance away. The points B, C and D lie in the same horizontal plane. The rabbit at D is 28.1 m away from the eagle and the angle of elevation from the rabbit at C to the eagle is 43°. $BC = 20.7\text{ m}$, $B\hat{C}D = 63^\circ$ and $C\hat{A}D = 36^\circ$.



a. Calculate the distance between the rabbit at C and the eagle at A.
b. Calculate the distance between the two rabbits.

The full solutions are at the end of the unit.

Unit 3: Solutions

Exercise 3.1

1.

$$q^2 = r^2 + p^2 - 2rp\cos Q$$
$$\therefore q^2 = 10^2 + 15^2 - 2 \cdot 10 \cdot 15 \cdot \cos 36^\circ$$
$$\therefore q = 9.07$$

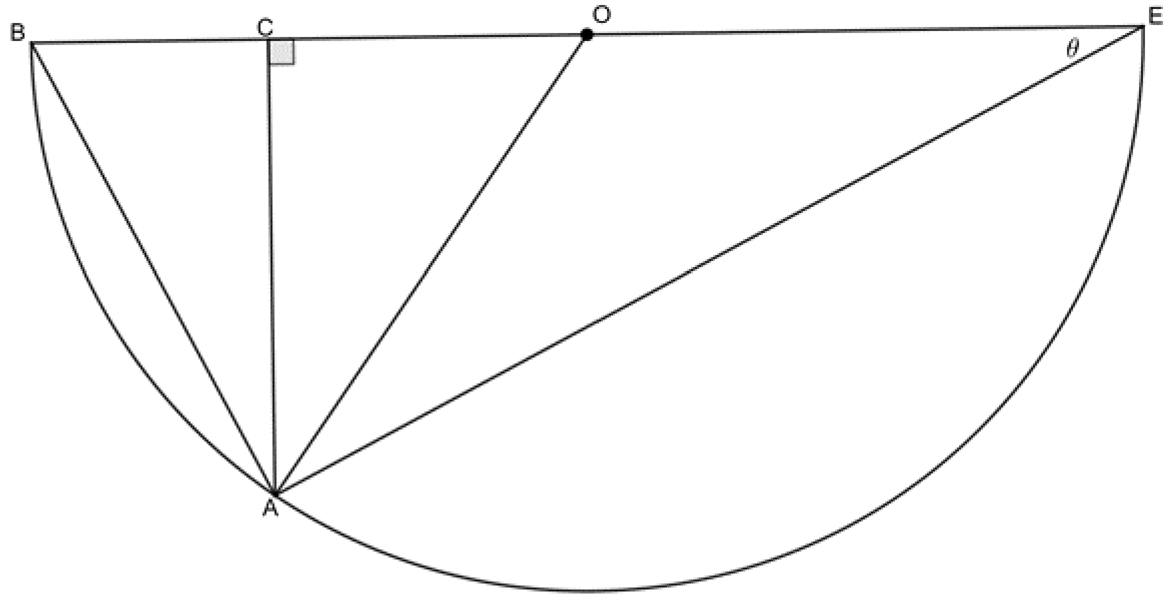
$$\frac{\sin P}{15} = \frac{\sin Q}{9.07}$$
$$\therefore \sin P = \frac{15 \times \sin Q}{9.07}$$
$$\therefore \hat{P} = 76.43^\circ$$

$$\hat{R} = 180^\circ - 36^\circ - 76.43^\circ \quad (\angle\text{s in } \Delta \text{ suppl})$$
$$\therefore \hat{R} = 67.57^\circ$$

2.

$$A\hat{C}B = 90^\circ \quad (\angle\text{s in a semi-circle})$$
$$\therefore C\hat{B}D = 90^\circ + \theta \quad (\text{ext } \angle \text{ of } \Delta = \text{opp int } \angle\text{s})$$
$$CD = x^2 + x^2 - 2 \cdot x \cdot x \cdot \cos C\hat{B}D$$
$$= 2x^2 - 2x^2\cos(90^\circ + \theta)$$
$$= 2x^2 - 2x^2(-\sin\theta)$$
$$= 2x^2 + 2x^2\sin\theta$$
$$= 2x^2(1 + \sin\theta)$$

3.



a.

$$AO = EO \quad \text{(radii)}$$
$$\therefore O\hat{A}E = \theta \quad \text{(isos } \Delta)$$
$$\therefore A\hat{O}C = 2\theta \quad (\text{ext } \angle \text{ of } \Delta = \text{opp int } \angle \text{s})$$

b. In ΔABE:
$\cos\theta = \dfrac{AE}{BE}$

c. In ΔACE:
$\sin\theta = \dfrac{AC}{AE}$

d. In ΔACO:
$\sin 2\theta = \dfrac{AC}{AO}$

e. $\sin 2\theta = \dfrac{AC}{AO}$

But $\sin\theta = \dfrac{AC}{AE}$. Therefore, $AC = AE\sin\theta$. Therefore, $\sin 2\theta = \dfrac{AE\sin\theta}{AO}$.

But $\cos\theta = \dfrac{AE}{BE}$. Therefore, $AE = BE\cos\theta$. Therefore, $\sin 2\theta = \dfrac{BE\cos\theta\sin\theta}{AO}$.

But $BE = 2AO$. Therefore, $\sin 2\theta = \dfrac{2AO\cos\theta\sin\theta}{AO} = 2\sin\theta\cos\theta$.

4.

a.

$$\sin 35^\circ = \frac{AC}{CE} = \frac{7}{CE}$$
$$\therefore CE = \frac{7}{\sin 35^\circ}$$

$$\sin 55^\circ = \frac{BD}{DE} = \frac{10}{DE}$$
$$\therefore DE = \frac{10}{\sin 55^\circ}$$

$$C\hat{E}D = 180^\circ - 35^\circ - 55^\circ = 90^\circ \quad (\angle\text{s on str line suppl})$$
$$\therefore CD^2 = CE^2 + DE^2 \quad (\text{Pythagoras})$$
$$\therefore CD^2 = \left(\frac{7}{\sin 35^\circ}\right)^2 + \left(\frac{10}{\sin 55^\circ}\right)^2$$
$$\therefore CD = 17.26$$

b.

$$\tan 35^\circ = \frac{AC}{AE} = \frac{7}{AE}$$
$$\therefore AE = \frac{7}{\tan 35^\circ}$$
$$\tan 55^\circ = \frac{BD}{BE} = \frac{10}{BE}$$
$$\therefore BE = \frac{10}{\tan 55^\circ}$$

$$\begin{aligned} AB &= AE + BE \\ &= \frac{7}{\tan 35^\circ} + \frac{10}{\tan 55^\circ} \\ &= 17.00 \end{aligned}$$

Back to Exercise 3.1

Exercise 3.2

1.

a.

$$\tan\beta = \frac{AD}{AB}$$
$$\therefore AD = AB \times \tan\beta$$

In ΔABC:

$$B\hat{A}C = \alpha \quad (\text{isos } \Delta)$$
$$\therefore A\hat{C}B = 180^\circ - 2\alpha \quad (\angle\text{s in } \Delta \text{ suppl})$$

$$\frac{AB}{\sin(180^\circ - 2\alpha)} = \frac{y}{\sin\alpha}$$
$$\therefore AB = \frac{y \times \sin 2\alpha}{\sin\alpha}$$
$$\therefore AD = \frac{y \times \sin 2\alpha}{\sin\alpha} \times \tan\beta$$

b. $y = 80 \text{ m}$, $\alpha = 89^\circ$, $\beta = 23^\circ$:

$$\begin{aligned} AD &= \frac{y \times \sin 2\alpha}{\sin\alpha} \times \tan\beta \\ &= \frac{80 \sin(2 \times 39^\circ)}{\sin 39^\circ} \times \tan 23^\circ \\ &= 52.78 \text{ m} \end{aligned}$$

2.

a. $p\tan\beta\left(\cos\theta - \frac{\sqrt{3}\sin\theta}{3}\right)$

$$\tan\beta = \frac{QT}{PQ}$$
$$\therefore QT = PQ\tan\beta$$

$$\begin{aligned}\hat{R} &= 180^\circ - 120^\circ - \theta \quad (\angle\text{s in } \Delta \text{ suppl})\\ &= 60^\circ - \theta\end{aligned}$$

$$\frac{p}{\sin 120^\circ} = \frac{PQ}{\sin R}$$
$$\therefore PQ = \frac{p\sin(60^\circ - \theta)}{\sin 120^\circ}$$

$$\begin{aligned}QT &= \frac{p\sin(60^\circ - \theta)}{\sin 120^\circ}\tan\beta\\ &= p\tan\beta\left(\frac{\sin(60^\circ - \theta)}{\sin 120^\circ}\right)\\ &= p\tan\beta\left(\frac{\sin 60^\circ\cos\theta - \cos 60^\circ\sin\theta}{\sin 60^\circ}\right)\\ &= p\tan\beta\left(\frac{\frac{\sqrt{3}}{2}\cos\theta - \frac{1}{2}\sin\theta}{\frac{\sqrt{3}}{2}}\right)\\ &= p\tan\beta\left(\frac{\frac{\sqrt{3}\cos\theta - \sin\theta}{2}}{\frac{\sqrt{3}}{2}}\right)\\ &= p\tan\beta\left(\frac{\sqrt{3}\cos\theta - \sin\theta}{2} \times \frac{2}{\sqrt{3}}\right)\\ &= p\tan\beta\left(\frac{\sqrt{3}\cos\theta - \sin\theta}{\sqrt{3}}\right) \qquad \text{Muliply by } \frac{\sqrt{3}}{\sqrt{3}}\\ &= p\tan\beta\left(\cos\theta - \frac{\sqrt{3}\sin\theta}{3}\right)\end{aligned}$$

b. $QR = 42 \text{ m}, \theta = 18^\circ, \beta = 30^\circ$:

$$\begin{aligned}QT &= p\tan\beta\left(\cos\theta - \frac{\sqrt{3}\sin\theta}{3}\right)\\ &= 42\tan 30^\circ\left(\cos 18^\circ - \frac{\sqrt{3}\sin 18^\circ}{3}\right)\\ &= 18.7 \text{ m}\end{aligned}$$

3.

In ΔBHS:

$$B\hat{S}H = \beta \quad (\text{isos } \Delta)$$

$$\therefore B\hat{H}S = 180^\circ - 2\beta \quad (\angle\text{s in } \Delta \text{ suppl})$$

$$\begin{aligned} SB^2 &= 200^2 + 200^2 - 2 \cdot 200 \cdot 200 \cdot \cos(180^\circ - 2\beta) \\ &= 2 \times 200^2 - 2 \times 200^2 \times (-\cos 2\beta) \\ &= 2 \times 200^2 + 2 \times 200^2 \times \cos 2\beta \\ &= 2 \times 200^2 (1 + \cos 2\beta) \\ &= 2 \times 200^2 (1 + 2\cos^2 \beta - 1) \\ &= 2 \times 200^2 (2\cos^2 \beta) \\ &= 4 \times 200^2 \cos^2 \beta \\ \therefore SB &= 2 \times 200 \times \cos \beta \\ &= 400 \cos \beta \end{aligned}$$

Back to Exercise 3.2

Unit 3: Assessment

1.

$$\begin{aligned} \tan 30^\circ &= \frac{AB}{BC} \\ \therefore AB &= BC \tan 30^\circ \\ BC^2 &= 30^2 + 20^2 - 2 \cdot 30 \cdot 20 \cdot \cos 80^\circ \\ \therefore BC &= 33.04 \\ \therefore AB &= 33.04 \tan 30^\circ \\ &= 19.1 \text{ m} \end{aligned}$$

2.

a.

$$\begin{aligned} \sin 35^\circ &= \frac{12}{BE} \\ \therefore BE &= \frac{12}{\sin 35^\circ} \\ &= 20.92 \text{ m} \end{aligned}$$

b.

$$\begin{aligned} \cos 30^\circ &= \frac{5}{AE} \\ \therefore AE &= \frac{5}{\cos 30^\circ} \\ &= 5.77 \text{ m} \end{aligned}$$

c.

$$\begin{aligned} AB^2 &= AE^2 + BE^2 - 2 \cdot AE \cdot BE \cdot \cos A\hat{E}B \\ &= 5.77^2 + 20.92^2 - 2 \cdot 5.77 \cdot 20.92 \cdot \cos 20^\circ \\ \therefore AB &= 15.62 \text{ m} \end{aligned}$$

3.

a.

$$\begin{aligned} \cos 43^\circ &= \frac{20.7}{AC} \\ \therefore AC &= \frac{20.7}{\cos 43^\circ} \\ &= 28.30 \text{ m} \end{aligned}$$

b.

$$\begin{aligned} CD^2 &= AD^2 + AC^2 - 2 \cdot AD \cdot AC \cdot \cos C\hat{A}D \\ &= 28.1^2 + 28.3^2 - 2 \cdot 28.1 \cdot 28.3 \cdot \cos 36^\circ \\ \therefore CD &= 17.43 \text{ m} \end{aligned}$$

Back to Unit 3: Assessment

Media Attributions

Unit 4: An introduction to radians

DYLAN BUSA

Unit 4 outcomes

By the end of this unit you will be able to:

- Define radian measure.
- Convert from degrees to radians.
- Convert from radians to degrees.

What you should know

Before you start this unit, make sure you can:

- Find the general solution for trigonometric equations. Refer to unit 2 of this subject outcome if you need help with this.

Introduction

Throughout levels 2, 3 and 4, whenever we have measured an angle or given its size, we have done so in degrees. We are very familiar with degrees. There are 360° in one full revolution. A half revolution is 180°. A quarter revolution is 90°, and so on.

So, in rotating through a revolution or circle, you have turned 360°. But have you ever asked yourself why there are 360° in one revolution? Why not 400° or 25°. The number 360 was chosen on purpose, although we are not quite sure why. Some theories are that it is based on old lunar calendars or the fact that the earth takes about 360days to travel around the sun (1° per day).

Now degrees work fine for some applications but it is not the best method because it is somewhat arbitrary. Mathematicians hate things being arbitrary, so they came up with a better way to measure angles. It is called **radian measure**. A radian is based on the **radius** of a circle.

What is a radian?

An angle of 1 radian (or rad) is made when we wrap a line the same length as the radius of a circle around the circumference of the circle. In other words, an angle of 1 radian is created by an arc length equal to the radius of a circle (see figure 1).

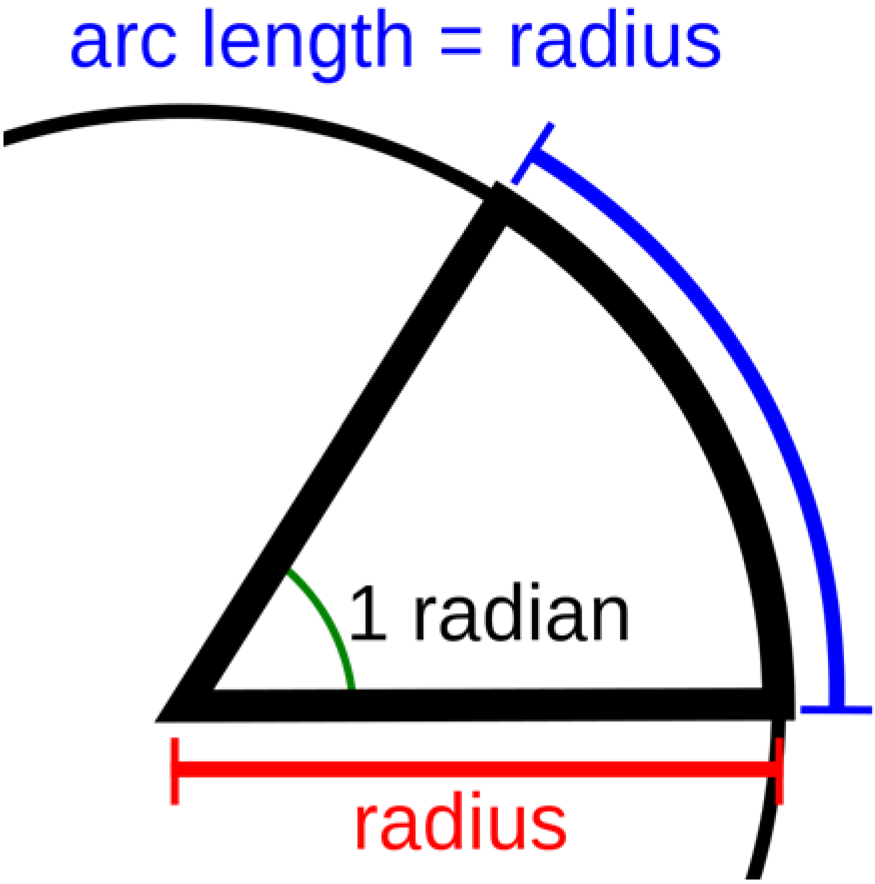


Figure 1: The definition of a radian

Note

If you have an internet connection, watch these excellent simple explanations of radians:

- a short animation
- the interactive simulation called "What is a Radian?"

For a fuller explanation, you can also watch the video called “What are Radians?”.

What are Radians (Duration: 05.39)

Unlike degrees, radians have no unit. Radians are pure numbers. Radians are also the official method of measuring angles.

Now, we know that the circumference of a circle is $2\pi r$. If to move through an angle of one radian, we have to move a distance around the circumference of r units, this means that there are 2π radians in one full revolution, or π radians in half a revolution (see figure 2). Click on figure 2 to play the animation.

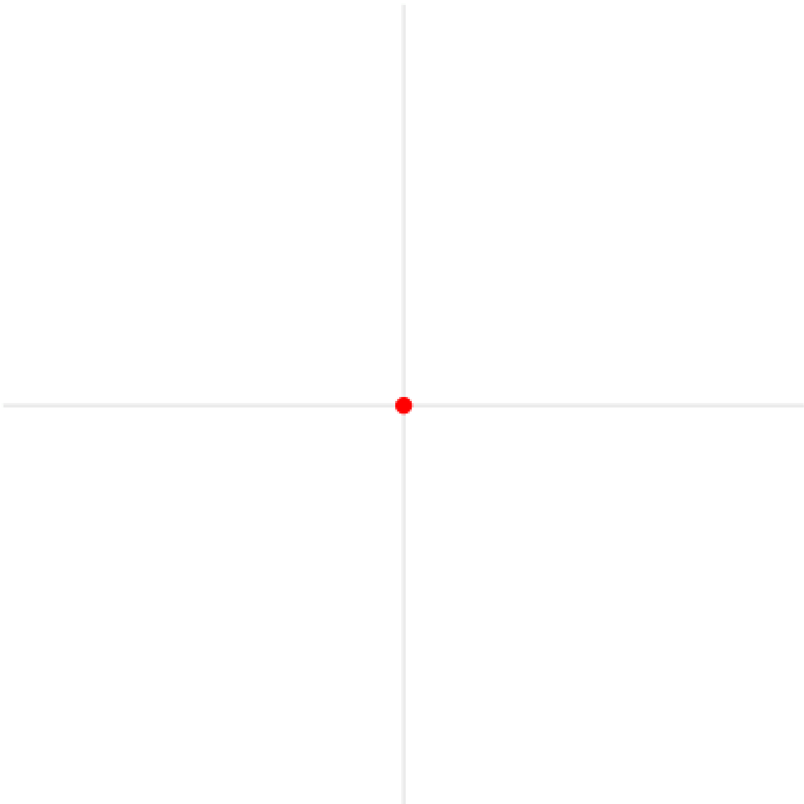


Figure 2: Radians

We can use this to calculate that there are $\frac{180^\circ}{\pi} \approx 57.2958^\circ$ in one radian.

Take note!

To convert from **degrees** to **radians**, multiply by π and divide by 180°.
To convert from **radians** to **degrees**, multiply by 180° and divide by π.

Degrees	**Radians**
30°	$\frac{\pi}{6}$
45°	$\frac{\pi}{4}$
60°	$\frac{\pi}{3}$
90°	$\frac{\pi}{2}$
180°	π
270°	$\frac{3\pi}{2}$
360°	2π

Example 4.1

1. Convert the following degrees to radians:
   a. 60°
   b. 50°
   c. 300°
   d. 272°
2. Convert the following radians to degrees:
   a. 2π
   b. $\frac{8\pi}{3}$
   c. 16π

d. 0.6512975

e. 3

Solutions

1. To convert from **degrees** to **radians**, we have to multiply the angle by π and then divide by $180°$.

   a. $60° = \frac{60° \times \pi}{180°} = \frac{\pi}{3}$

   b. $50° = \frac{50° \times \pi}{180°} = 0.278\pi \approx 0.873$

   c. $300° = \frac{300° \times \pi}{180°} = \frac{5\pi}{3} \approx 5.236$

   d. $272° = \frac{272° \times \pi}{180°} = \frac{68\pi}{45} \approx 4.747$

2. To convert from **radians** to **degrees**, we have to multiply by $180°$ and then divide by π.

   a. $2\pi = \frac{2\pi \times 180°}{\pi} = 360°$

   b. $\frac{8\pi}{3} = \frac{8\pi \times 180°}{3\pi} = 480°$

   c. $16\pi = \frac{16\pi \times 180°}{\pi} = 2\ 880°$

   d. $0.6512975 = \frac{0.6512975 \times 180°}{\pi} \approx 37.317°$

   e. $3 = \frac{3 \times 180°}{\pi} \approx 171.887°$

Exercise 4.1

1. Convert the following angles from degrees to radians:

   a. $30°$

   b. $75°$

   c. $135°$

   d. $215°$

2. Convert the following angles from radians to degrees:

   a. 5π

   b. $\frac{\pi}{6}$

   c. $\frac{7\pi}{2}$

   d. π^2

The full solutions are at the end of the unit.

Working with radians

Everything that we have learnt to do in degrees, we can do in radians. Some of it is even easier to do in radians. However, switching from degrees to radians can sometimes be a little confusing. Therefore, if you are ever asked to give your final answer in radians, you can work in degrees and then simply convert your final answer to radians.

Example 4.2

Solve for θ in radians if $0 \leq \theta \leq 6.282$ and $2\cos\theta - 1 = 0$.
(Note: This interval restriction is very often given as $0 \leq \theta \leq 2\pi$.)

Solution

We have been asked to solve for θ in radians. If we like, we can work in degrees and convert the final answer. However, let's try work in radians.

Firstly, we need to recognise that the interval for θ is one revolution. $6.282 \approx 2 \times \pi$ or one revolution. Therefore, this is the same interval as $0^\circ \leq \theta \leq 360^\circ$.

$2\cos\theta - 1 = 0$

$\therefore \cos\theta = \frac{1}{2}$

Ref angle: $\theta = \frac{\pi}{3}$

Cosine is positive in the first and fourth quadrants.

$\theta = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$

General solution: $\theta = \frac{\pi}{3} + k.2\pi$ or $\theta = \frac{5\pi}{3} + k.2\pi, k \in \mathbb{Z}$

In the interval $0 \leq \theta \leq 6.282$: $\theta = \frac{\pi}{3}$ or $\theta = \frac{5\pi}{3}$

Alternatively, you could have done all your working in degrees and converted your final answers of $\theta = 60^\circ$ or $\theta = 300^\circ$ to radians.

Take note!

If you would like to work in radians, you need to change your calculator to work in **radian mode**. To change a Casio calculator, follow these steps:

1. If you see a little **D** or **DEG** at the top of your screen, it means that your calculator is working in degree mode.
2. Press the **SHIFT** key and then the **MODE/SETUP** key. You will see a full list of the available modes. Radians is normally option **4**, so press 4 (or whatever option your calculator says is radians).
3. After this, you will see a little **R** or **RAD** at the top of your screen.
4. Go through the same process to change your calculator back to degree mode.

Exercise 4.2

In each of the following equations, determine the value(s) of θ in radians if $0 \leq \theta \leq 2\pi$:

1. $\frac{1}{2}\cos\theta = 0.435$
2. $\tan\left(\theta - \frac{\pi}{6}\right) = 1.57$
3. $2\sin^2\theta + \sin\theta = 1$

The full solutions are at the end of the unit.

Summary

In this unit you have learnt the following:

- That one radian is the angle created by moving one radius arc length along the circumference of a circle.
- To convert from degrees to radians, multiply by π and divide by $180°$.
- To convert from radians to degrees, multiply by $180°$ and divide by π.

Unit 4: Assessment

Suggested time to complete: 15 minutes

Question 1 and 2 adapted from NC(V) Level 4 Paper 2 November 2011 question 3.2 and November 2012 question 3.2 respectively

In each of the following, determine the value(s) of θ in radians if $0 \leq \theta \leq 6.282$.

1. $6\cos\theta - 5 = \frac{4}{\cos\theta}$
2. $4\sin\theta + 3\tan\theta = \frac{3}{\cos\theta} + 4$
3. $\cos 2\theta - \cos\theta + 1 = 0$
4. $\cos 2\theta = 1 - 3\cos\theta$

The full solutions are at the end of the unit.

Unit 4: Solutions

Exercise 4.1

1.

   a. $30^\circ = \dfrac{30^\circ \times \pi}{180^\circ} = \dfrac{\pi}{6}$

   b. $75^\circ = \dfrac{75^\circ \times \pi}{180^\circ} = \dfrac{5\pi}{12}$

   c. $135^\circ = \dfrac{135^\circ \times \pi}{180^\circ} = \dfrac{3\pi}{4}$

   d. $215^\circ = \dfrac{215^\circ \times \pi}{180^\circ} = \dfrac{43\pi}{36} = 1.194\pi$

2.

   a. $5\pi = \dfrac{5\pi \times 180^\circ}{\pi} = 900^\circ$

   b. $\dfrac{\pi}{6} = \dfrac{\pi \times 180^\circ}{6\pi} = 30^\circ$

   c. $\dfrac{7\pi}{2} = \dfrac{7\pi \times 180^\circ}{2\pi} = 630^\circ$

   d. $\pi^2 = \dfrac{\pi^2 \times 180^\circ}{\pi} = \pi \times 180^\circ = 565.49^\circ$

Back to Exercise 4.1

Exercise 4.2

1.

   $\dfrac{1}{2}\cos\theta = 0.435$
   $\therefore \cos\theta = 0.87$
   Ref angle: 0.516 (or $\theta = 29.54^\circ$)
   Cosine is positive in the first and fourth quadrants.
   $\theta = 2\pi - 0.516 = 5.767$ radians (or $\theta = 360^\circ - 29.54^\circ = 330.46^\circ$)
   Therefore, in the interval $0 \leq \theta \leq 6.282$, $\theta = 0.516 \text{ or } \theta = 5.767$ radians.

2.

   $\tan\left(\theta - \dfrac{\pi}{6}\right) = 1.57$
   Ref angle: $\theta - \dfrac{\pi}{6} = 1.004$ (or $\theta = 57.51^\circ$)
   Tangent is positive in the first and third quadrants.
   $\theta - \dfrac{\pi}{6} = \pi + 1.004 = 4.146$ (or $\theta = 237.51^\circ$)
   $\theta - \dfrac{\pi}{6} = 4.146 + k.\pi, k \in \mathbb{Z}$
   $\theta = 4.146 + \dfrac{\pi}{6} + k.\pi, k \in \mathbb{Z}$
   $\therefore \theta = 4.669 + k.\pi, k \in \mathbb{Z}$
   In the interval $0 \leq \theta \leq 6.282$: $\theta = 1.528 \text{ or } \theta = 4.670$

3.

$$2\sin^2\theta + \sin\theta = 1$$
$$\therefore 2\sin^2\theta + \sin\theta - 1 = 0$$
$$\therefore (2\sin\theta - 1)(\sin\theta + 1) = 0$$
$$\therefore \sin\theta = \frac{1}{2} \quad \text{or} \quad \sin\theta = -1$$

$\sin\theta = \frac{1}{2}$:

Ref angle: $\theta = \frac{\pi}{6}$

Sine is positive in the first and second quadrants.

$\theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$

General solution: $\theta = \frac{\pi}{6} + k.2\pi \text{ or } \theta = \frac{5\pi}{6} + k.2\pi, k \in \mathbb{Z}$

In the interval $0 \leq \theta \leq 2\pi$: $\theta = \frac{\pi}{6} \text{ or } \theta = \frac{5\pi}{6}$

$\sin\theta = -1$:

Ref angle: $\theta = -\frac{\pi}{2}$

General solution: $\theta = -\frac{\pi}{2} + k.2\pi, k \in \mathbb{Z}$

In the interval $0 \leq \theta \leq 2\pi$: $\theta = \frac{3\pi}{2}$

Back to Exercise 4.2

Unit 4: Assessment

1.

$$6\cos\theta - 5 = \frac{4}{\cos\theta} \qquad \cos\theta \neq 0 \therefore \theta \neq \frac{\pi}{2} + k.\pi, k \in \mathbb{Z}$$
$$\therefore 6\cos^2\theta - 5\cos\theta - 4 = 0$$
$$\therefore (3\cos\theta - 4)(2\cos\theta + 1) = 0$$
$$\therefore \cos\theta = \frac{4}{3} \text{ or } \cos\theta = -\frac{1}{2}$$

$\cos\theta = \frac{4}{3}$: no solution

$\cos\theta = -\frac{1}{2}$:

Ref angle $\theta = \frac{2\pi}{3}$

Cosine is negative in the second and third quadrants.

$\theta = 2\pi - \frac{2\pi}{3} = \frac{4\pi}{3}$

For the interval $0 \leq \theta \leq 6.282$: $\theta = \frac{2\pi}{3} \text{ or } \theta = \frac{4\pi}{3}$

2.

$$4\sin\theta + 3\tan\theta = \frac{3}{\cos\theta} + 4 \qquad \cos\theta \neq 0 \therefore \theta \neq \frac{\pi}{2} + k.\pi, k \in \mathbb{Z}$$
$$\therefore 4\sin\theta + \frac{3\sin\theta}{\cos\theta} = \frac{3}{\cos\theta} + 4$$
$$\therefore 4\sin\theta\cos\theta + 3\sin\theta = 3 + 4\cos\theta$$
$$\therefore 4\sin\theta\cos\theta + 3\sin\theta - 3 - 4\cos\theta = 0$$
$$\therefore 4\cos\theta(\sin\theta - 1) + 3(\sin\theta - 1) = 0$$
$$\therefore (\sin\theta - 1)(4\cos\theta + 3) = 0$$
$$\therefore \sin\theta = 1 \text{ or } \cos\theta = -\frac{3}{4}$$

$\sin\theta = 1$:

Ref angle: $\theta = \dfrac{\pi}{2}$

General solution: $\theta = \dfrac{\pi}{2} + k.2\pi, k \in \mathbb{Z}$

$\cos\theta = -\dfrac{3}{4}$

Ref angle: $\theta = 2.419$

Cosine is negative in the second and third quadrants.

$\theta = 2\pi - 2.419 = 3.864$

General solution: $\theta = 2.419 + k.2\pi \text{ or } \theta = 3.864 + k.2\pi, k \in \mathbb{Z}$

For the interval $0 \leq \theta \leq 6.282$: $\theta = \dfrac{\pi}{2} \text{ or } \theta = 2.419 \text{ or } \theta = 3.864$

3.

$$\begin{aligned} \cos 2\theta - \cos\theta + 1 &= 0 \\ \therefore 2\cos^2\theta - 1 - \cos\theta + 1 &= 0 \\ \therefore 2\cos^2 - \cos\theta &= 0 \\ \therefore \cos\theta(2\cos\theta - 1) &= 0 \\ \therefore \cos\theta = 0 \text{ or } \cos\theta &= \frac{1}{2} \end{aligned}$$

$\cos\theta = 0$:

Ref angle: $\theta = \dfrac{\pi}{2}$

General solution: $\theta = \dfrac{\pi}{2} + k.\pi, k \in \mathbb{Z}$

$\cos\theta = \dfrac{1}{2}$:

Ref angle: $\theta = \dfrac{\pi}{3}$

Cosine is positive in the first and fourth quadrants.

$\theta = 2\pi - \dfrac{\pi}{3} = \dfrac{5\pi}{3}$

General solution: $\theta = \dfrac{\pi}{3} + k.2\pi \text{ or } \theta = \dfrac{5\pi}{3} + k.2\pi, k \in \mathbb{Z}$

For the interval $0 \leq \theta \leq 6.282$: $\theta = \dfrac{\pi}{2} \text{ or } \theta = \dfrac{3\pi}{2} \text{ or } \theta = \dfrac{\pi}{3} \text{ or } \theta = \dfrac{5\pi}{3}$

4.

$$\begin{aligned} \cos 2\theta &= 1 - 3\cos\theta \\ \therefore 2\cos^2\theta - 1 - 1 + 3\cos\theta &= 0 \\ \therefore 2\cos^2\theta + 3\cos\theta - 2 &= 0 \\ \therefore (2\cos\theta - 1)(\cos\theta + 2) &= 0 \\ \therefore \cos\theta = \frac{1}{2} &\text{ or } \cos\theta = -2 \end{aligned}$$

$\cos\theta = \dfrac{1}{2}$:

Ref angle: $\theta = \dfrac{\pi}{3}$

Cosine is positive in the first and fourth quadrants.

$\theta = 2\pi - \dfrac{\pi}{3} = \dfrac{5\pi}{3}$

General solution: $\theta = \dfrac{\pi}{3} + k.2\pi \text{ or } \theta = \dfrac{5\pi}{3} + k.2\pi, k \in \mathbb{Z}$

$\cos\theta = -2$: No solution

For the interval $0 \leq \theta \leq 6.282$: $\theta = \dfrac{\pi}{3} \text{or } \theta = \dfrac{5\pi}{3}$

Back to Unit 4: Assessment

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SUBJECT OUTCOME XI

DATA HANDLING: REPRESENT, ANALYSE AND INTERPRET DATA USING VARIOUS TECHNIQUES

Subject outcome

Subject outcome 4.1: Represent, analyse and interpret data using various techniques

Learning outcomes

- Identify situations or issues that can be dealt with through statistical methods.
  Range: Data given should include problems relating to health, social, economic, cultural, political and environmental issues.
  Note: Not for examination purposes but for class activities only.
- Discuss the use of appropriate and efficient methods to record, organise and interpret given data by making use of:
  - Manageable data sample sizes (less than or equal to 10) and which are representative of the population.
  - Graphical representations and numerical summaries which are consistent with the data, and clear and appropriate to the situation and target audience.
    Note: Discussion only, not expected to draw again.
  - Compare different representations of given data.
- Justify and apply statistics to answer questions about problems.
- Discuss new questions that arise from the modelling of data.
- Take a position on an issue by comparing different representations of given data.

Unit 1 outcomes

By the end of this unit you will be able to:

- Identify situations or issues that can be dealt with through statistical methods.
- Make resolutions to maximise efficiency from given data which has been organised and graphically represented.

Unit 1: Use various techniques for data collection, representation and interpretation

NATASHIA BEARAM-EDMUNDS

Unit outcomes

By the end of this unit you will be able to:

- Identify situations or issues that can be dealt with through statistical methods.
- Make resolutions to maximise efficiency from given data which has been organised and graphically represented.

What you should know

Before you start this unit, make sure you can:

- understand basic principles of statistics. You can revise the following statistics units from the previous levels:
  - level 2 subject outcome 4.1
  - level 2 subject outcome 4.2
  - level 3 subject outcome 4.1
  - level 3 subject outcome 4.2.

Introduction

Social media has a huge impact on the way people interact and the decisions they make. It can also influence many decisions and highlight issues of public importance. A major environmental issue is the need to reduce plastic waste.

This has been widely publicised and the negative effect of plastic on the ocean is well documented. But, what if we wanted to find out how social media reacts to plastic pollution? How could we assess the reaction? To make a conclusion about this question we need to collect information.

The calculation of statistics always starts with collecting information. Why do you think it is important to get information and analyse statistics about plastic waste?

If consumers are becoming more concerned about plastic waste this will influence decisions about the type of products they buy. Businesses need to pay attention to the growing environmental concerns so they can adapt and change their focus to avoid financial pitfalls. In fact, a survey has been done on this very topic. It concluded that people are talking more about the plastics problem on social media, and they are googling the topic more, too. Below is a snapshot of that survey.

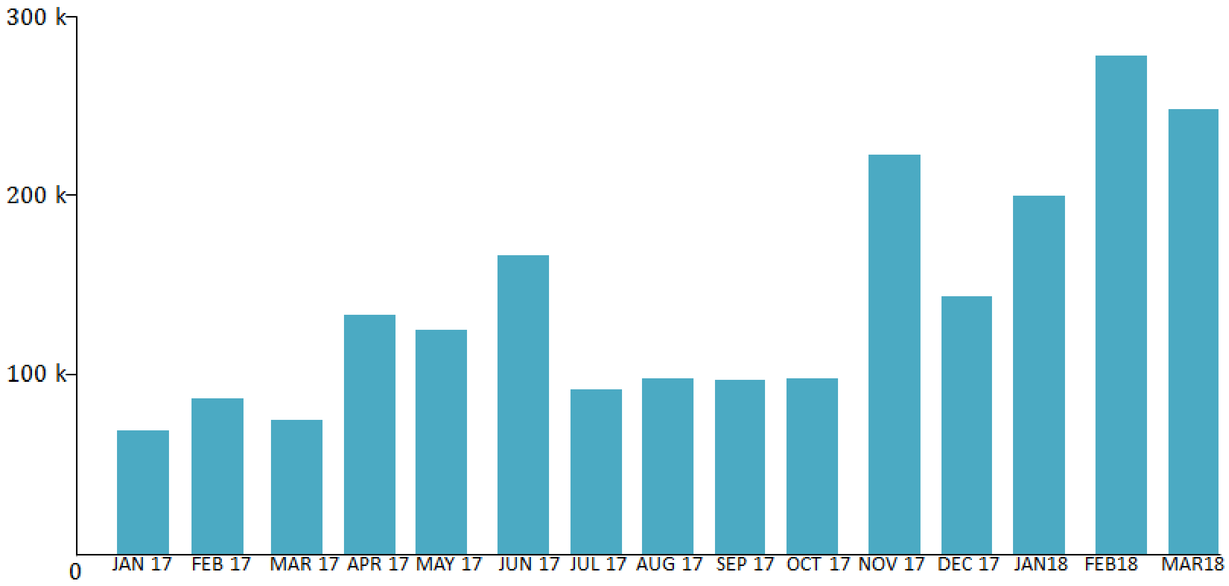


Figure 1: Tweets about plastic waste between 1 January 2017 and 31 March 2018

From environmental research to sports, statistical calculations are used in almost every field. As you will no doubt encounter statistics somewhere, it is important to be able to analyse statistics and understand how they are computed.

Ask the right questions

To make an informed decision about a current problem, such as plastic pollution, you will need to research the problem and compute statistics. Statistics are calculated for research purposes in many fields. But, not every question justifies the cost and effort of performing statistical research.

Think about situations around you that possibly need further research. So, how do we know if a problem warrants further study? The following will guide your decision to conduct statistical research.

- Can the issue be studied and what is the purpose of the investigation?
- Does it justify further research?
- Is it worth the time, money and effort that will go into the research?
- Is there money available to investigate the issue?
- Are there people with the right skills available to conduct the research?
- Is the size of the sample reasonable to investigate?
- Can you formulate the hypothesis?

A **hypothesis** is a statement that must be proved or tested through research (observation or experimentation). It is an educated guess and expresses the supposed relationship between two variables. Remember that a variable is something that changes and can have different values or conditions.

For example, if you suspect that time watching TV negatively influences exam results, then your hypothesis could be that the more time you spend watching TV the worse you perform at exams. The variables are time spent watching TV and exam performance.

A hypothesis test involves collecting data from a sample and evaluating the data. Then, a statistician makes a decision whether or not there is sufficient evidence, based on analysis of the data, to reject or accept the hypothesis.

Note

Hypothesis testing is not examinable but it is the basis for most statistical calculations. For your own interest you can learn more about hypothesis testing by watching this video when you have internet access, "Simple hypothesis testing".

Simple hypothesis testing (Duration: 06.25)

When an issue needs further statistical research we must collect, record, organise and interpret the data using the methods discussed in detail in levels 2 and 3. We will revise those methods next.

Data collection

These are the different types of data that we have worked with so far.

Qualitative data deals with descriptions that can be observed but not measured. For example colours, size, tastes, and appearance.

Categorical data are qualitative. For example hair colour of people at a shopping mall.

Quantitative data deals with numerical data that can be measured. For example length, height, weight, time, cost, and number of people.

Quantitative data are divided into discrete and continuous.

Discrete data are whole number values. For example the number of people attending a maths course.

Continuous data are values that can be measured. For example the heights of learners in an NCV level 4 maths class.

Data sources are varied and include the internet, surveys, censuses and existing records. Often questionnaires, observations and interviews are used to collect data.

In statistics, we generally want to study a **population**. You can think of a population as a large collection of persons, things, or objects under study. To study the population, we select a **sample**. The idea of sampling is to select a portion (or subset) of the larger population and study that portion (the sample) to gain information about the population. Data are the result of sampling from a population.

Because it takes a lot of time and money to examine an entire population, sampling is a very practical technique. From the sample data, we can calculate a **statistic**. A statistic is a number that represents a property of the sample.

The statistic is an estimate of a population **parameter**. A parameter is a numerical characteristic of the whole population that can be estimated by a statistic.

Data can be collected by sampling in many ways. The simplest way is direct observation.

For example, if you want to find out how many bicycles pass a busy intersection during rush hour traffic, you can stand close to the intersection and count the number of bicycles that pass by in that interval.

Statistics can be a powerful tool in research. Unfortunately, statistics can also have faults. Sample bias is one such fault. Bias is deliberate favouritism when collecting data, resulting in lopsided, misleading results. Bias can occur in the way the sample is chosen and the way the data are collected.

For example, if we wanted to find out how many learners played sport at a college and chose only the male learners to be part of the survey. This will result in misleading results as we have not chosen a sample that is representative of the entire college population, which includes females.

It is important to keep in mind that sampling bias refers to the method of sampling, not the sample itself.

Avoiding bias when selecting a sample

The methods used to collect data must ensure that the data is reliable. This means that it is data that we can trust. Data cannot be trusted unless it has been collected in a way that makes sure that every member of the population under investigation has the same chance of being selected in the sample.

Sample bias occurs when a particular group of the population from which the sample is drawn does not represent that population. The way to avoid sample bias is to take a **random sample**. A sample is random if every member of the population has an equal chance of being selected.

In addition to the sample being random it must be of an **adequate size**. The bigger the sample size the more accurate the results.

Example 1.1

Identify the bias in the example below:

Sibusiso collected data from a sample of grade 12 boys at his school to find out how many learners play soccer.

Solution

Since the sample is not random, some individuals are more likely than others to be chosen. Always think very carefully about which individuals are being favoured and how that will influence the results. Sibusiso's sample is restricted to boys only and is more likely to get a favourable result and skew data. The sample must include girls as well to be a true reflection of the learners at the school.

Organising data

Data is often recorded electronically by using spreadsheets, computer software, scanners and online surveys. The data can then be sorted and organised by:

- grouping using frequency tables
- tallies on tally tables
- stem and leaf diagrams.

Once data are organised it can be summarised so that it can be better analysed.

Summarising data

In levels 2 and 3 we discussed single numerical values that gave us information about the data; measures of central tendency and dispersion. The measures we have already learnt about are the:

- mean
- median
- mode
- range
- lower quartile
- upper quartile
- interquartile range
- semi-interquartile range
- variance and
- standard deviation.

You must be able to calculate the above measures for ungrouped and grouped data, where applicable.

The following formulae are used to calculate the estimated mean, median and mode for grouped data.

Mean:

$$\bar{x} = \frac{\sum f_i x_i}{n}$$

$f_i x_i$ is the class midpoint multiplied by the frequency
n is the number of observations

Median:

$$\mathrm{M}_e = l + \frac{\left(\frac{n}{2} - F\right)}{f} \times c$$

l is the lower limit of the median class
n is the number of observations
F is cumulative frequency of the class before the median class
f is the frequency of the median class
c is the class width

Mode:

$$M_o = l + \frac{f_m - f_{m-1}}{2f_m - f_{m-1} - f_{m+1}} \times c$$

l is the lower limit of the modal class
f_m is the frequency of the modal class
f_{m-1} is the frequency of the class before the modal class
f_{m+1} is the frequency of the class after the modal class
c is the class width

Example 1.2

The frequency distribution shows the pulse rates of a group of women.

Pulse rates of women	**Frequency**
$60 - 69$	12
$70 - 79$	14
$80 - 89$	11
$90 - 99$	1
$100 - 109$	1
$110 - 119$	0
$120 - 129$	1

Use the table to find:

1. The average pulse rate for the women.
2. If these pulse rates are observed in a sample of women admitted to a private hospital is this a good indication of the average pulse rate of all patient admissions?
3. Find the median pulse rate.

Solutions

1. Find the class midpoints to apply the formula.

Pulse rates of women	**Class midpoint**	**Frequency**
$60 - 69$	64.5	12
$70 - 79$	74.5	14
$80 - 89$	84.5	11
$90 - 99$	94.5	1
$100 - 109$	104.5	1
$110 - 119$	114.5	0
$120 - 129$	124.5	1

$$\begin{aligned}\bar{x} &= \frac{\sum f_i x_i}{n} \\ &= \frac{64.5 \times 12 + 74.5 \times 14 + 84.5 \times 11 + 94.5 + 104.5 + 124.5}{40} \\ &= \frac{3\ 070}{40} \\ &= 76.75\end{aligned}$$

The average pulse rate for women is 76.75.

2. No it is not a good representation of the entire population. Male pulse rates are excluded, and the sample size is very small, making this an unreliable sample.
3. The median class is $70 - 79$.

$$\begin{aligned} M_e &= l + \frac{\left(\frac{n}{2} - F\right)}{f} \times c \\ &= 70 + \frac{\left(\frac{40}{2} - 12\right)}{14} \times 10 \\ &= 75.71 \end{aligned}$$

The median pulse rate is 75.71.

Example 1.3

On a timed maths test, the lower quartile for time it took to finish the exam was at 35 minutes. Interpret the first quartile in the context of this situation.

Solution

This means that 25% of learners finished the exam in less than 35 minutes, or we can say 75% of learners finished the exam in more than 35 minutes.

Representing data

We have used different types of graphs to represent data. Graphs represent data well because they give a picture of the data that is easy to interpret.

Some graphs are better for displaying certain kinds of information than others. The type of graph depends mostly on the type of data that needs to be represented.

Representation	Advantages
Stem-and-leaf diagram	Used to plot data and look at the distribution. All data values within a class are visible.
Box-and-whisker diagram	Used to organise data visually. Easy to see the five-number-summary.
Bar graph	Used for showing discrete quantitative data or data in categories. Bar graphs allow us to compare the quantities of different categories, for example, the exam results of different subjects. They are a really good way to show relative sizes.
Compound bar graph	Used to compare two or more characteristics for each category. For example, we could use a double-bar graph to compare the differences between male and female preferences for sport to watch.
Histogram	Used to represent continuous data that is grouped into equal class intervals, for example height, weight, etc. Histograms are useful to show the way the data is spread out.
Pie chart	Used to show a whole divided into parts. They show how the parts relate to each other and how the parts relate to a whole. They do not show the quantities involved. You can use pie charts to show the relative sizes of many things, such as what type of phone people prefer, etc.
Broken line graph	Used to show trends or changes in quantities over time, where the categories are related to each other or follow on from each other. For example the categories might be consecutive times, days, months, or years.
Ogive (cumulative frequency graph)	Used to determine how many data values lie above or below a particular value in a data set. Ogives are useful for determining the median, percentiles and five number summary of data.
Scatter plot	Used to graph data points that have two values associated with them. Data values have two independent measurements, for example, maths marks and science marks.

You do not need to draw the statistical graphs again in level 4 but you are expected to interpret given graphs and answer questions based on the graphs.

Note

For more information on choosing the correct graph to represent data, you can read about data representations and try examples online.

Example 1.4

The stem-and-leaf diagram shows Drew's calculus test marks (in percentages) for the year.

3	5
4	3 9
5	8 9 9
6	5 7
7	2 5 5 5
8	1 4

1. How many calculus tests did Drew write?
2. What is his highest mark?
3. What is the modal mark?
4. Calculate the mean mark to the nearest percent.

Solution

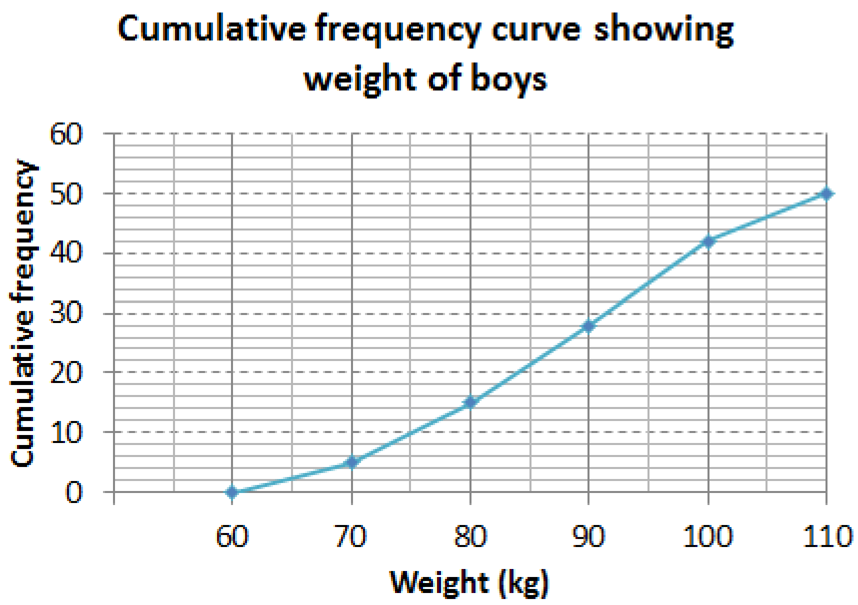
1. Remember: The stem and-leaf diagram is a good choice when the data sets are small. To create the diagram, divide each observation of data into a stem and a leaf. The leaf consists of a final significant digit. For example, 35 has stem 3 and leaf 5. The decimal 8.7 has stem 8 and leaf 7. To draw the stem-and-leaf diagram, list the stems vertically from smallest to largest. Draw a vertical line to the right of the stems. Then write the leaves in increasing order next to their corresponding stem.
2. There are 14 marks listed, so Drew wrote 14 tests.
3. His highest mark is 84%.
4. The modal mark is the one that occurs most often. His modal mark is 75% as it appears three times.
5. Mean mark:

$$\begin{aligned}\bar{x} &= \frac{\sum x}{n}\\ &= \frac{35+43+49+58+59+59+65+67+72+3(75)+81+84}{14}\\ &= \frac{897}{14}\\ &= 64\%\end{aligned}$$

Example 1.5

Question adapted from Siyavula Maths Grade 11

The weights of a random sample of boys from a sports club were recorded. The cumulative frequency graph (ogive) below represents the recorded weights.



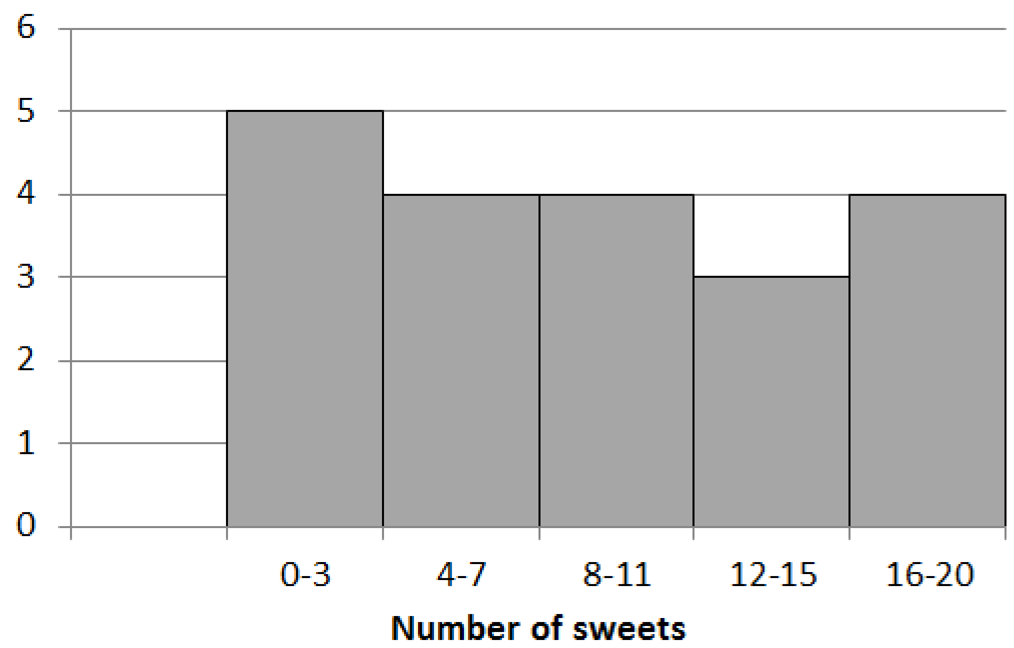
1. How many of the boys weighed between 90 and 100 kilograms?
2. Estimate the median weight of the boys.
3. If there were 250 boys in the club, estimate how many of them would weigh less than 80 kilograms?
4. Which other graph(s) could have been used to represent the data?

Solutions

1. $42 - 28 = 14$ weighed between 90 and 100 kilograms.
2. Approximately 88 kg.
3. 15 out of 50 boys weigh less than 80 kilograms so 75 boys $\left(\frac{15}{50} \times 250 = 75\right)$ out of the total of 250 would weigh less than 80 kilograms.
4. A histogram would be an appropriate way to represent the data.

Example 1.6

A group of learners count the number of sweets they each have. This is a histogram describing the data they collected.



A cat jumps onto the table and all their notes land on the floor, mixed up, by accident! Help them find which of the following data sets match the above histogram:

Data set A

2	1	20	10	5	3	10	2	6	1
2	2	17	3	18	3	7	10	8	18

Data set B

2	9	12	10	5	9	10
13	6	5	11	10	7	2

Data set C

3	12	16	10	15	17	18	2	3	7
11	12	8	2	7	17	3	11	4	4

Solution

Count the number of values in each range of the drawn histogram and compare that to the given tables of data.

Data set A has eight values in the $0-3$ range but the histogram has five values in that range so A does not match the histogram.

Data set B has one value in the $0-3$ range so it is not the right match for the histogram.

Data set C has five values in the $0-3$ range and the number of values in each of the other ranges matches too. Therefore, data set C matches the given histogram.

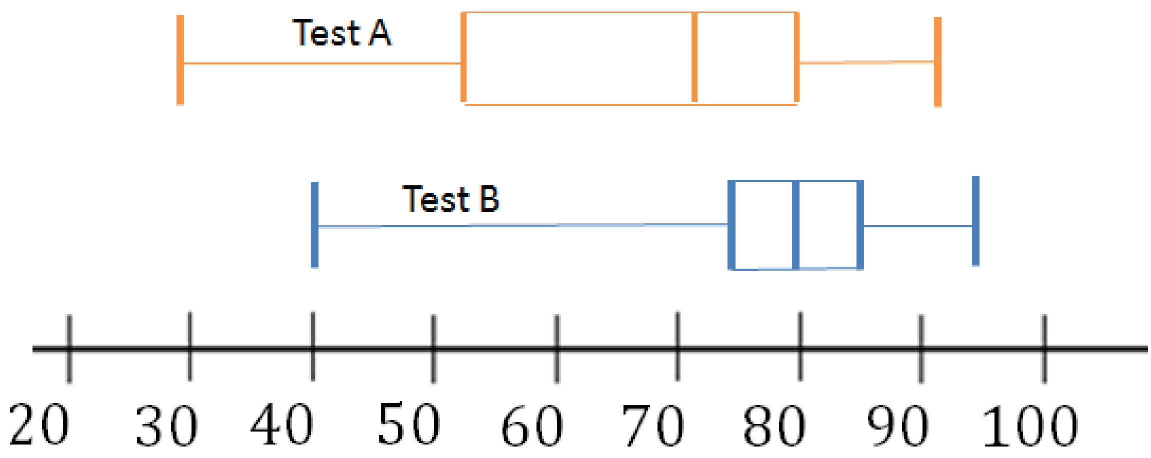
Note

To learn more about statistical studies watch the video “Types of statistical studies” when you have access to the internet.

Types of statistical studies (Duration: 10.31)

Exercise 1.1

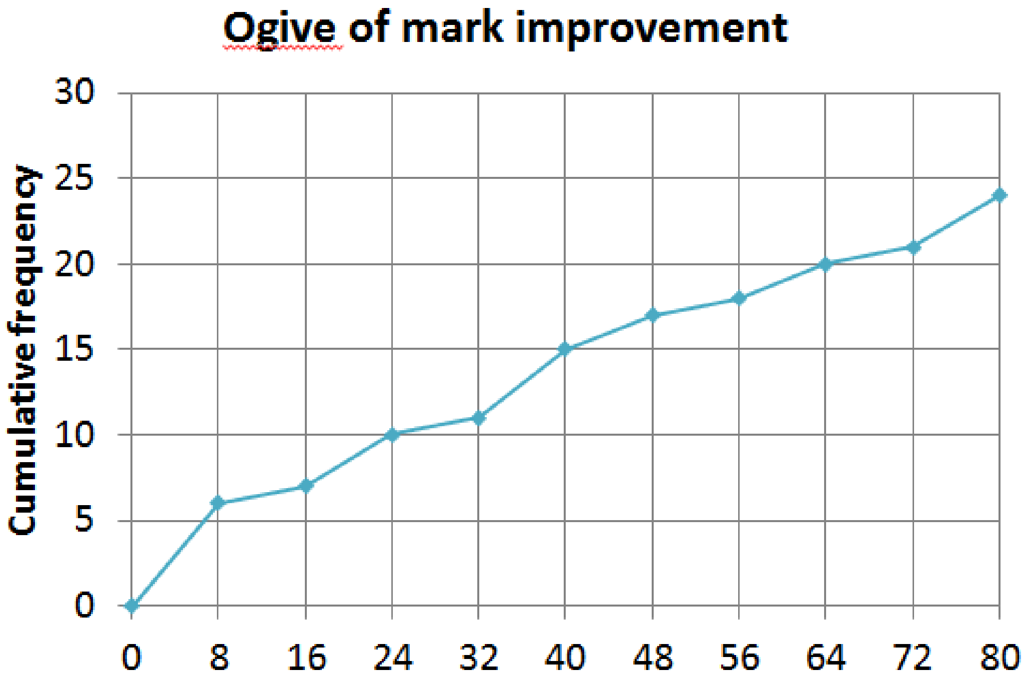
1. The box-and-whisker diagrams (plots) show the maths test results in percentages for two tests that learners wrote.



   a. What is the highest mark in test A?
   b. What is the lowest mark in test B?
   c. What is the median mark in test B?
   d. Between what values do 50% of the marks lie in test A?
   e. What mark did 25% of learners get less than in test B?
   f. What mark did $75\%$of learners get more than in test A?
   g. Which other graph(s) could have been used to represent the data?

2. The cumulative frequency curve shows the percentage improvement in marks of a group of

learners after they attended a maths camp.



a. How many learners attended the maths camp?

b. How many learner' marks improved by 24 to 40%?

c. How many learners' marks increased by 64% or more?

d. Would a box-and-whisker diagram be an appropriate representation for the type of information we are looking for in this case?

The full solutions are at the end of the unit.

Summary

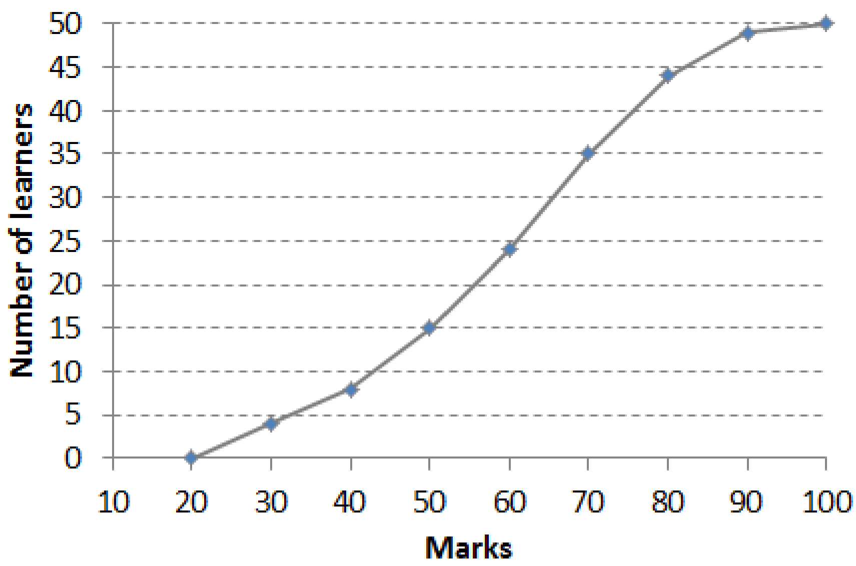
In this unit you have learnt the following:

- How to test if issues warrant further scientific research.
- How to identify graphs that best represent a given data set.
- How to compare different data representations.

Unit 1: Assessment

Suggested time to complete: 20 minutes

1. The following ogive shows the test results, in percentages, for a class.



a. How many learners are in the class?

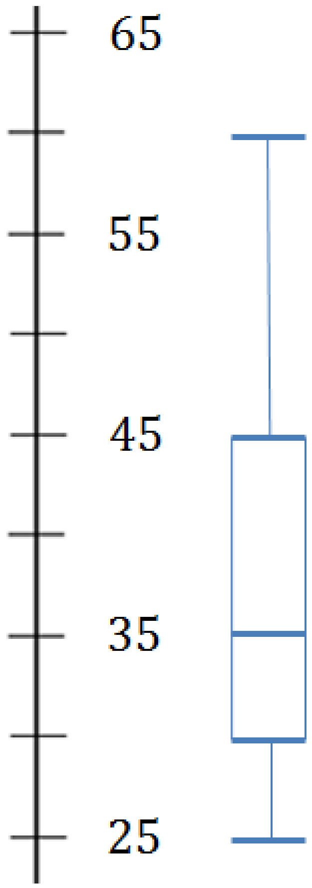
b. How many learners got 70% or less?

c. $30\%$of learners got less than what mark?

d. If the pass mark is 50%, how many learners passed?

e. What other graph(s) could have been used to represent the data?

2. The box-and-whisker plot shows the ages of members at a sports club.



a. How old is the youngest member?

b. What is the median age?
c. Between what ages do the middle 50% of data values lie?
d. Below what age do 100% of data values lie?
e. 75% of the club membership is older than what age?
f. What other graph(s) could have been used to represent the data?

The full solutions are at the end of the unit.

Unit 1: Solutions

Exercise 1.1

1.
   a. The highest mark in test A is 90%.
   b. The lowest mark in test B is 40%.
   c. The median mark in test B is 80%.
   d. In test A 50% of the marks lie between 70% and 90% (or between 30% and 70%).
   e. 25% of learners got less than 75% in test B.
   f. $75\%$of learners got more than 50% in test A.
   g. Ogives or compound bar graphs could have been used to represent and compare the data.
2.
   a. 24
   b. $15 - 10 = 5$
   c. 4
   d. No a box-and-whisker diagram would not be an adequate representation in this case. The ogive is also known as the 'less than' graph and we can easily see what percentages/values are below or above a certain point.

Back to Exercise 1.1

Unit 1: Assessment

1.
   a. 50
   b. 35
   c. 50%
   d. 35 learners passed.
   e. A box-and-whisker diagram or bar graph could have been used.
2.
   a. The youngest member is 25 years old.
   b. 35 is the median age.
   c. The middle 50% of data values lie between 30 and 45.
   d. 60

e. 75% of the club membership is older than 30.

f. A bar graph or ogive could have been used.

Back to Unit 1: Assessment

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SUBJECT OUTCOME XII

DATA HANDLING: USE VARIANCE AND REGRESSION ANALYSIS TO INTERPOLATE AND EXTRAPOLATE BIVARIATE DATA

Subject outcome

Subject outcome 4.2: Use variance and regression analysis to interpolate and extrapolate bivariate data

Learning outcomes

- Calculate:
  - variance, and
  - standard deviation manually for small sets of data only.
- Interpret the meaning of variance and standard deviation for small sets of data only.
- Represent bivariate numerical data as a scatter plot.
- Identify intuitively whether a linear, quadratic or exponential function would best fit the data.
- Draw the intuitive line of best fit.
  Range:
  - Data given should include problems related to health, social, economic, cultural, political and environmental issues.
  - For small sets of data only (limited to 8).
- Use least squares regression method to determine a function which best fits a given set of bivariate data.
- Use the regression line to predict the outcome of a given problem.

Unit 1 outcomes

By the end of this unit you will be able to:

- Calculate variance for ungrouped data manually.
- Calculate standard deviation for ungrouped data manually.
- Interpret variance and standard deviation.

Unit 2 outcomes

By the end of this unit you will be able to:

- Draw a scatter plot.
- Understand when it is appropriate to use a scatter plot.
- Draw an intuitive line of best fit.

Unit 3 outcomes

By the end of this unit you will be able to:

- Determine the linear regression equation $\hat{y} = a + bx$.
- Use the regression line to predict the outcome of a given problem.

Unit 1: Calculate variance and standard deviation

GILL SCOTT

Unit outcomes

By the end of this unit you will be able to:

- Calculate variance for ungrouped data manually.
- Calculate standard deviation for ungrouped data manually.
- Interpret variance and standard deviation.

What you should know

Before you start this unit, make sure you can:

- Calculate measures of central tendency of a data set, such as the mean, median and mode, of both ungrouped and grouped data, and interpret what these tell you about a data set. To revise this, you can work through:
  - level 2, subject outcome 4.1, units 2 and 3
  - level 3, subject outcome 4.1, unit 1 and
  - level 3, subject outcome 4.2 unit 4.
- Calculate the range of data.

Introduction

Measures of central tendency of data sets, the mean, median and mode, give a first impression of the characteristics of a data set. From the work that you have already done, you saw that although these measures can be useful, they can also be misleading. So, it is necessary to investigate how the data in any set is spread, scattered or dispersed in order to have a complete picture of the data set.

The range is a measure of dispersion , being the spread of data from smallest to largest. The interquartile range (IQR) is a better measure of dispersion than the range. It gives the range of spread around the *median*, so 50% of the data set. However, the mean is often a better measure of central tendency than the median is, and in this unit we will investigate how data is spread around the *mean*. The measures of dispersion around the mean are the **variance** and the **standard deviation**.

Variance

Suppose that two groups of nine learners wanted to see how long they could balance on a slackline, with each member recording how many seconds passed before they fell off.

The nine members of group A balanced for:

320sec; 250sec; 183sec; 41sec; 335sec; 78sec; 142sec; 210sec; 115sec.

The nine members of group B balanced for:

185sec; 188sec; 183sec; 191sec; 185sec; 179sec; 192sec; 184sec; 187sec;

The mean of each group was calculated:
Group A's mean:
$$\bar{x}_A = \frac{320+250+183+41+335+78+142+210+115}{9} = \frac{1674}{9} = 186 \text{ sec}$$
Group B's mean:
$$\bar{x}_B = \frac{185+188+183+191+185+179+192+184+187}{9} = \frac{1674}{9} = 186 \text{ sec}$$

The means of the two groups are the same but as you can see the recorded data values are very different. The mean does not provide enough information to make a useful comparison of the data sets.

The deviation of each value from the mean for the groups was tabulated:

Group A	
Time x	**Deviation from the mean** $x - \bar{x}_A$
320	$320 - 186 = 134$
250	$250 - 186 = 64$
183	$183 - 186 = -3$
41	$41 - 186 = -145$
335	$335 - 186 = 149$
78	$78 - 186 = -108$
142	$142 - 186 = -44$
210	$210 - 186 = 24$
115	$115 - 186 = -71$

Group B	
Time x	**Deviation from the mean** $x - \bar{x}_B$
185	$185 - 186 = -1$
188	$188 - 186 = 2$
183	$183 - 186 = -3$
191	$191 - 186 = 5$
185	$185 - 186 = -1$
179	$179 - 186 = -7$
192	$192 - 186 = 6$
184	$184 - 186 = -2$
187	$187 - 186 = 1$

The table shows that although the means for both groups were the same, the times for group A are much more widely dispersed about the mean than the times for group B. We need to investigate the dispersions. Suppose we find the total deviations from the mean for each group:

Sum of Group A's deviations: $\sum(x-\bar{x}) = 134+64-3-145+149-108-44+24-71=0$
Sum of Group B's deviations: $\sum(x-\bar{x}) = -1+2-3+5-1-7+6-2+1=0$

In each group, the negative values cancel out the positive values giving the total of 0 (this will happen for any group; can you see why?). However, the extent of dispersion of data around the mean gives a good idea of how representative the mean is of the data set. Squaring the distance from the mean for each data element gives a positive value for each, and so enables us to look at total spread about the mean, although this value is squared. Thus, the next step is to square each of the deviations from the mean, and to calculate the sum of the squared values:

Group A:

$$\begin{aligned}\sum(x-\bar{x})^2 &= (134)^2+(64)^2+(-3)^2+(-145)^2+(149)^2+(-108)^2+(-44)^2+(24)^2+(-71)^2\\ &= 17\ 956+4\ 096+9+21\ 025+22\ 201+11\ 664+1\ 936+576+71\\ &= 84\ 504\end{aligned}$$

Group B:

$$\begin{aligned}\sum(x-\bar{x})^2 &= (-1)^2+(2)^2+(-3)^2+(5)^2+(-1)^2+(-7)^2+(6)^2+(-2)^2+(1)^2\\ &= 1+4+9+25+1+49+36+4+1\\ &= 130\end{aligned}$$

The **variance**, σ, is defined as the average, or mean, of the squared deviations, so the sum for each must be divided by the number of data elements:

$$\text{Variance for group A} = \frac{\sum(x-\bar{x})^2}{n} = \frac{84\ 504}{9} = 9\ 389.33$$

$$\text{Variance for group B} = \frac{\sum(x-\bar{x})^2}{n} = \frac{130}{9} = 14.44$$

The variance of a data set is the average $\bar{x}$ of the squared deviations of each of the n elements x of the set from the mean for the set:

$$\text{Variance} = \frac{\sum(x-\bar{x})^2}{n}$$

Notice that the units of variance are squared units.

Standard deviation

From the definition above, you can see that the variance is a squared value, which is not a very useful measure as the data values given are not squared. The other measure of dispersion, the standard deviation, represented by the Greek letter σ(lower case 'sigma'), is the square root of the variance.

Continuing the example above:

$$\text{Standard deviation for group A} = \sqrt{\text{variance}} = \sqrt{\frac{\sum(x-\bar{x})^2}{n}} = \sqrt{9\ 389.33} = 96.90$$

$$\text{Standard deviation for group B} = \sqrt{\text{variance}} = \sqrt{\frac{\sum(x-\bar{x})^2}{n}} = \sqrt{14.44} = 3.80$$

The much larger standard deviation for group A indicates that the data for group A is much more widely distributed around the mean than that for group B. There is greater dispersion in the distribution of data for group A than that for group B. This shows that the mean for group B is much less representative of the data elements than that of group A. The more varied the data values are, the less reliable they are as a means of prediction.

Standard deviation may serve as a measure of uncertainty – or accuracy. It gives an idea of how much variation there is from the mean. The standard deviation is the square root of the average distance of the values in the data set from their mean. The standard deviation is always a positive value, and is always measured in the same units as the data elements of the set.

Standard deviation σ of n elements x of data in a set:

$$\sigma = \sqrt{\frac{\sum (x - \bar{x})^2}{n}}$$

Note that the standard deviation is always positive, and the units of the standard deviation are the same as the units of the data elements.

Note

For other explanations of variance and standard deviation watch "Variance of a population",

Variance of a population (Duration: 08.05)

or read through "Describing Variability".

Take note!

For a fairly normal distribution that is not too skewed by having some very large or very small values:

- about 67% of the elements of the data set will lie within one standard deviation of the mean
- about 95% of the elements of the data set will lie within two standard deviation of the mean.

Example 1.1

Eight cupcakes from a batch were weighed and their masses recorded as follows:
$23\text{ g}; 37\text{ g}; 25\text{ g}; 28\text{ g}; 33\text{ g}; 31\text{ g}; 29\text{ g}; 26\text{ g}.$

1. Find the range of the masses.
2. Calculate the mean.
3. Calculate the variance.
4. Calculate the standard deviation.

Solutions

1. Arrange the masses in order: $23\text{ g}; 25\text{ g}; 26\text{ g}; 28\text{ g}; 29\text{ g}; 31\text{ g}; 33\text{ g}; 37\text{ g}.$
   Subtract the smallest mass from the largest: Range $= 37 - 23 = 14\text{ g}$
2. Divide the sum of all the masses by the number of cupcakes:
   Mean:
   $$\begin{aligned}\bar{x} &= \frac{\sum x}{n}\\ &= \frac{232}{8}\\ &= 29\text{ g}\end{aligned}$$
3. To find the variance, find the deviation of each mass from the mean, and square that.

Mass x	Deviation from the mean $x - \bar{x}$	(Deviations)^2 $(x - \bar{x})^2$
23	$23 - 29 = -6$	36
37	$37 - 29 = 8$	64
25	$25 - 29 = -4$	16
28	$28 - 29 = -1$	1
33	$33 - 29 = 4$	16
31	$31 - 29 = 2$	4
29	$29 - 29 = 0$	0
26	$26 - 29 = -3$	9

$$\begin{aligned}\text{Variance} &= \frac{\sum (x - \bar{x})^2}{n}\\ &= \frac{36 + 64 + 16 + 1 + 16 + 4 + 0 + 9}{8}\\ &= \frac{146}{8}\\ &= 18.25\end{aligned}$$

4. Standard deviation is the square root of the variance:

$$\sigma = \sqrt{\frac{\sum (x - \bar{x})^2}{n}}$$
$$= \sqrt{\frac{146}{8}}$$
$$= \sqrt{18.25}$$
$$= 4.27 \text{ g}$$

You will notice that tabulating the data and the calculations simplifies the application of the formulae.

Activity 1.1: Working with temperatures

Time required: 12 minutes

What you need:

- a pen and paper
- a calculator

What to do:

The maximum daily temperatures in Johannesburg in the second week of April 2021 are recorded and tabulated below, alongside those of the second week of January of the same year.

April	Temperature x	Deviation from the mean $x - \bar{x}$	(Deviation)^2 $(x - \bar{x})^2$
11^{th}	21°		
12^{th}	26°		
13^{th}	23°		
14^{th}	19°		
15^{th}	25°		
16^{th}	26°		
17^{th}	27°		

January	Temperature x	Deviation from the mean $x - \bar{x}$	(Deviation)² $(x - \bar{x})^2$
10^{th}	25°		
11^{th}	27°		
12^{th}	27°		
13^{th}	25°		
14^{th}	24°		
15^{th}	26°		
16^{th}	28°		

1. Work out:
    a. The mean temperature for the week in April (correct to one decimal place).
    b. The mean temperature for the week in January (correct to one decimal place).
2. Copy and complete the table above for both months.
3. Work out the variance for April and for January (correct to one decimal place).
4. Work out the standard deviation for April and for January (correct to one decimal place).
5. On what percentage of days in each of the months was the maximum temperature within one standard deviation of the mean?
6. What do the two standard deviations and your calculations show about the spread of data around the respective means?

What did you find?

1.
    a. April $\text{Mean } = \bar{x} = \frac{\sum x}{7} = \frac{167}{7} = 23.9^\circ$
    b. January $\text{Mean } = \bar{x} = \frac{\sum x}{7} = \frac{182}{7} = 26^\circ$
2. Table for April

April	Temperature x	Deviation from the mean $x - \bar{x}$	(Deviation)² $(x - \bar{x})^2$
11^{th}	21°	$21^\circ - 23.9 = -2.9$	8.4
12^{th}	26°	$26^\circ - 23.9 = 2.1$	4.4
13^{th}	23°	$23^\circ - 23.9 = -0.9$	0.8
14^{th}	19°	$19^\circ - 23.9 = -4.9$	24.0
15^{th}	25°	$25^\circ - 23.9 = 1.1$	1.2
16^{th}	26°	$26^\circ - 23.9 = 2.1$	4.4
17^{th}	27°	$27^\circ - 23.9 = 3.1$	9.6

Table for January

January	Temperature x	Deviation from the mean $x - \bar{x}$	(Deviation)^2 $(x - \bar{x})^2$
10^{th}	25°	$25^\circ - 26 = -1$	1
11^{th}	27°	$27^\circ - 26 = 1$	1
12^{th}	27°	$27^\circ - 26 = 1$	1
13^{th}	25°	$25^\circ - 26 = -1$	1
14^{th}	24°	$24^\circ - 26 = -2$	4
15^{th}	26°	$26^\circ - 26 = 0$	0
16^{th}	28°	$28^\circ - 26 = 2$	4

3. April:

Formula does not parse

$$\begin{aligned}\text{Variance} &= \frac{\sum (x - \bar{x})^2}{n} \\ &= \frac{8.4 + 4.4 + 0.8 + 24.0 + 1.2 + 4.4 + 9.6}{7} \\ &= \frac{52.8}{7} \\ &= 7.5\end{aligned}$$

Notice that we leave out the units for variance: the 'square' of degrees is not helpful here.

January:

$$\begin{aligned}\text{Variance} &= \frac{\sum (x - \bar{x})^2}{n} \\ &= \frac{1 + 1 + 1 + 1 + 4 + 0 + 4}{7} \\ &= \frac{12}{7} \\ &= 1.7\end{aligned}$$

4. Standard deviation for April:

$$\begin{aligned}\sigma &= \sqrt{\frac{\sum (x - \bar{x})^2}{n}} \\ &= \sqrt{\frac{52.8}{7}} \\ &= \sqrt{7.5} \\ &= 2.74^\circ\end{aligned}$$

Standard deviation for January:

$$\begin{aligned}\sigma &= \sqrt{\frac{\sum (x - \bar{x})^2}{n}} \\ &= \sqrt{\frac{12}{7}} \\ &= \sqrt{1.7} \\ &= 1.3^\circ\end{aligned}$$

5. April:

$$\begin{aligned}\text{One standard deviation from the mean} &= \bar{x} \pm \sigma \\ &= 23.9 \pm 2.74\end{aligned}$$

So the interval is Formula does not parse.

The maximum temperatures on 12, 13, $15$and 16April fall within this interval.

$\frac{4}{7} \times 100\% = 57.14\%$

So the maximum temperature on 57.14% of the days of the week in April fall within one standard deviation of the mean.

January:

$$\begin{aligned}\text{One standard deviation from the mean } &= \bar{x} \pm \sigma \\ &= 26 \pm 1.3\end{aligned}$$

So the interval is $[26 - 1.3; 26 + 1.3] = [24.7; 27.3]$.

The maximum temperatures on 10, 11, 12, 13 and 15 January fall within this interval.

$\frac{5}{7} \times 100\% = 71.43\%$

So the maximum temperature on 71.43% of the days of the week in January falls within one standard deviation of the mean.

6. The temperatures were more consistent, with fewer fluctuations, in the week in January than the week in April.

Exercise 1.1

1. World Health Organisation data for 2018 reported numbers of tuberculosis cases per $100\ 000$ in the population for some countries in Southern and Eastern Africa as follows:

Country	**Number per** $100\ 000$
Angola	355
Botswana	275
Kenya	292
Lesotho	659
Malawi	153
Mozambique	361
Namibia	524
South Africa	677
Zimbabwe	210
Uganda	200
United Republic of Tanzania	253
Zambia	346

   a. What is the range of tuberculosis incidence per $100\ 000$ in the populations across these countries?
   b. What is the mean for the entire region?

c. What is the standard deviation of numbers of reported tuberculosis cases per $100\ 000$ for the entire region?

d. What percentage of countries' tuberculosis incidence falls within one standard deviation from the mean?

2. World Health Organisation estimated data for 2016 country death rates due to road traffic injuries per $100\ 000$ population are as follows:

Country	**Number per** $100\ 000$
Angola	23.6
Botswana	23.8
Kenya	27.8
Lesotho	28.9
Malawi	31
Mozambique	30.1
Namibia	30.4
South Africa	25.9
Zimbabwe	34.7
Eswatini	26.9
United Republic of Tanzania	29.2
Zambia	20.9

a. What is the range of road traffic death rates per $100\ 000$ in the populations for each of these countries?

b. What is the mean for the region?

c. What is the standard deviation of numbers of deaths per $100\ 000$ for the region?

d. What percentage of countries' road traffic death rates falls within one standard deviation from the mean?

3. A manufacturer checks the width of a number of roller bearings from the production line in order to control quality. The following widths were measured, in micrometres (thousandth of a millimetre):
$15\ 015;\ 15\ 101;\ 15\ 089;\ 15\ 062;\ 15\ 111;\ 15\ 054;\ 15\ 028;\ 15\ 137;\ 15\ 009;\ 15\ 096$

a. Calculate the range.

b. Calculate the mean.

c. Calculate the standard deviation.

d. What percentage of the measurements are within one standard deviation of the mean?

The full solutions are at the end of the unit.

Summary

In this unit you have learnt the following:

- How to calculate the variance of a data set.
- How to calculate the standard deviation of a data set.

- How to interpret the results of calculations of standard deviation of a data set

Unit 1: Assessment

Suggested time to complete: 25 minutes

1. World Health Organisation data for 2018 reported numbers of malaria cases per $1\ 000$ in the population for some countries in Southern and Eastern Africa as follows:

Country	**Number per** $100\ 000$
Angola	227.36
Botswana	0.59
Kenya	60.05
Malawi	207.33
Mozambique	314.66
Namibia	31.68
South Africa	1.65
Zimbabwe	55.97
Eswatini	0.97
Uganda	262.69
Zambia	157.5

   a. What is the range of numbers of cases per $1\ 000$ for these countries?
   b. What is the mean incidence of malaria for the region?
   c. What is the standard deviation?

2. A potential car-buyer investigated the prices of eight cars on a car sales website, and wrote the following prices down (prices are in rands).
$103\ 125;\ 129\ 900;\ 87\ 900;\ 99\ 900;\ 85\ 000;\ 120\ 000;\ 95\ 000;\ 88\ 000$
   a. What is the range of prices?
   b. What is the mean?
   c. What is the standard deviation?

Question 3 adapted from the NC(V) level 4 Mathematics second paper of November 2017

3. The following represents the marks (percentages) of the learners who wrote the examination in Pattern Maker's Theory:

Scores	66	59	43	72	57	47	81	54	92	61

   Calculate the standard deviation of the marks of the 10 learners.

4. The table below shows the number of minutes taken by seven mechanic apprentices each, to replace a control arm bushing on a vehicle:

Minutes	84	150	54	126	78	102	108

Calculate the standard deviation of the time taken by the seven apprentices.

The full solutions are at the end of the unit.

Unit 1: Solutions

Exercise 1.1

1. Tuberculosis incidence 2018:

Country	**Number per** $100\ 000$	$x-\bar{x}$	$(x-\bar{x})^2$
Angola	355	$355-358.75=-3.75$	14.06
Botswana	275	$275-358.75=-83.75$	$7\ 014.06$
Kenya	292	$292-358.75=-66.75$	$4\ 455.56$
Lesotho	659	$659-358.75=300.25$	$90\ 150.06$
Malawi	153	$153-358.75=-205.75$	$42\ 333.06$
Mozambique	361	$361-358.75=2.25$	5.06
Namibia	524	$524-358.75=165.25$	$27\ 307.56$
South Africa	677	$677-358.75=318.25$	$101\ 283.10$
Zimbabwe	210	$210-358.75=-148.75$	$22\ 126.56$
Uganda	200	$200-358.75=-158.75$	$25\ 201.56$
United Republic of Tanzania	253	$253-358.75=-105.75$	$11\ 183.06$
Zambia	346	$346-358.75=-12.75$	162.56

a. $\text{Range } = 677-153=524$ cases per $100\ 000$.

b. Mean:
$$\begin{aligned}\bar{x}=\frac{\sum x}{n} &= \frac{355+275+292+659+153+361+524+677+210+200+253+346}{12}\\ &= \frac{4305}{12}\\ &= 358.75\end{aligned}$$

c. Standard deviation:
$$\begin{aligned}\sigma &= \sqrt{\frac{\sum(x-\bar{x})^2}{n}}\\ &= \sqrt{\frac{331236.3}{12}}\\ &= \sqrt{227603.025}\\ &= 166.14\end{aligned}$$

d. Percentage of countries in the region falling within one standard deviation of the mean:
$$\begin{aligned}\text{One standard deviation from the mean} &= \bar{x}\pm\sigma\\ &= 358.75\pm166.14\end{aligned}$$
So the interval is $[358.75-166.14;358.75+166.14]=[192.61;524.89]$.

Three countries (Malawi, Lesotho, South Africa) fall outside this interval, so $\frac{9}{12}\times100\%=75\%$ fall within one standard deviation of the mean.

2. Road traffic death rates:

Country	**Number per** $100\ 000$	$x - \bar{x}$	$(x - \bar{x})^2$
Angola	23.6	$23.6 - 27.77 = -4.17$	17.39
Botswana	23.8	$23.8 - 27.77 = -3.97$	15.76
Kenya	27.8	$27.8 - 27.77 = 0.03$	0.00
Lesotho	28.9	$28.9 - 27.77 = 1.13$	1.28
Malawi	31	$31 - 27.77 = 3.23$	10.43
Mozambique	30.1	$30.1 - 27.77 = 2.33$	5.43
Namibia	30.4	$30.4 - 27.77 = 2.63$	6.92
South Africa	25.9	$25.9 - 27.77 = -1.87$	3.50
Zimbabwe	34.7	$34.7 - 27.77 = 6.93$	48.02
Eswatini	26.9	$26.9 - 27.77 = -0.87$	0.76
United Republic of Tanzania	29.2	$29.2 - 27.77 = 1.43$	2.04
Zambia	20.9	$20.9 - 27.77 = -6.87$	47.20

a. $\text{Range} = 34.7 - 20.9 = 13.8$ road traffic deaths per $100\ 000$.

b. Mean:

$$\begin{aligned}\bar{x} = \frac{\sum x}{n} &= \frac{23.6 + 23.8 + 27.8 + 28.9 + 31 + 30.1 + 30.4 + 25.9 + 34.7 + 26.9 + 29.2 + 20.9}{12} \\ &= \frac{333.2}{12} \\ &= 27.77\end{aligned}$$

c. Standard deviation:

$$\begin{aligned}\sigma &= \sqrt{\frac{\sum (x - \bar{x})^2}{n}} \\ &= \sqrt{\frac{158.73}{12}} \\ &= \sqrt{13.2275} \\ &= 3.64\end{aligned}$$

d. Percentage of countries in the region falling within one standard deviation of the mean:

$$\begin{aligned}\text{One standard deviation from the mean} &= \bar{x} \pm \sigma \\ &= 27.77 \pm 3.64\end{aligned}$$

So the interval is $[27.77 - 3.64; 27.77 + 3.64] = [24.13; 31.41]$.

Four countries (Angola, Botswana, Zimbabwe, Zambia) fall outside this interval, so $\frac{8}{12} \times 100\% = 66.67\%$ fall within one standard deviation of the mean.

3.

Width (mm)	**Deviation from the mean** $x-\bar{x}$	$(\text{Deviation})^2$ $(x-\bar{x})^2$
$15\ 015$	-55.2	$3\ 047.04$
$15\ 101$	30.8	948.64
$15\ 089$	18.8	353.44
$15\ 062$	-8.2	67.24
$15\ 111$	40.8	$1\ 664.64$
$15\ 054$	-16.2	262.44
$15\ 028$	-42.2	$1\ 780.84$
$15\ 137$	66.8	$4\ 462.24$
$15\ 009$	-61.2	$3\ 745.44$
$15\ 096$	25.8	665.64

a. $\text{Range } = 15\ 137 - 15\ 009 = 128 \text{ mm}$

b. Mean:

$$\begin{aligned}\bar{x} &= \frac{\sum x}{n}\\ &= \frac{150702}{10}\\ &= 15\ 070.2\end{aligned}$$

c. Standard deviation:

$$\begin{aligned}\sigma &= \sqrt{\frac{\sum (x-\bar{x})^2}{n}}\\ &= \sqrt{\frac{16\ 997.6}{10}}\\ &= \sqrt{1\ 699.76}\\ &= 41.228\end{aligned}$$

d. Percentage of bearings falling within one standard deviation of the mean:

$$\begin{aligned}\text{One standard deviation from the mean } &= \bar{x} \pm \sigma\\ &= 15\ 070.2 \pm 41.2\end{aligned}$$

So the interval is $[15\ 070.2 - 41.2; 15\ 070.2 + 41.2] = [15\ 029; 15\ 111.4]$.

Four bearings fall outside this interval ($15\ 015;\ 15\ 028;\ 15\ 137;\ 15\ 009$), so $\frac{6}{10} = 60\%$ fall within one standard deviation from the mean.

Back to Exercise 1.1

Unit 1: Assessment

1.

Country	Number malaria cases per $1\ 000$	Deviation from mean $x-\bar{x}$	(Deviation)² $(x-\bar{x})^2$
Angola	227.36	107.32	$11\ 517.39$
Botswana	0.59	-119.45	$14\ 268.52$
Kenya	60.05	-59.99	$3\ 598.91$
Malawi	207.33	87.29	$7\ 619.39$
Mozambique	314.66	194.62	$37\ 876.59$
Namibia	31.68	-88.36	$7\ 807.65$
South Africa	1.65	-118.39	$14\ 016.41$
Zimbabwe	55.97	-64.07	$4\ 105.08$
Eswatini	0.97	-119.07	$14\ 177.88$
Uganda	262.69	142.65	$20\ 348.76$
Zambia	157.5	37.46	$1\ 403.18$

a. $\text{Range} = 314.66 - 0.59 = 314.07$

b. Mean:

$$\text{Mean } = \bar{x} = \frac{\sum x}{11} = \frac{1320.45}{11} = 120.04$$

c. Standard deviation:

$$\begin{aligned}\sigma &= \sqrt{\frac{\sum(x-\bar{x})^2}{n}}\\ &= \sqrt{\frac{136\ 739.8}{11}}\\ &= \sqrt{12\ 430.89}\\ &= 111.49\end{aligned}$$

2.

Price in Rands	Deviation from mean $x-\bar{x}$	(Deviation)² $(x-\bar{x})^2$
$103\ 125$	$2\ 021.88$	$4\ 087\ 978.52$
$129\ 900$	$28\ 796.88$	$829\ 260\ 009.77$
$87\ 900$	$-13\ 203.13$	$174\ 322\ 509.77$
$99\ 900$	$-1\ 203.13$	$1\ 447\ 509.77$
$85\ 000$	$-16\ 103.13$	$259\ 310\ 634.77$
$120\ 000$	$18\ 896.88$	$357\ 091\ 884.77$
$95\ 000$	$-6\ 103.13$	$37\ 248\ 134.77$
$88\ 000$	$-13\ 103.13$	$171\ 691\ 884.77$

a. $\text{Range} = 129\ 900 - 85\ 000 = \text{ R}44\ 900$

b.

$$\text{Mean } = \bar{x} = \frac{\sum x}{8} = \frac{808\ 825}{8} = 101\ 103.10$$

The mean price is $\text{R}101\ 103.10$.

c. Standard deviation:

$$\sigma = \sqrt{\frac{\sum (x-\bar{x})^2}{n}}$$
$$= \sqrt{\frac{1\ 834\ 460\ 546.88}{8}}$$
$$= \sqrt{229\ 307\ 568.40}$$
$$= 15\ 142.90$$

The standard deviation is $\text{R}15\ 142.90$.

3. Table of values:

Mark	Deviation from mean $x-\bar{x}$	(Deviation)^2 $(x-\bar{x})^2$
66	2.8	7.84
59	-4.2	17.64
43	-20.2	408.04
72	8.8	77.44
57	-6.2	38.44
47	-16.2	262.44
81	17.8	316.84
54	-9.2	84.64
92	28.8	829.44
61	-2.2	4.84

Standard deviation of the marks of the 10 learners:

$$\sigma = \sqrt{\frac{\sum (x-\bar{x})^2}{n}}$$
$$= \sqrt{\frac{2\ 047.6}{10}}$$
$$= \sqrt{204.76}$$
$$= 14.31$$

4. Table of values:

Minutes taken	Deviation from mean $x-\bar{x}$	(Deviation)^2 $(x-\bar{x})^2$
84	-16.29	265.36
150	49.71	2471.08
54	-46.29	2142.76
126	25.71	661.00
78	-22.29	496.84
102	1.71	2.92
108	7.71	59.44

Standard deviation of the times taken:

$$\begin{aligned}\sigma &= \sqrt{\frac{\sum(x-\bar{x})^2}{n}}\\ &= \sqrt{\frac{6\ 099.43}{7}}\\ &= \sqrt{204.76}\\ &= 29.52\end{aligned}$$

Back to Unit 1: Assessment

Unit 2: Represent data using a scatter plot

GILL SCOTT

Unit outcomes

By the end of this unit you will be able to:

- Draw a scatter plot.
- Understand when it is appropriate to use a scatter plot.
- Draw an intuitive line of best fit.

What you should know

Before you start this unit, make sure you can:

- Work with measures of central tendency and dispersion. To revise this refer to level 3 subject outcomes 4.1 and 4.2.
- Work with graphs of linear, quadratic and exponential functions. To revise this you can refer back to level 2 subject outcome 2.1, and level 3 subject outcome 2.1.

Introduction

Very often the purpose of an investigation is to find a relationship between two variables. For example height and mass; a taller person is likely to weigh more than a shorter person does. The aim is to find a mathematical expression of the relationship between these variables, because this would help to predict values for data elements that may not have been included in the set. In this unit, we will plot points to find the mathematical relationship between the variables of each element of data in a set.

Each element of **univariate** data has only ***one*** variable.

Each element of **bivariate** data has ***two*** variables.

Drawing scatter plots

When each element of data in a dataset consists of two parts, for example height and mass, it is called bivariate data, to indicate that it consists of two variables. A first step in analysing bivariate data is to visualise it by plotting the data elements on an x-y coordinate system, with each axis representing one of the variables. The plotting of the data points is then analysed to see if the plotted points (the 'scatter plot') approximate the graph of one of the functions we have seen before.

Example 2.1

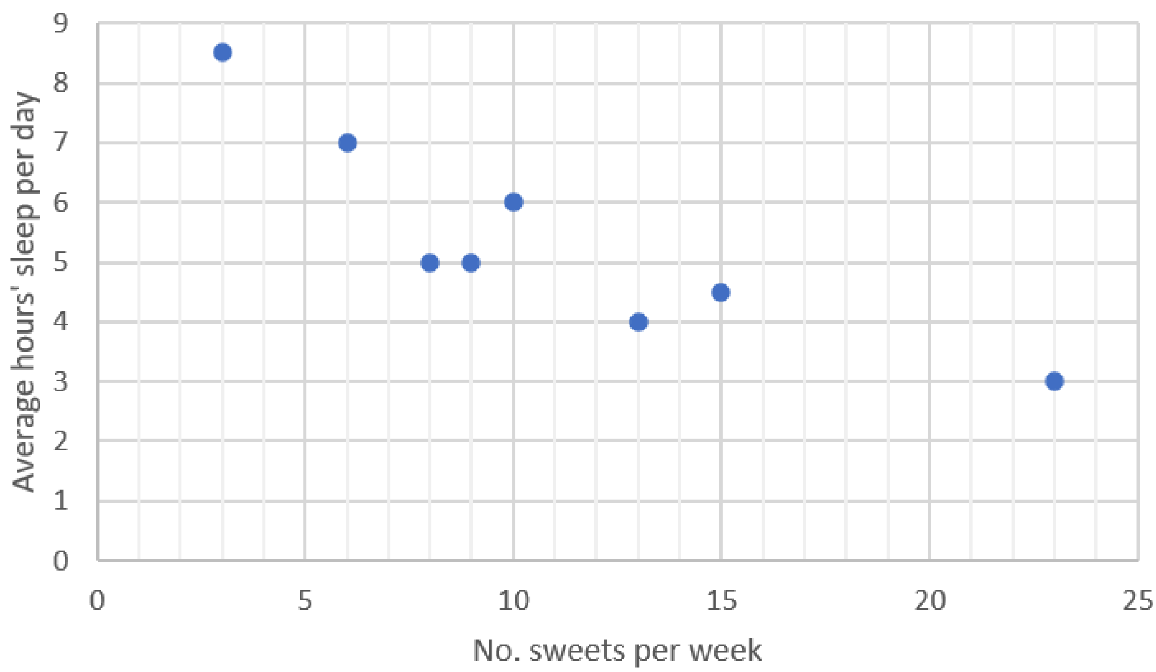
Example adapted from Siyavula Grade 11 Mathematics p. 480

Eight children's sweets consumption and sleeping habits were recorded as in the table below. Draw a scatter plot of the data by plotting the independent variables on the x-axis, and the dependent variables on the y-axis. Explain what you find.

No. sweets per week	15	9	10	6	23	8	13	3
Average hours sleep per day	4.5	5	6	7	3	3	4	8.5

Solution

The data can be plotted as follows:



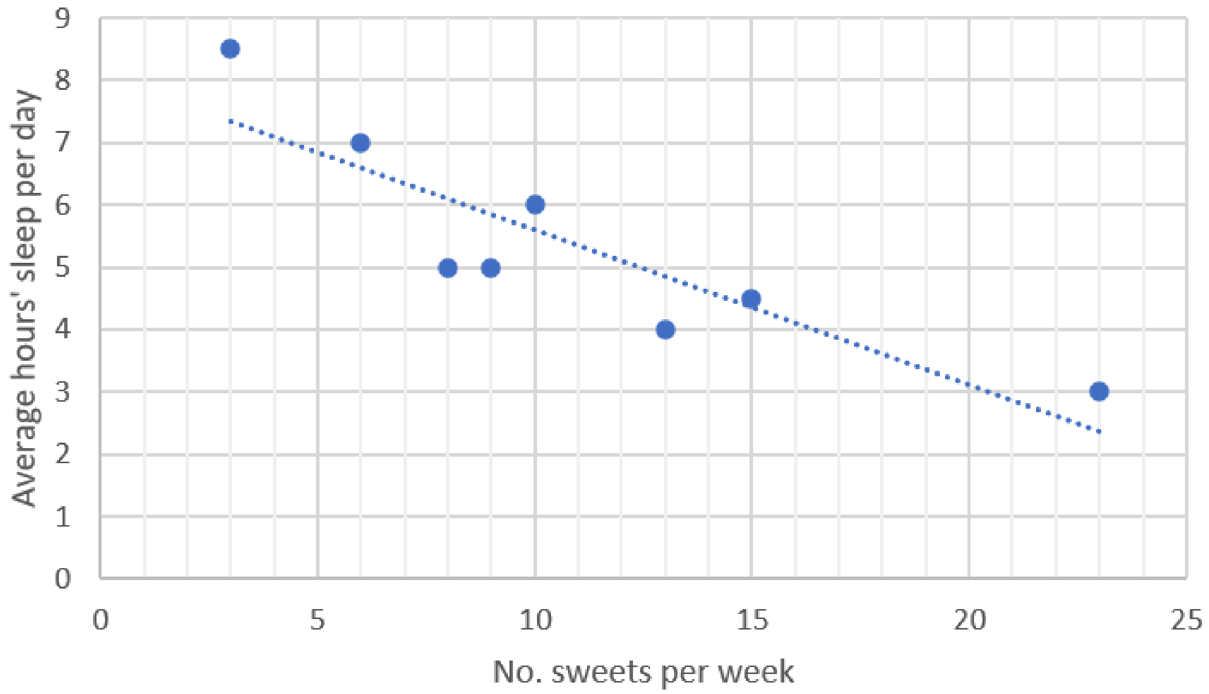
Looking at the scatter plot above, it appears that the points approximate a straight line, rather than any curve, although they clearly do not fit exactly to one straight line. It is also clear that the points in general have a negative relationship (the line has a negative gradient), with a lower consumption of sweets linked to higher average hours of sleep per day.

Take note!

When the function is a straight line, if increased values of the one variable correspond to increased values of the other variable, the relationship is said to be '*positive*'; this corresponds to the line having a positive slope. Similarly, if the line has a negative slope, and low values of one variable correspond to high values of the other variable, the relationship is said to be '*negative*'.

A straight line can be drawn approximating the points in the scatter plot in example 2.1, more or less as follows:

Once the line has been drawn, the normal processes can be used to find its equation.



In the above graph:

- $y\text{-intercept} = c = 8$
- The straight line passes through the point $(20, 3)$.

$$
\begin{aligned}
y &= mx + c \\
y &= mx + 8 \\
3 &= 20m + 8 \\
20m &= -5 \\
m &= -\frac{1}{4}
\end{aligned}
$$

So the equation of the line is $y = -\frac{1}{4}x + 8$.

Take note!

In drawing the line:

- Take care that it follows the general direction followed by the points.
- If one or two points clearly do not follow the general direction, they are probably 'outliers', and should be ignored.
- Of those points that do not lie *on* the line, there should be more or less the same number *above* the line as *below* it.

- Clusters of points above and below the line should not occur at the ends of the line.
- Taking the above guidelines into account, the more points that lie *on* the line, the better.

Activity 2.1: Draw a scatter plot, and intuitively fit a curve

Activity adapted from an example in Siyavula, Grade 12 Mathematics Chapter 9 p 385

Time required: 15 minutes

What you need:

- a pen or pencil
- paper on which to draw a graph

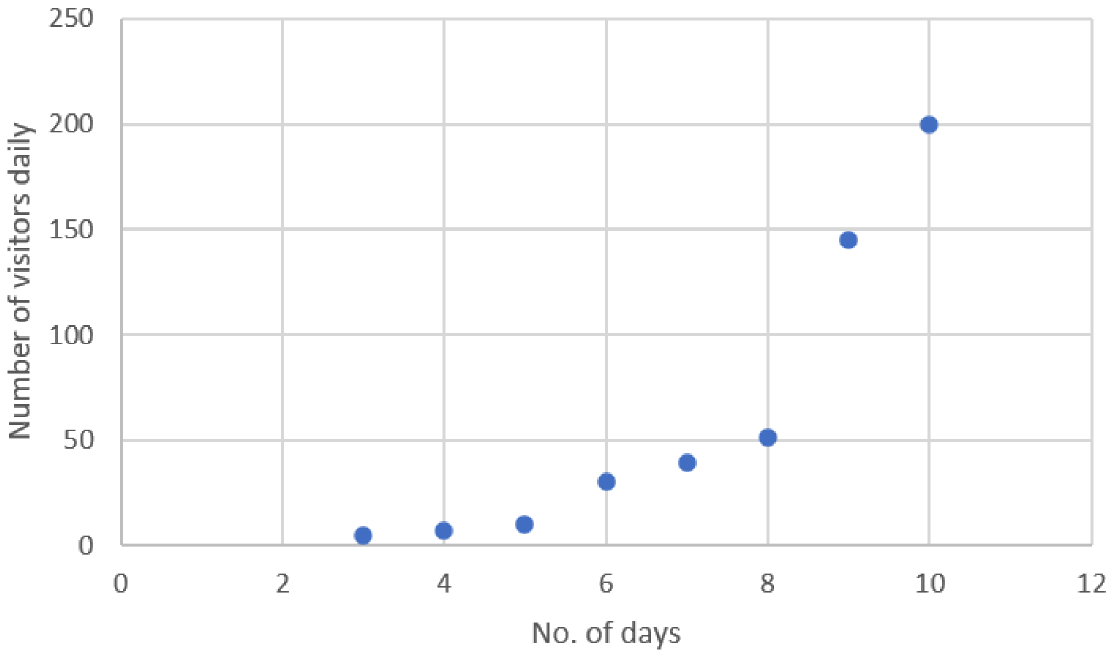
What to do:

The following points represent numbers of visitors to a website (y) on the xth day since the establishment of the website.
$(3,5)$; $(4,7)$; $(5,10)$; $(6,30)$, $(7,39)$; $(8,51)$; $(9,145)$; $(10,200)$

1. Plot the points on an x-y coordinate system.
2. Answer the following questions:
   a. What are the two variables being compared?
   b. What type of function best fits the data?
   c. Is the relationship between the two variables strong or weak?
   d. Is the relationship between the two variables positive or negative?
3. Using the answers above, describe the relationship between the two variables in one sentence.

What did you find?

1. Plot the points on an x-y coordinate system:



2.

a. The variables being compared are the number of daily visitors and the number of days since establishment of the website.

b. The data fit an exponential function.

c. The data points do not fit the curve very closely, so the relationship can be described as weak.

d. As time increases, the number of visitors increases, so the relationship can be described as positive.

3. There is a weak, positive exponential relationship between the number of visitors to the website and the number of days since its establishment.

Note

For more explanations of drawing scatter plots and lines of best fit, have a look at the following sites when you have access to the internet:

MooMath and Science: Creating a scatterplot and drawing a line of best fit (Duration: 03:42)

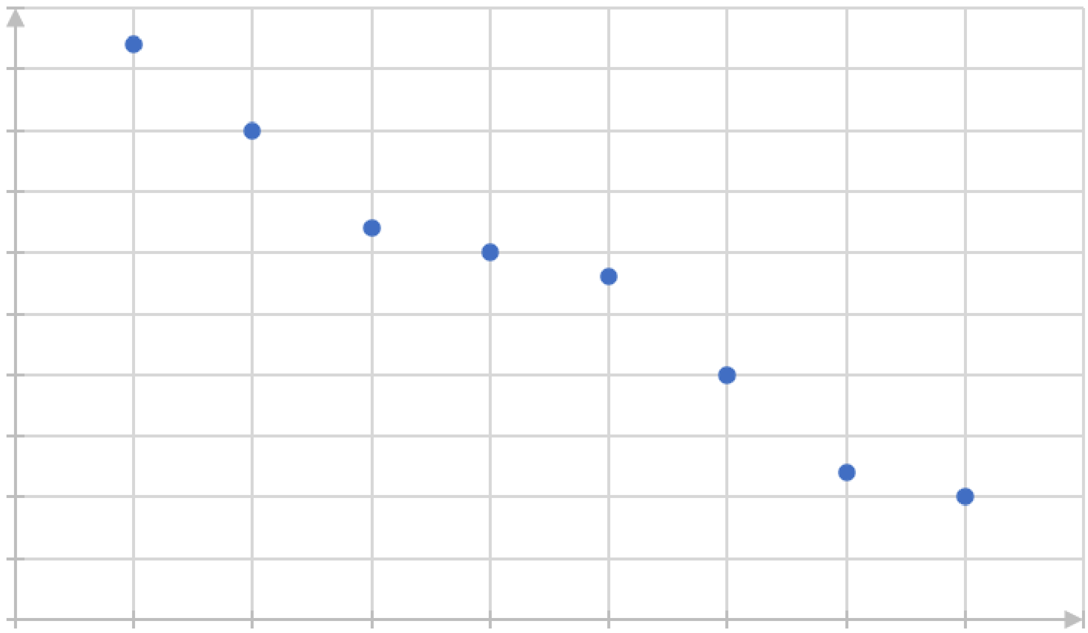
[Maths Tutorial: Interpreting scatter plots (VCE Further maths Tutorials)](#) (Duration: 14:55)

Exercise 2.1

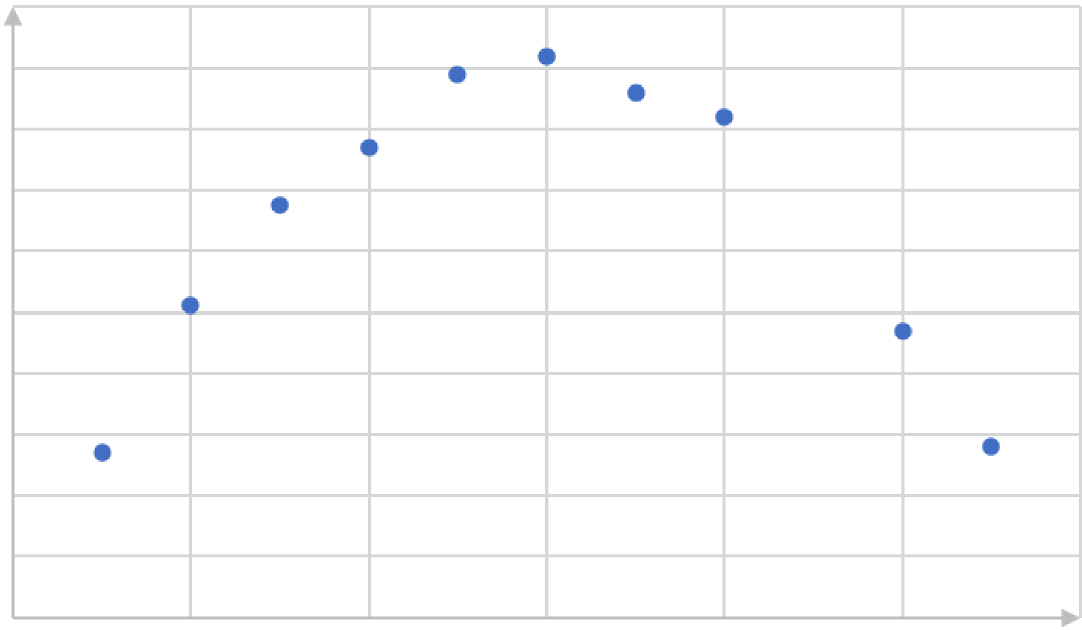
Question 1 adapted from Siyavula Grade 12 Mathematics Exercise 9-2

1. Identify the function (linear, exponential or quadratic) which would best fit the data in each of the scatter plots below. Describe the relationship (positive or negative) where possible, and the strength of the fit:

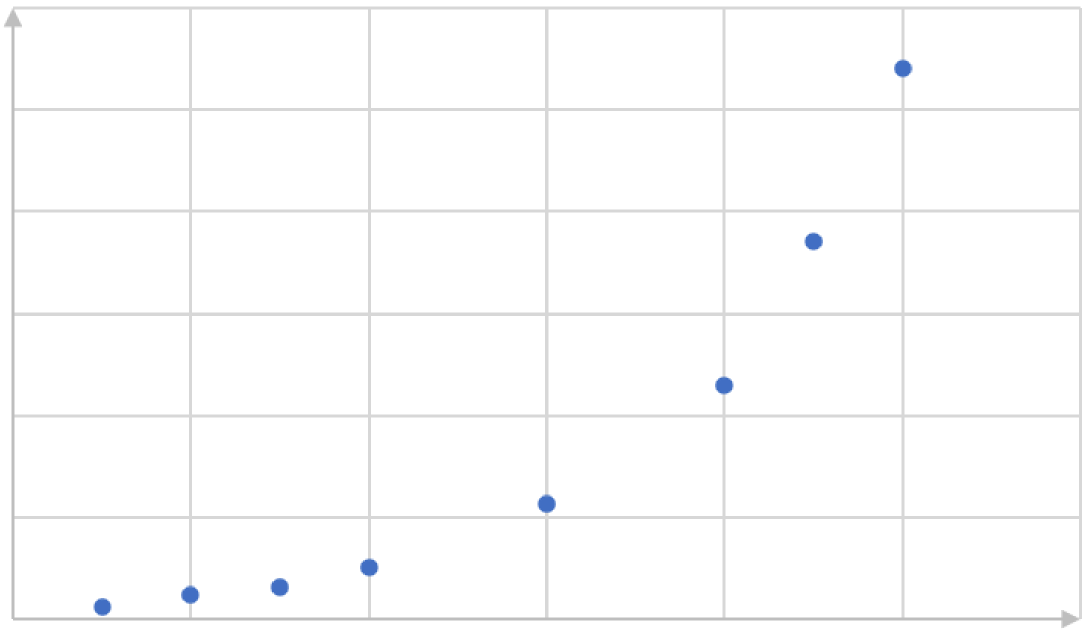
   a.



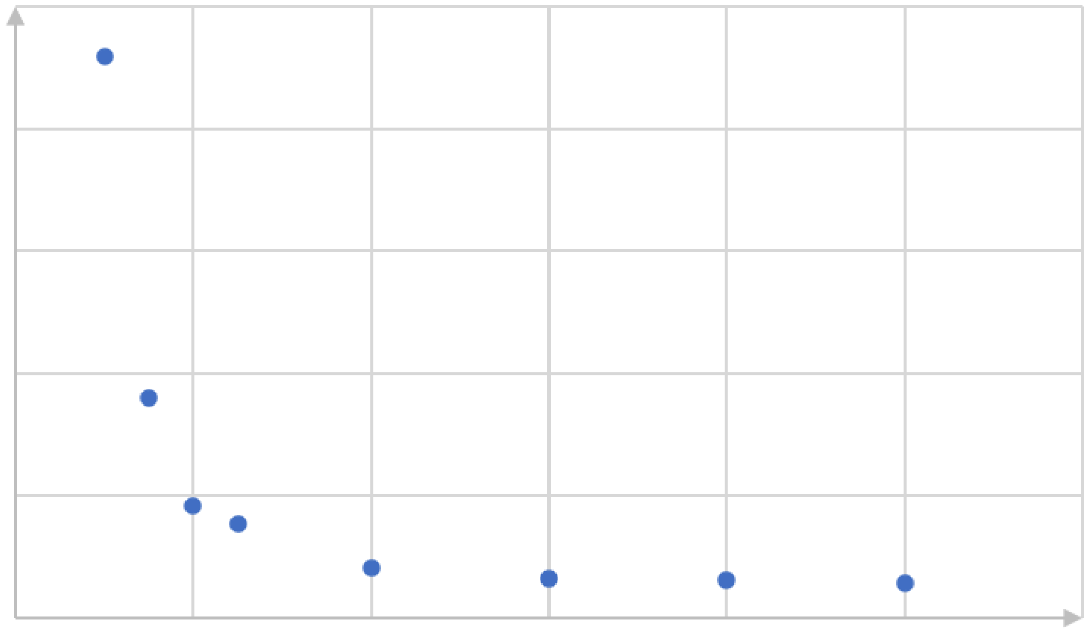
   b.



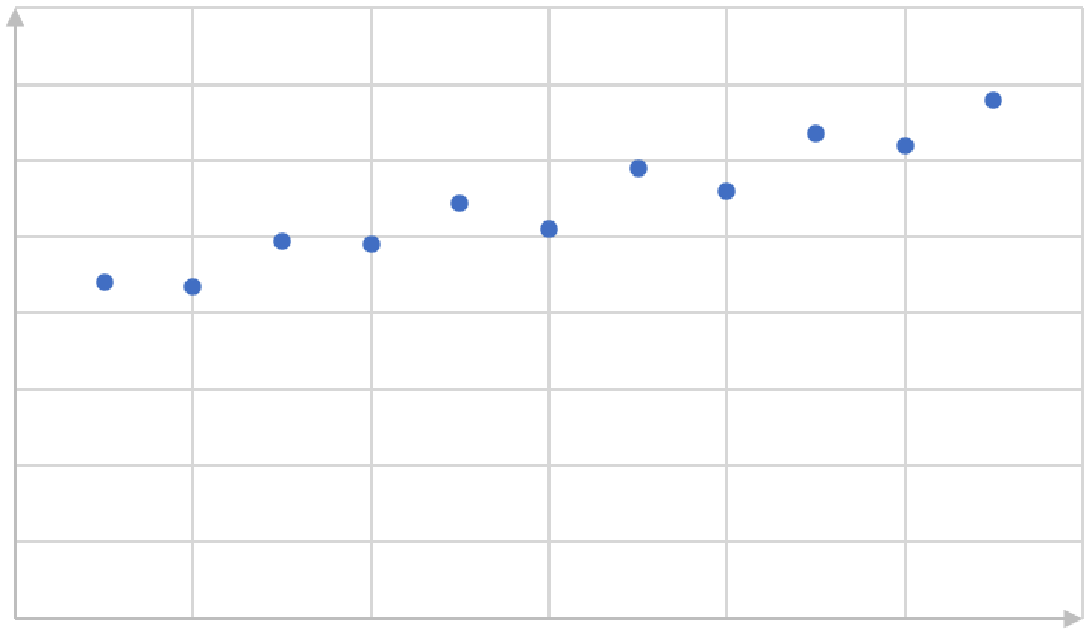
c.



d.



e.



Question 2 adapted from Siyavula Grade 12 Mathematics Exercise 9-2

2. Dr Dandara is a scientist trying to find a cure for a disease that has an 80% mortality rate. This means that 80% of people who get the disease will die. He knows of a plant which is used in traditional medicine to treat the disease. He extracts the active ingredient from the plant and tests different dosages (measured in milligrams) on different groups of patients. Examine the data below and complete the questions that follow.

Dosage (mg)	0	25	50	75	100	125	150	175	200
Mortality rate (%)	80	73	63	49	42	32	25	11	5

a. Draw a scatter plot of the data.

b. Which function would best fit the data? Draw in the line of best fit, and find its equation.

c. Describe the fit in terms of strength and direction.

3. The enrolment of learners for NC(V) programmes at TVET colleges was reported as follows:

Year	2010	2011	2012	2013	2014	2015	2016
NC(V) enrolment (in thousands)	130	124	140	154	166	165	177

a. Draw a scatter plot with the year on the horizontal axis and the enrolment on the vertical axis.

b. Draw a best fit line through the points and find its equation.

c. What is the meaning of the gradient of the line?

d. According to your model, what was the approximate enrolment in the year 0. Do you think this answer is realistic? Discuss.

Question 4 adapted from Siyavula Grade 12 Mathematics Exercise 9-2

4. Different climate conditions, such as temperature and rainfall, have significant effects on the yield of vegetable and other crops. Farmers need information of this type to work out the best time for planting. The following table matches average temperatures over a 12-month period in different parts of a country against average crop yields recorded in tonnes per hectare.

Average monthly temperature	8	10	13	15	18	20	21	19
Tonnes per hectare produced	10	16	22	24	23	21	18	20

a. Draw a scatter plot of the data.

b. Which function would best fit the data? Draw in the line of best fit, and give the general form of its equation.

c. Describe the fit in terms of strength.

The full solutions are at the end of the unit.

Summary

In this unit you have learnt the following:

- How to draw a scatter plot.
- When it is appropriate to draw a scatter plot.
- About drawing lines of best fit intuitively.

Unit 2: Assessment

Suggested time to complete: 40 minutes

Keep your solutions to these questions for referring to again in unit 3.

Question 1 adapted from NC(V) Mathematics Level 4 examination, November 2017

1. A study was done to compare electricity usage of geysers that are inside or outside the house. The table below shows the electricity usage (in kilowatt hours) for equivalent water consumption for matched households that have geysers inside the house, and those with geysers outside the house. Nine houses of each type were considered in the study.

Inside the house (kWh)	29	31	20	40	26	39	32	34	35
Outside the house (kWh)	19	23	13	32	17	28	25	24	28

   a. Draw a scatter plot of the data.
   b. Draw a line of best fit.
   c. Find the equation of the line.
   d. Describe the relationship between the electricity usage when the geyser is inside the house and when it is outside.

2. Tobacco smoking is still one of the world's largest health problems, although the prevalence of smoking is generally decreasing. The table below shows numbers of deaths (in thousands) in South Africa from smoking, in recent years.

Year	2010	2011	2012	2013	2014	2015	2016	2017
Deaths $('000)$	38.0	35.8	34.1	32.6	31.8	31.5	31.3	29.9

   a. Draw a scatter plot of the data.
   b. Draw the line of best fit.
   c. Find the equation of the line.
   d. Describe the relationship between the year and the number of deaths from smoking.

3. A college helps learners to complete their national diplomas by negotiating with employers in the region with the aim of placing the learners for work experience. Over recent years they have tracked their engagements with employers against numbers of learners placed in work experience as follows:

No. employers engaged	15	45	65	35	38	25	40	30
No. learners placed	40	90	128	90	95	60	140	75

   a. Draw a scatter plot of the data.
   b. Draw the line of best fit.
   c. Find the equation of the line.
   d. Describe the relationship between the numbers of employers engaged and the numbers of learners placed.

Question 4 taken from NC(V) Mathematics Level 4 examination, November 2019

4. The data below shows the mathematics marks of 10 learners at a college for the internal examinations and the external examinations.

Internal examinations (x)	80	68	94	72	74	83	56	68	65	75
External examinations (x)	72	71	96	77	82	72	58	83	78	80

a. Draw a scatter plot of the marks in the above table on an x-y plane, with each axis showing values from 50 to 96.

b. Draw the line of best fit.

c. Find the equation of the line.

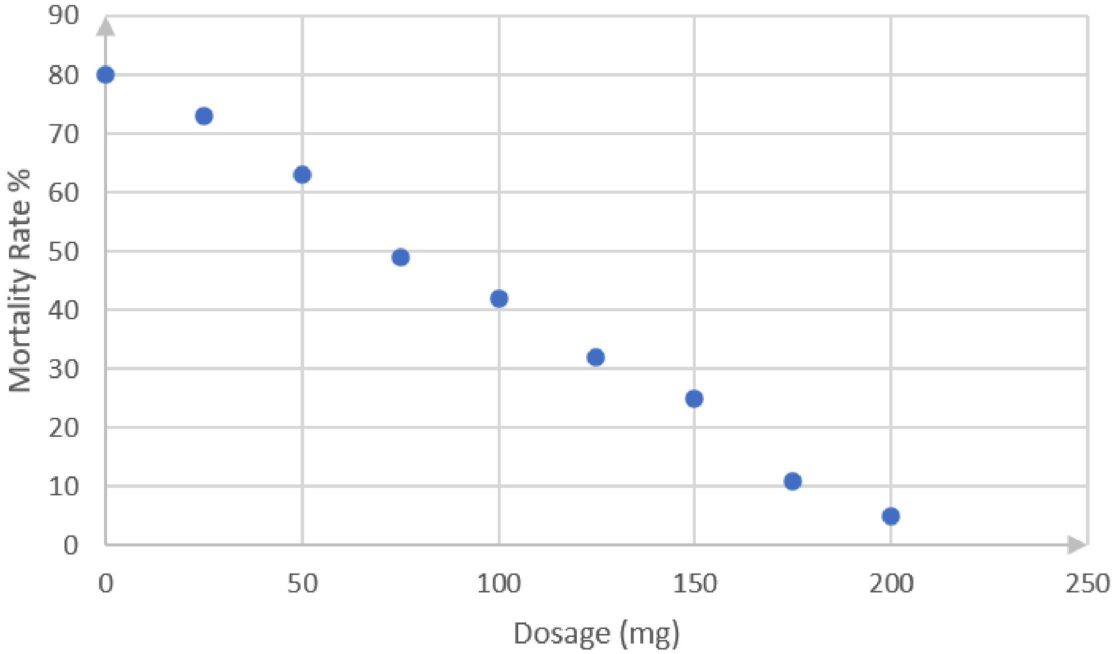
d. Describe the relationship between the internal and external examination marks.

The full solutions are at the end of the unit.

Unit 2: Solutions

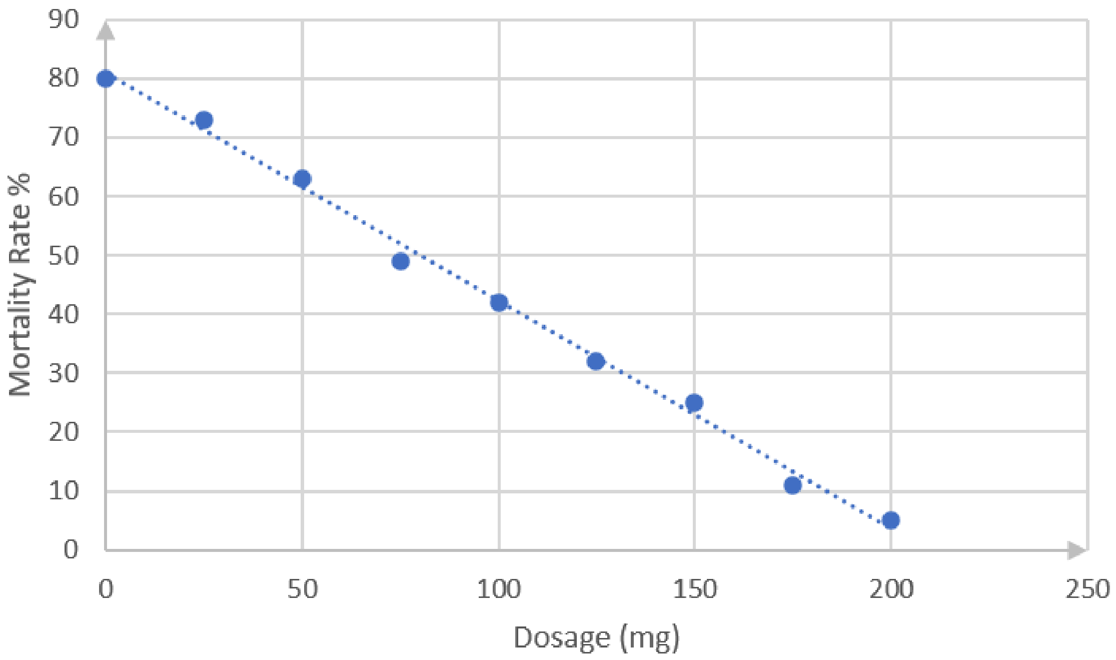
Exercise 2.1

1.
   a. Weak, negative linear relationship
   b. Fairly strong quadratic relationship
   c. Strong positive, exponential relationship
   d. Negative exponential relationship
   e. Weak positive linear relationship
2.
   a.



The data points approximate a straight line.

   b.



The best-fit line passes through points $(0, 80)$and $(200, 5)$. (You might have drawn a slightly different straight line, in which case your equation would be a bit different from that calculated below. Check that your line complies with the guidelines in the notes before activity 2.1.)
Using the method you learnt in level 3 subject outcome 3.2 unit 1, or any other method, the equation of this straight line can be found as follows.

$$\frac{y-y_1}{x-x_1} = \frac{y_2-y_1}{x_2-x_1}$$

$$\frac{y-80}{x-0} = \frac{5-80}{200-0}$$

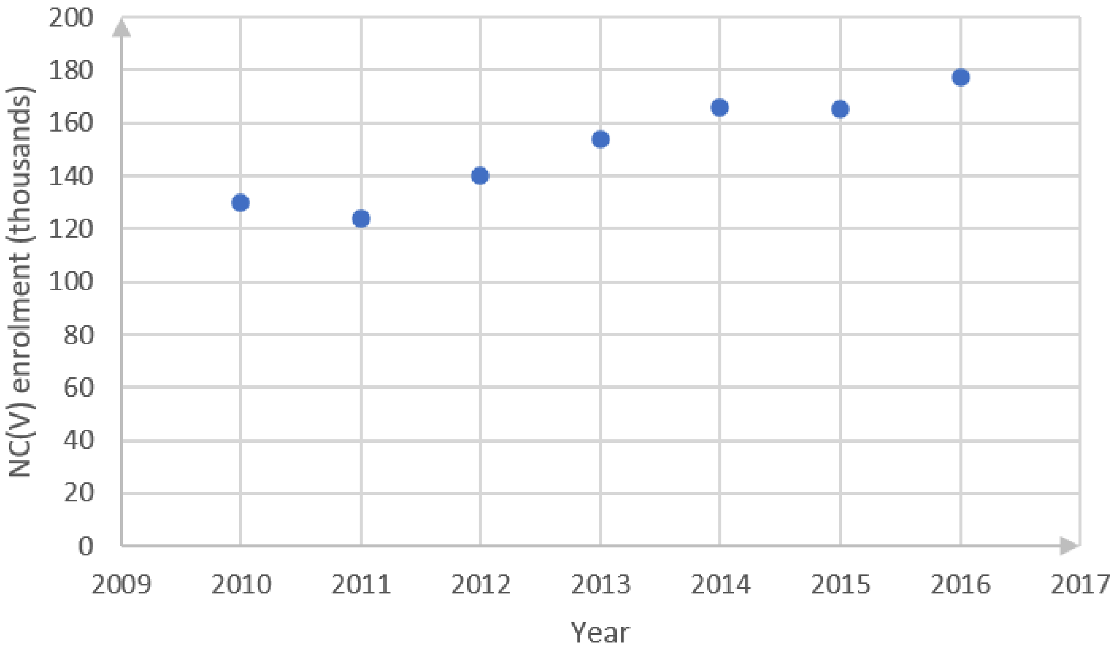
$$y - 80 = -\frac{3x}{8}$$

$$y = -\frac{3x}{8} - 80$$

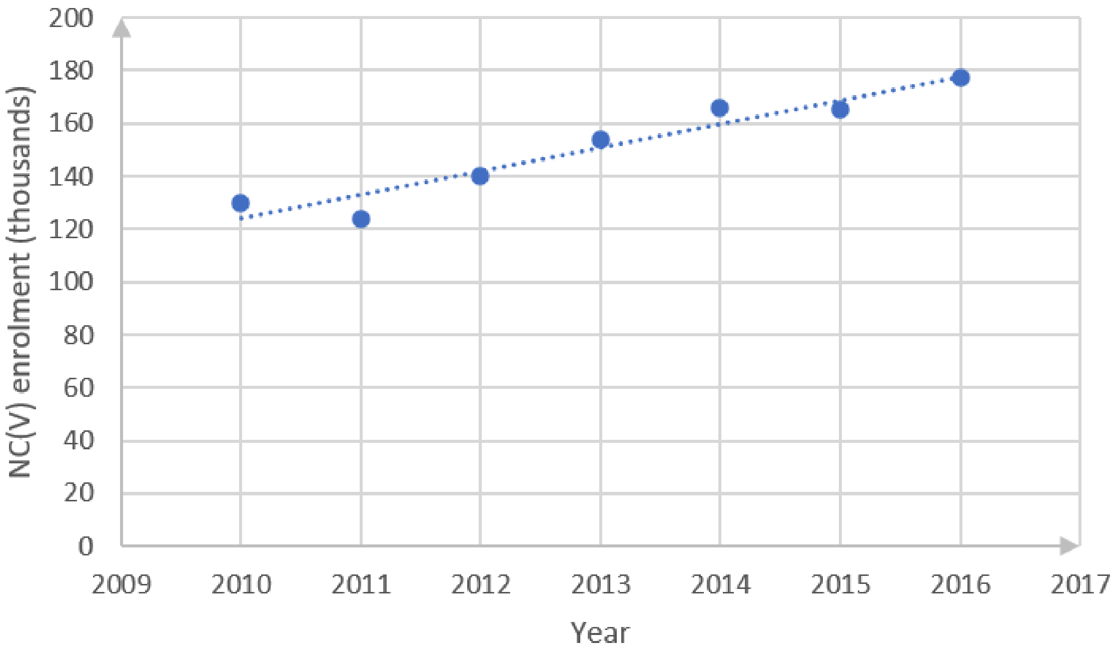
c. There is a strong negative relationship between an increase in the dosage and a decrease in the mortality rate.

3.

a.



b.



(You might have drawn a slightly different straight line, in which case your equation would be different from that calculated below. Check that your line complies with the guidelines in the notes before activity 2.1.)

Best-fit line passes through point $(2014, 160)$and $(2016, 180)$.

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\frac{y - 160}{x - 2\ 014} = \frac{180 - 160}{2\ 016 - 2\ 014}$$

$$y - 160 = \frac{20}{2}(x - 2\ 014)$$

$$y = 10x - 20\ 140 + 160$$

$$y = 10x - 19\ 980$$

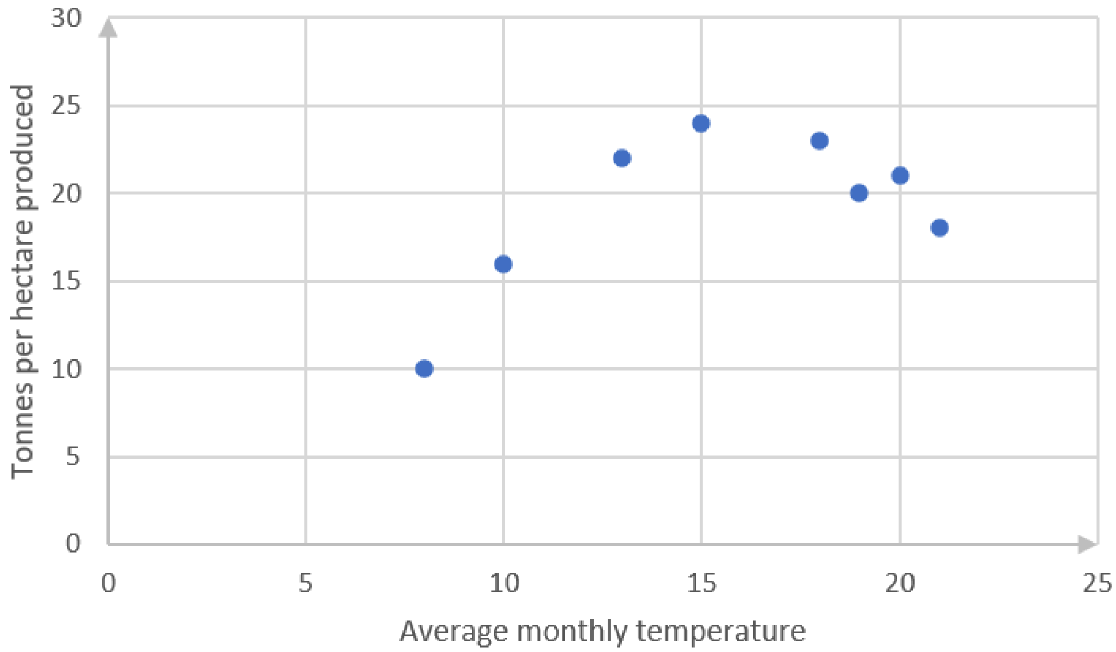
c. With the gradient of the line being 10, this means that the enrolment is increasing every year by a factor of 10.

d. The straight-line graph indicates a negative enrolment of $-19\ 980$ in year 0. This is not realistic, since it would need the college to have been enrolling NC(V) learners for more than $2\ 000$ years, while the curriculum was only introduced in 2007.

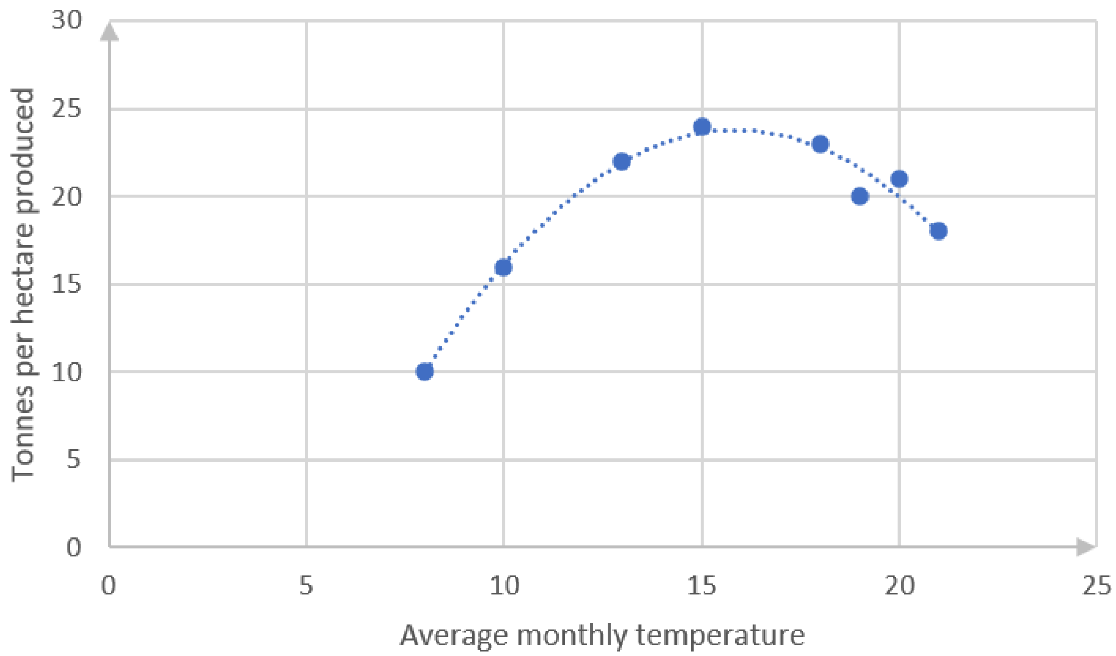
4.

Average monthly temperature	8	10	13	15	18	20	21	19
Tonnes per hectare produced	10	16	22	24	23	21	18	20

a.



b. The scatter plot seems to indicate a quadratic equation.



$y = -ax^2 + bx - c$

The coefficients of x^2 and the constant are indicated as negative because the curve is concave facing down, and the y-intercept will clearly be negative if the curve is continued to that point.

c. There is a strong fit between the data and the curve.

Note: You are not required to find equations of best fit lines for scatter plots that do not satisfy or approximate straight lines, but you should be able to recognise the type of graph that best fits the data.

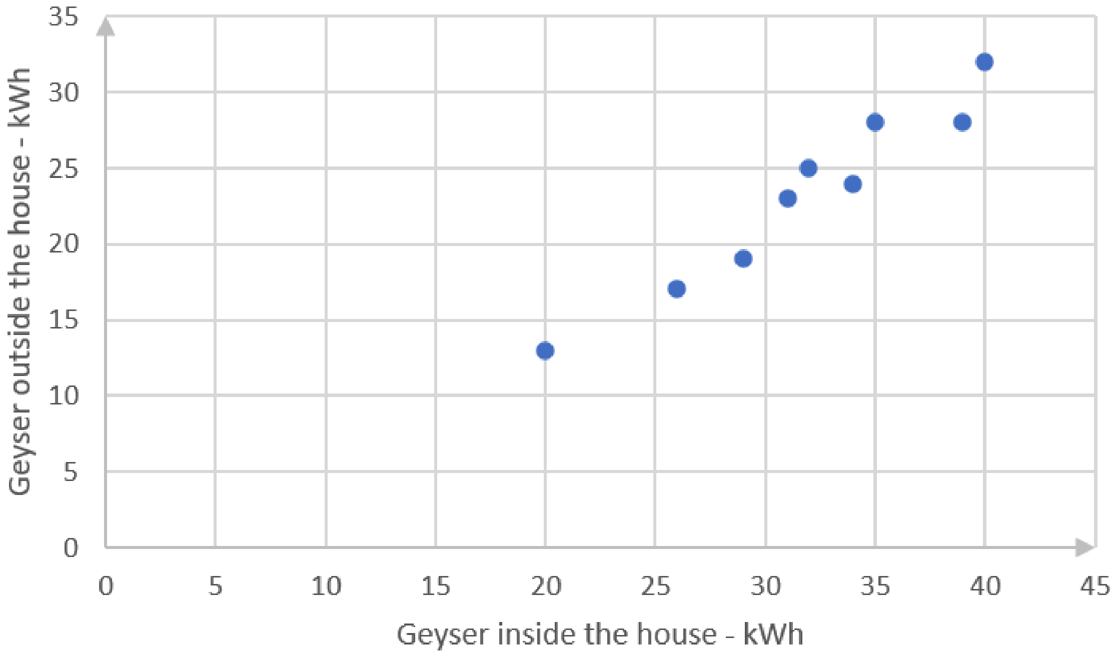
Back to Exercise 2.1

Unit 2: Assessment

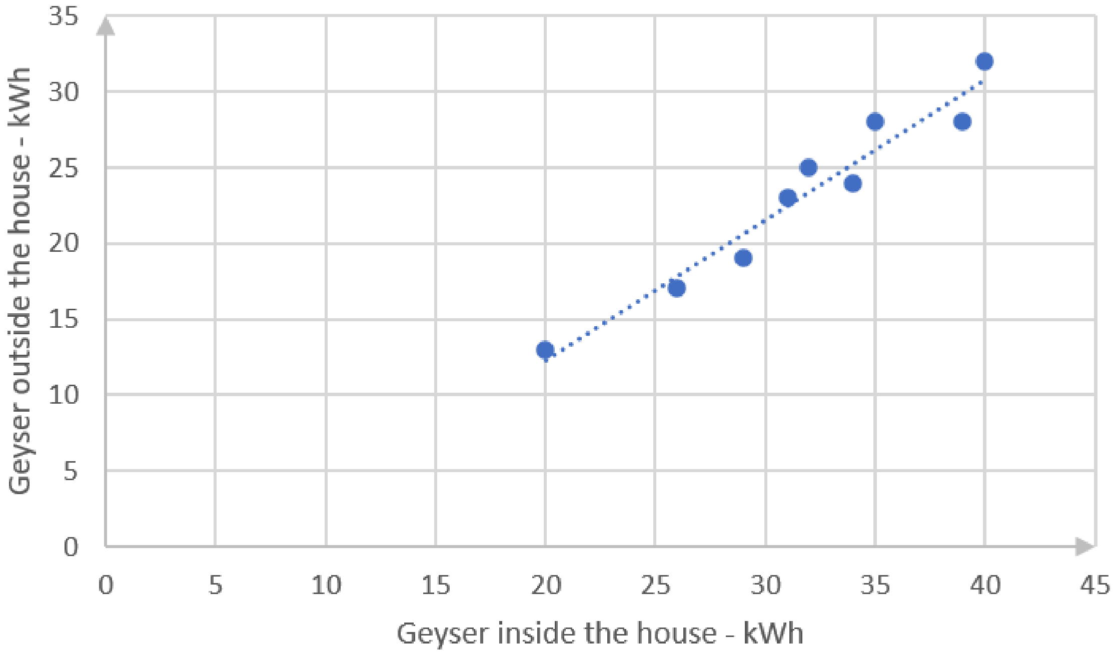
1.

Inside the house (kWh)	29	31	20	40	26	39	32	34	35
Outside the house (kWh)	19	23	13	32	17	28	25	24	28

a.



b.



(You might have drawn a slightly different straight line, in which case your equation would be different from that calculated in c. below. Check that your line complies with the guidelines in the notes before activity 2.1.)

c. The straight line passes through $(20, 13)$ and $(35, 26)$.

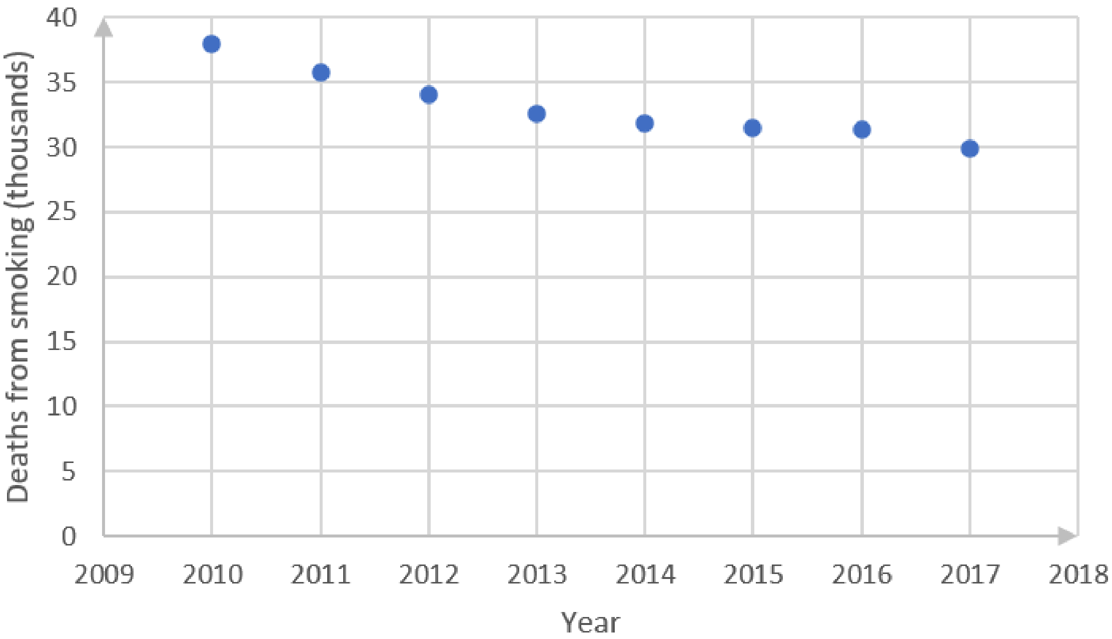
$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$
$$\frac{y - 13}{x - 20} = \frac{26 - 13}{35 - 20}$$
$$y - 13 = \frac{13}{15}(x - 20)$$
$$y = \frac{13x}{15} - \frac{52}{3} + \frac{39}{3}$$
$$y = \frac{13x}{15} - \frac{13}{3}$$

d. There is a strong, positive relationship between electricity usage when the geyser is inside the house and when it is outside.

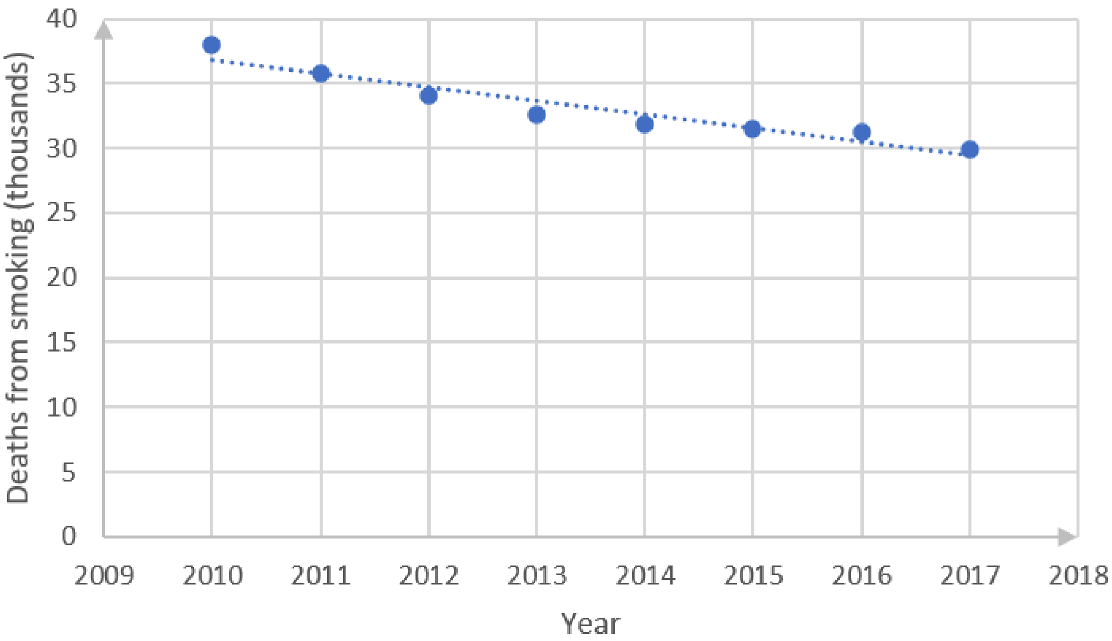
2.

Year	2010	2011	2012	2013	2014	2015	2016	2017
Deaths $('000)$	38.0	35.8	34.1	32.6	31.8	31.5	31.3	29.9

a.



b.



c. The straight line passes more or less through the points $(2016, 31)$ and $(2012, 34)$.
(You might have drawn a slightly different straight line, in which case your equation would be a bit different from that calculated below. Check that your line complies with the guidelines in the notes before activity 2.1.)
The equation of the line is the following:

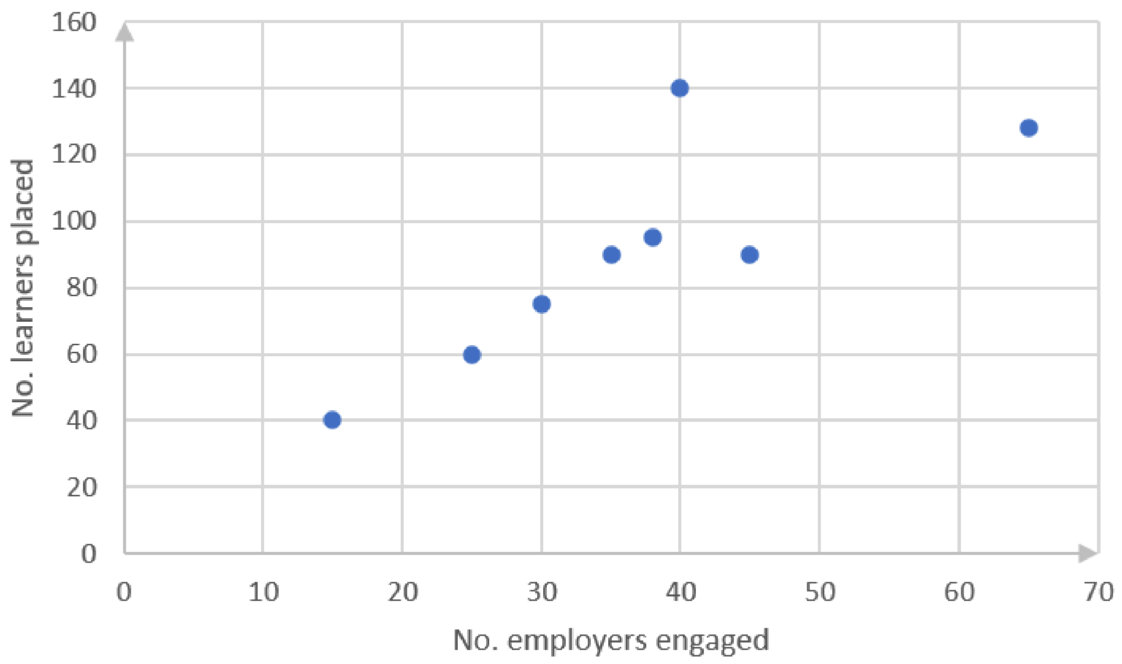
$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$
$$\frac{y - 31}{x - 2016} = \frac{34 - 31}{2012 - 2016}$$
$$y - 31 = -\frac{3}{4}(x - 2016)$$
$$y = -\frac{3x}{4} + 1512 + 31$$
$$y = -\frac{3x}{4} + 1543$$

d. The relationship between the year and the numbers of deaths from smoking is a strong negative one.

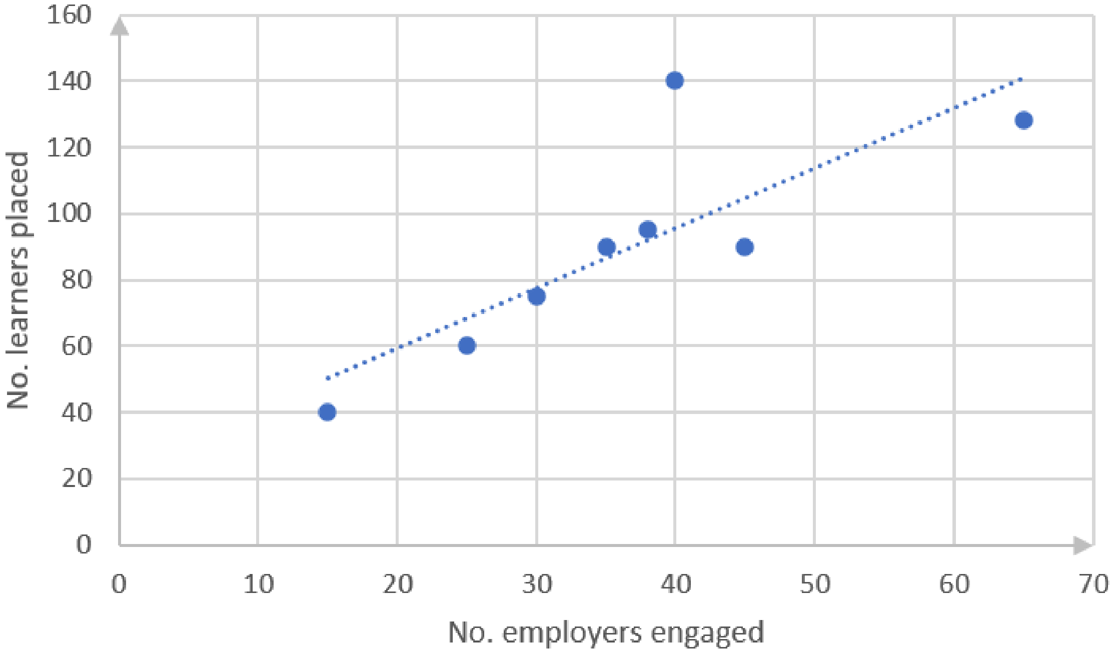
3.

No. employers engaged	15	45	65	35	38	25	40	30
No. learners placed	40	90	128	90	95	60	140	75

a.



b.



c. The straight line passes more or less through $(20, 60)$ and $(60, 132)$. (You might have drawn a slightly different straight line, in which case your equation would be a bit different from that calculated below. Check that your line complies with the guidelines in the notes before Activity 2.1.) The equation of the line is the following:

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\frac{y - 60}{x - 20} = \frac{132 - 60}{60 - 20}$$

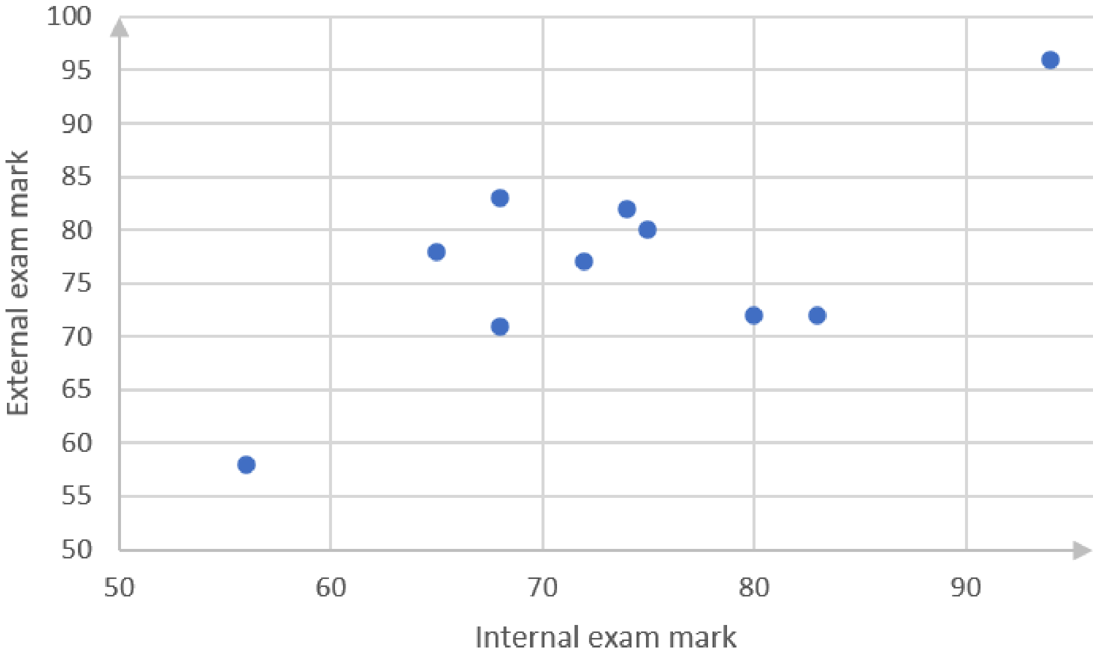
$$y - 60 = \frac{72}{40}(x - 20)$$

$$y = 1.8x - 24$$

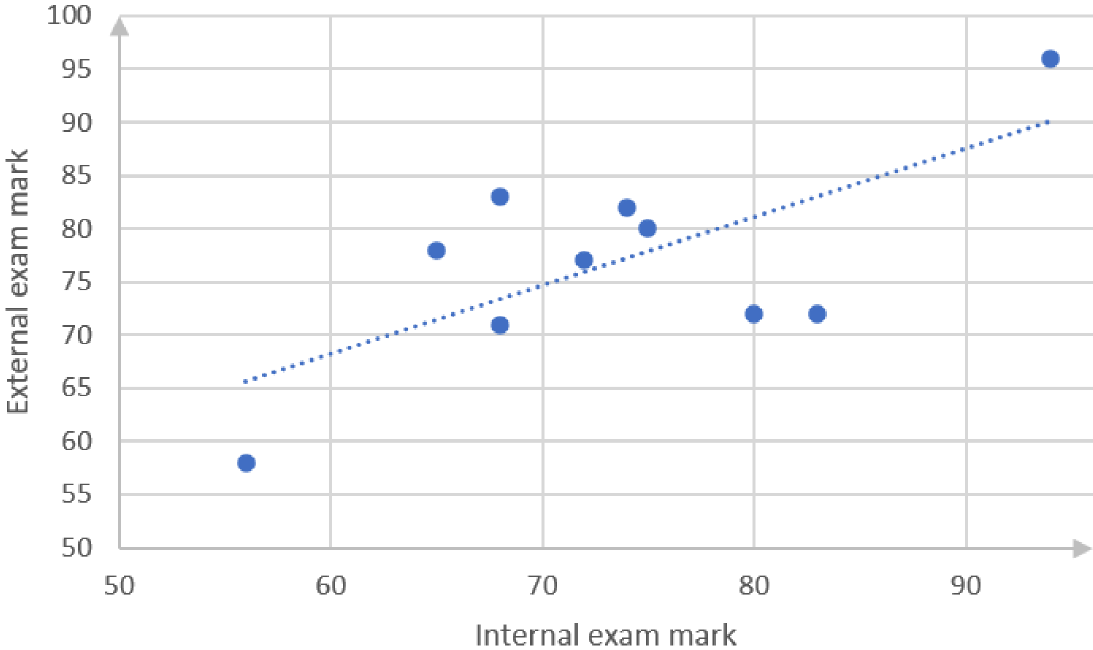
d. There is a positive relationship between numbers of employers engaged and numbers of learners placed, but this is not a very strong relationship.

4.

a.



b.



c. The straight line passes through points $(70, 75)$ and $(90, 87)$. (You might have drawn a slightly different straight line, in which case your equation would be a bit different from that calculated below. Check that your line complies with the guidelines in the notes before activity 2.1.)
The equation of the line is the following:

$$\frac{y-y_1}{x-x_1}=\frac{y_2-y_1}{x_2-x_1}$$
$$\frac{y-75}{x-70}=\frac{87-75}{90-70}$$
$$y-75=\frac{12}{20}(x-70)$$
$$y=\frac{3x}{5}-33$$

d. There is a weak, positive relationship between internal and external examinations.

Back to Unit 2: Assessment

Media Attributions

Unit 3: Linear regression analysis

GILL SCOTT

Unit outcomes

By the end of this unit you will be able to:

- Determine the linear regression equation $\hat{y} = a + bx$.
- Use the regression line to predict the outcome of a given problem.

What you should know

Before you start this unit, make sure you can:

- Calculate measures of central tendency of a data set, such as the mean, median and mode, and interpret what these tell you about a data set. To revise this, you can work through:
  - level 2 subject outcome 4.1 units 2 and 3
  - level 3 subject outcome 4.2 unit 1.
- Calculate and interpret the variance and standard deviation of a set of data. Refer to unit 1 of this subject outcome to revise this.
- Represent bivariate data as a scatter plot, and identify intuitively the best fit linear function to this data. You can refer to unit 2 of this subject outcome to revise this.

Introduction

Unit 2 of this subject outcome focused on drawing scatter plots of given bivariate data sets, and intuitively drawing lines of best fit between the two components of the plotted points. The obvious next step in this process is to use a more reliable system than intuition to find the lines of best fit. This unit explains the 'least squares regression' method of determining a straight-line function that best fits a given set of bivariate data.

Residuals

In example 2.1 from unit 2 of this subject outcome we drew an intuitive line of best fit of data measuring the number of sweets children ate per week against their average number of hours' sleep per day. The data and the scatter plot with best fit line are repeated below:

No. sweets per week	15	9	10	6	23	8	13	3
Average hours' sleep per day	4.5	5	6	7	3	5	4	8.5

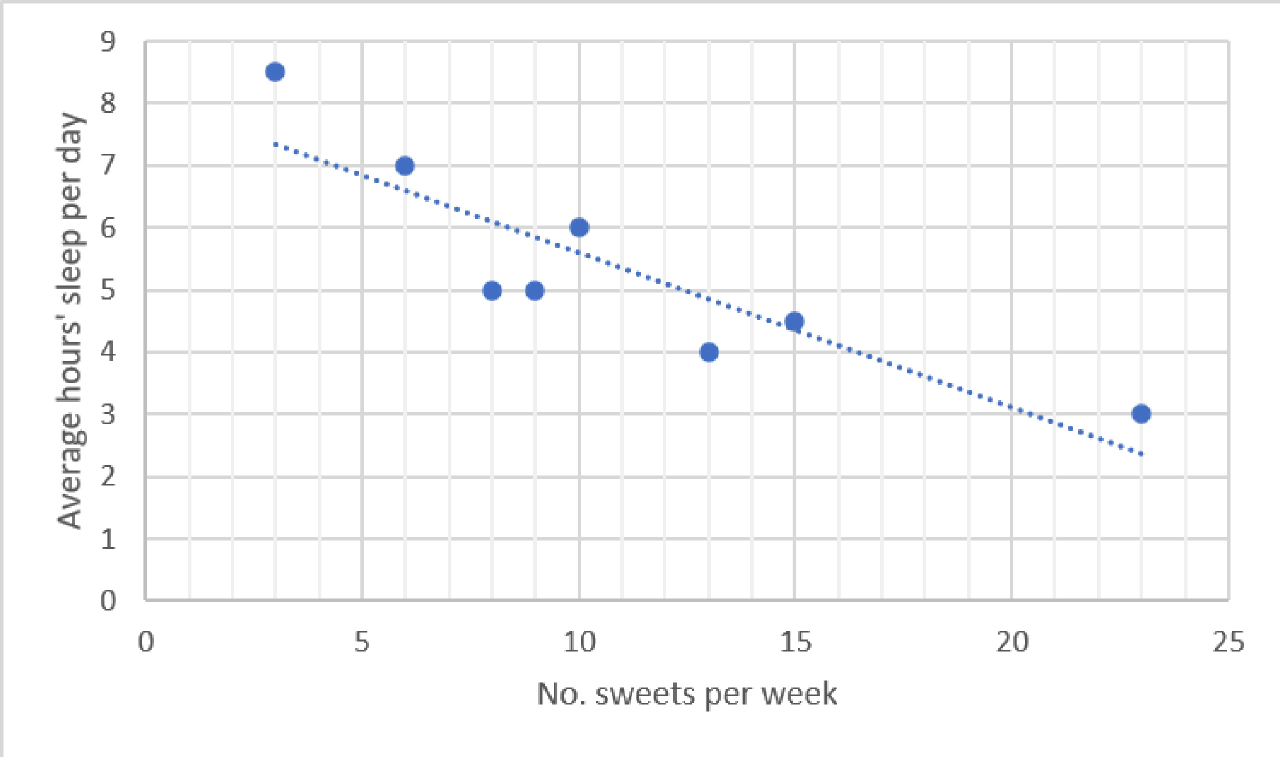


Figure 1: Scatter plot and best fit line from example 2.1 from unit 2 of this subject outcome

In this example, we intuitively found the line of best fit and we saw that the equation of the line would be different for different people, depending on which coordinates we chose to use. For mathematical accuracy and consistency, we need a method that will give one single equation of the line of best fit for a given dataset. This method is called linear regression.

The form of the linear regression equation preferred by statisticians for the best fit straight line, is $\hat{y} = a + bx$. We say 'y hat': statistics uses the 'hat' operator '^' to indicate that a value is an estimation. The intercept on the y-axis is a, and b is the slope of the line. This equation is a variant of the familiar $y = mx + c$ straight-line equation, with a instead of c, and b instead of m.

Where x is the independent variable, the best-fit straight line is the one where the sum of the deviations of y-distances from each given point to the line is a minimum. This distance, $y - \hat{y}$, is called the residual. The strategy of squaring, as used in calculating standard deviation, solves the problem of negative residuals cancelling out those that are positive.

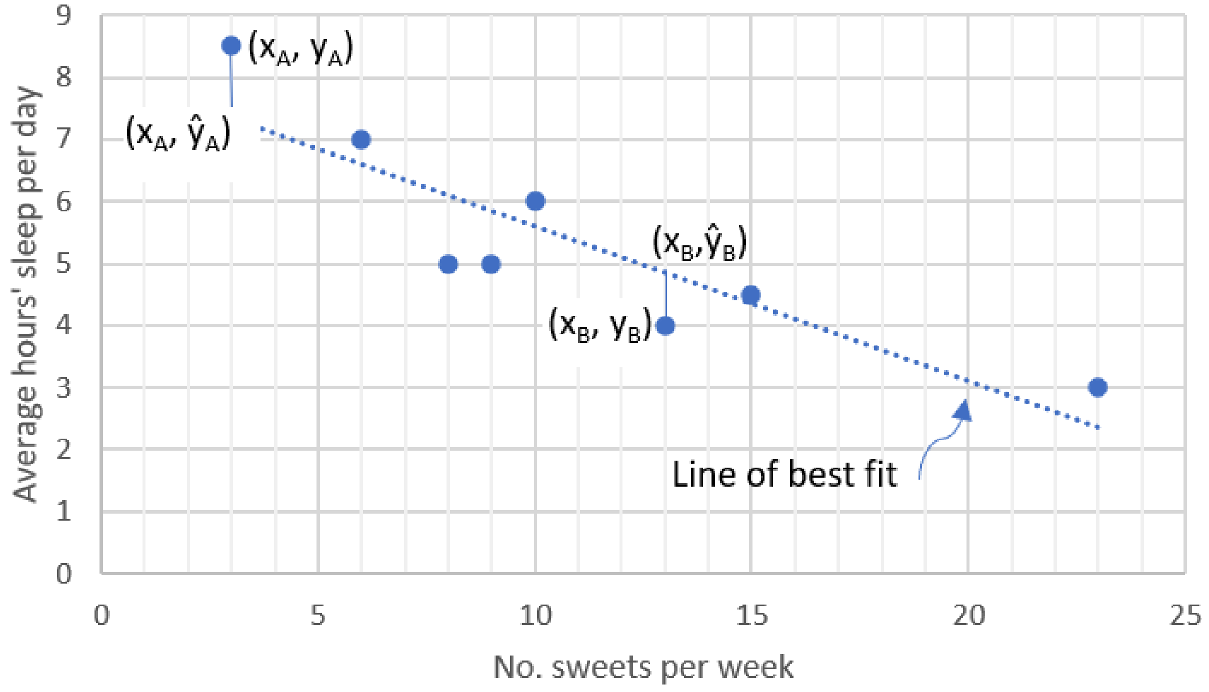


Figure 2: Scatter plot with estimated line of best fit from example 2.1 from unit 2 of this subject outcome

Activity 3.1: Calculating the distances from data points to $\hat{y}$, the line of best fit

Time required: 20 minutes

What you need:

- a pen
- paper (preferably graph paper)
- a ruler

What to do:

1. With $\hat{y}$ being the line of best fit (the dotted line in the graph above):
   a. Read off the $\hat{y}$ coordinate for each x value plotted from the given data set. Note it down in a copy of the table below.
   b. Measure as accurately as you can the vertical distance from each plotted data point to the best fit line. Note this in the $y - \hat{y}$ column with the appropriate sign, whether positive or negative.
2. Calculate the 'residual' $y - \hat{y}$ for each x value, noting it in the table.
3. Calculate the squared error for each given data point and the best fit straight line. Note these in the appropriate column.
4. Calculate the sum of the residuals and note it in the table.
5. Write down what the sum of residuals says about how well $\hat{y}$ fits to the data.

6. Calculate the sum of the squared error values, and note this in the table.
7. Discuss what the value of the squared error says about the goodness of fit of $\hat{y}$ to the data.

No. sweets per week x	Average hours' sleep per night y	$\hat{y}$	Residual $y - \hat{y}$	Squared error $(y - \hat{y})^2$
15	4.5			
9	5			
10	6			
6	7			
23	3			
8	5			
13	4			
3	8.5			
Sums				

What did you find?

Your $\hat{y}$ values may differ slightly from those inserted in the table below. You will probably find that it is difficult to be completely accurate.

Answers to questions 1. to 4. and 6. are in the table below.

No. sweets per week x	Average hours' sleep per night y	$\hat{y}$	Residual $y - \hat{y}$	Squared error $(y - \hat{y})^2$
15	4.5	4.4	0.1	0.01
9	5	5.8	-0.8	0.64
10	6	5.6	0.4	0.16
6	7	6.7	0.3	0.09
23	3	2.3	0.7	0.49
8	5	6.1	-1.1	1.21
13	4	4.9	-0.9	0.81
3	8.5	7.4	1.1	1.21
Sums			-0.2	4.62

5. The sum of residuals is not a good indicator of how well $\hat{y}$ fits to the data. Although the -0.2 sum is quite a small value, this could easily be the result of large positive and negative differences cancelling each other out.
6. The sum of squared error does relate to how well $\hat{y}$ fits to the data: the smaller this sum, the closer the line is to each of the data points. However, this value in itself does not seem to be helpful in the actual plotting of the line of best fit.

Finding the equation $\hat{y} = a + bx$ of a straight line of best fit

The formula for finding the gradient of the line of best fit of a set of bivariate data points uses the mean and standard deviation of all the $x\text{-values}$, and the mean and standard deviation of the given $y\text{-values}$.

Take note!

The derivation of the formulae used for finding the equation of the line is fairly complicated and is not required at this level.

Note

If you would like to investigate the derivation of the formulae, when you have access to the internet, watch the series of videos starting at "Introduction to residuals and least squares regression".

Introduction to residuals and least squares regression (Duration: 07.39)

You can also read "Introduction to residuals" by Khan Academy.

The equation $\hat{y} = a + bx$ of the straight line of best fit satisfies the following conditions:

- The formulae are based on the fact that x is the independent variable, and y is the dependent variable (this is an important requirement).
- The point $(\bar{x},\ \bar{y})$ lies on the line of best fit, where $\bar{x}$ is the mean of all the given $x\text{-values}$ in the dataset, and $\bar{y}$ is the mean of all the given $y\text{-values}$.
- The gradient: $b = \frac{\sum(x-\bar{x})(y-\bar{y})}{\sum(x-\bar{x})^2}$
  You will remember that the standard deviation also used the difference between data values and their means.
- The y-intercept: $a = \frac{\sum y - b\sum x}{n} = \bar{y} - b\bar{x}$ where b is the gradient of the best-fit line.
- The sum of residuals is 0.
- The mean of residuals is 0.
- The sum of squares of residuals is a minimum.

The first bullet point above is a requirement, and the next three bullets are the tools we use to find the equation of the line of best fit.

Example 3.1

This example further develops question 2 from exercise 2.1 of unit 2 in this subject outcome.
Example adapted from Siyavula Grade 12 Mathematics Exercise 9-2

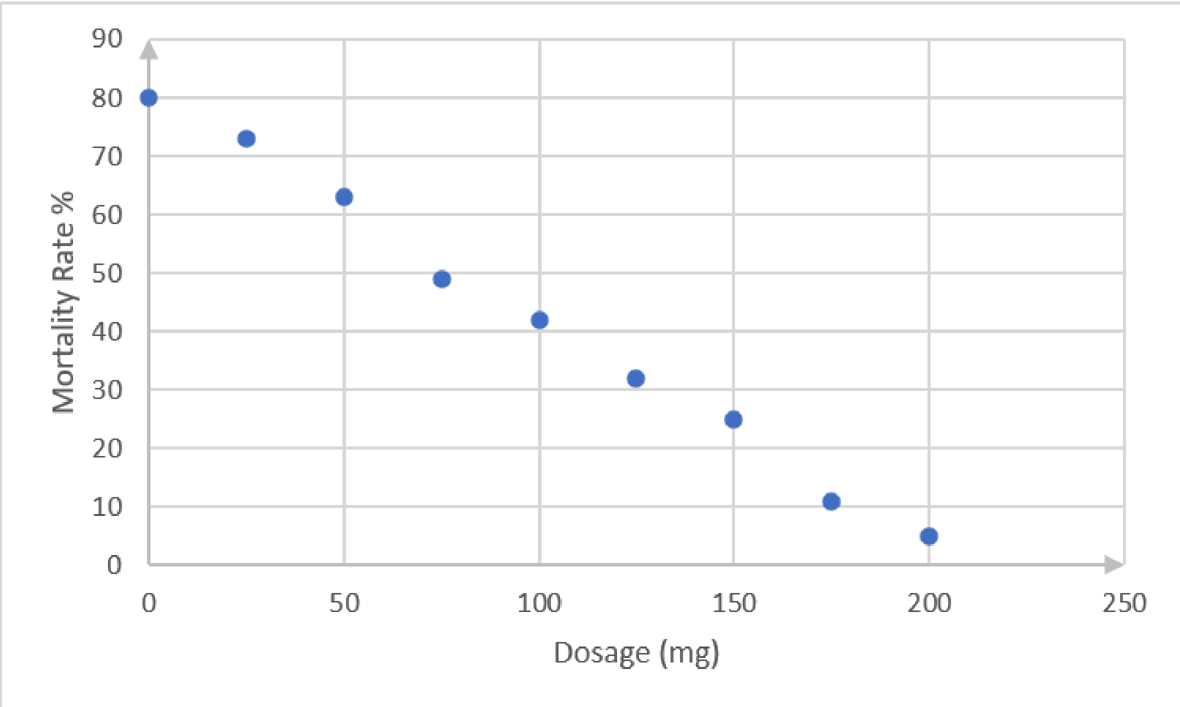
Dr Dandara is a scientist trying to find a cure for a disease which has an 80% mortality rate. That means that 80% of people who get the disease will die. He knows of a plant which is used in traditional medicine to treat the disease. He extracts the active ingredient from the plant and tests different dosages (measured in milligrams) on different groups of patients. Examine the data below and complete the questions that follow.

Dosage (mg)	0	25	50	75	100	125	150	175	200
Mortality rate (%)	80	73	63	49	42	32	25	11	5

1. Draw the scatter plot of the data on graph paper.
2. Use the formulae for b and a to find the regression equation for the line of best fit.
3. Draw the line of best fit on the scatter plot. Compare this line to the best fit line you drew in unit 2.
4. Use the equation of the line of best fit to estimate the dosage required for a 0% mortality rate.

Solution

1.



2. It is best to tabulate the data in these questions to be able to check your answers. Also, the Assessment Guidelines require that these values should be calculated ‘manually’ – that is without using your calculator’s statistical function. Examination questions also sometimes stipulate that calculator statistical functions should not be used. For this reason, this approach has been

adopted in this unit.

	Dosage (mg) x	Mortality rate (%) y	$x-\bar{x}$	$y-\bar{y}$	$(x-\bar{x})(y-\bar{y})$	$(x-\bar{x})^2$
	0	80	-100	37.8	$-3\ 777.8$	$10\ 000$
	25	73	-75	30.8	$-2\ 308.3$	$5\ 625$
	50	63	-50	20.8	$-1\ 038.9$	$2\ 500$
	75	49	-25	6.8	-169.4	625
	100	42	0	-0.2	0	0
	125	32	25	-10.2	-255.6	626
	150	25	50	-17.2	-861.1	$2\ 500$
	175	11	75	-31.2	$-2\ 341.7$	$5\ 625$
	200	5	100	-37.2	$-3\ 722.2$	$10\ 000$
Sums	900	380			$-14\ 475$	$37\ 500$
Mean	100	42.2				

Gradient:

$$\begin{aligned} b &= \frac{\sum (x-\bar{x})(y-\bar{y})}{\sum (x-\bar{x})^2} \\ &= \frac{-14\ 475}{37\ 500} \\ &= -0.386 \end{aligned}$$

y-intercept:

$$\begin{aligned} a &= \frac{\sum y - b\sum x}{n} = \bar{y} - b\bar{x} \\ &= 42.2 - (-0.386)(100) \\ &= 42.2 + 38.6 \\ &= 80.8 \end{aligned}$$

Regression equation: substituting for b and a in $\hat{y} = a + bx$:

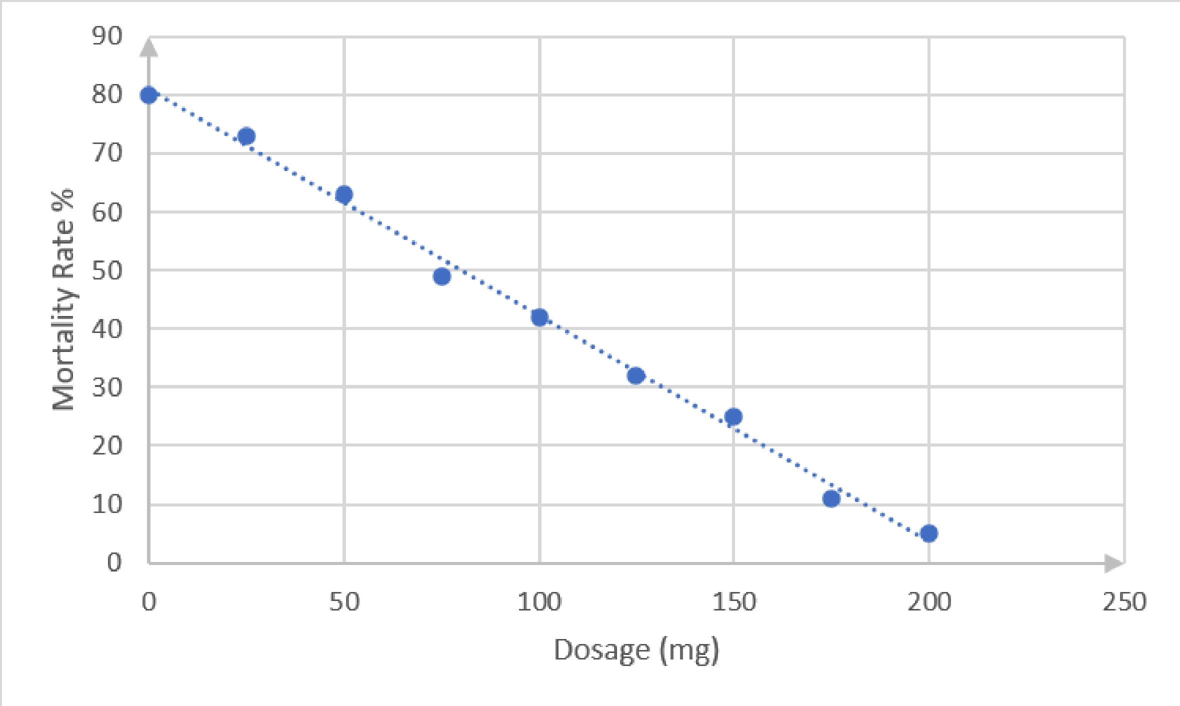
$\hat{y} = a + bx$

$\hat{y} = 80.8 - 0.386x$

Check that the point $(\bar{x},\ \bar{y}) = (100, 42.2)$ lies on the line.

Substituting $x = 100$ into the regression equation gives $\hat{y} = 80.8 - 0.386x = 80.8 - 38.6 = 42.2$, so this point lies on the line.

3.



4. To find the dosage for a 0% mortality rate, substitute $\hat{y} = 0$ in the regression equation, therefore:

$$\hat{y} = 80.8 - 0.386x$$
$$0.386x = 80.8$$
$$x = 209.33$$

Exercise 3.1

Question 1 follows on from exercise 2.1 number 3 of unit 2 in this subject outcome.

1. The enrolment of learners for NC(V) programmes at TVET colleges was reported as follows:

Year	2010	2011	2012	2013	2014	2015	2016
NC(V) enrolment (in thousands)	130	124	140	154	166	165	177

   a. Draw a scatter plot of this data with the year on the horizontal axis and the enrolment on the vertical axis.
   b. Use the formulae for b and a to find the regression equation for the line of best fit.
   c. Draw the line of best fit on the scatter plot. Compare this line to the best fit line you drew in unit 2, exercise 2.1 question 3.
   d. According to the equation, what enrolment could be expected in the year 2022?

Question 2 taken from Siyavula Grade 12 Mathematics Exercise 9-3

2. For each of the following data sets:

a.

x	10	4	9	11	11	6	8	18
y	1	0	6	3	9	5	9	8

b.

x	8	12	12	7	6	14	8	14
y	-5	4	3	-3	-5	-6	-2	0

c.

x	1.9	1.1	-1.5	1.3	0.95	8.25	10.6	6.2
y	7	8.45	0.9	0.1	2.45	4.35	2.2	1.4

i. Draw a scatter plot of the data.
ii. Use a table to determine the values of b and a in order to find the least squares regression equation for each line of best fit. Round b and a off to two decimal places where necessary.
iii. Draw the line of best fit on the scatter plot.
iv. Use your equation in each case to predict the value of y when $x = 25$.

Question 3 follows on from question 3 of unit 2 assessment in this subject outcome

3. A college assists learners to complete their national diplomas by negotiating with employers in the region with the aim of placing the learners for work experience. Over recent years they have tracked their engagements with employers against numbers of learners placed in work experience as follows:

No. employers engaged	15	45	65	35	38	25	40	30
No. learners placed	40	90	128	90	95	60	140	75

a. Draw a scatter plot of the data.
b. Use a table to determine the values of b and a in order to find the least squares regression equation for the line of best fit. Round b and a off to two decimal places where necessary.
c. Draw the line of best fit on the scatter plot. Compare this line to the best fit line you drew in unit 2.
d. Use the equation of the line of best fit to estimate the number of employers that would need to be engaged in order to place 175 learners in work placements.

The full solutions are at the end of the unit.

Summary

In this unit you have learnt the following:

- How to determine the linear regression equation $\hat{y} = a + bx$ for a set of bivariate data.
- How to use the regression line to predict the outcome of a given problem.

Unit 3: Assessment

Suggested time to complete: 55 minutes

See question 1 of unit 2 assessment; question adapted from NC(V) Mathematics Level 4 examination, November 2017

1. A study was done to compare electricity usage of geysers that are inside or outside the house. The table below shows the electricity usage (in kilowatt hours) for equivalent water consumption for matched households that have geysers inside the house, and those with geysers outside the house. Nine houses of each type were considered in the study.

Inside the house (kWh)	29	31	20	40	26	39	32	34	35
Outside the house (kWh)	19	23	13	32	17	28	25	24	28

   a. Draw a scatter plot of the data.
   b. Using the information above, find the sample regression equation using the method of least squares.
   c. If the geyser fitted outside the house uses 40kWh, what will the usage be with the geyser inside the house?

Question 2 taken from NC(V) Mathematics Level 4 examination, November 2015

2. ATA Consultants is a company that offers tuition for learners from grade 8 to grade 12. For the past 5 years they have distributed flyers to learners and have enrolled numbers of learners according to the table given below.

Number of flyers distributed (x)	Number of learners enrolled (y)
50	15
250	45
200	40
350	65
150	35

   a. Draw a scatter plot showing on the x-axis the number of flyers distributed and on the y-axis the number of learners enrolled.
   b. Using the information above find the simple regression equation by the method of least squares.
   c. Use the regression equation to determine the number of learners that would be enrolled if 500 flyers were sent out.

Question 3 taken from NC(V) Mathematics Level 4 examination, November 2019

3. The data below shows the mathematics marks of ten learners at a college for the internal examinations and the external examinations.

Internal examinations (x)	80	68	94	72	74	83	56	68	65	75
External examinations (x)	72	71	96	77	82	72	58	83	78	80

a. Make a scatter plot of the marks in the above table on an x-y plane, with each axis showing values from 50 to 96.

b. Calculate the equation of the least squares regression line for the data. No marks will be awarded if answers are taken directly from a calculator. Complete all the calculations in a table that shows EITHER (x) (y) xy x2
OR (x) (y) $(x-\bar{x})(y-\bar{y})$ $(x-\bar{x})^2$

c. Draw the least squares regression line on the x-y plane.

d. Calculate the predicted final examination mark for a learner who scores 70 in the internal examination.

See unit 2 assessment question 2

4. Tobacco smoking is still one of the world's largest health problems, although prevalence of smoking is generally decreasing. The table below shows numbers of deaths (in thousands) in South Africa from smoking, in recent years.

Year	2010	2011	2012	2013	2014	2015	2016	2017
Deaths $('000)$	38.0	35.8	34.1	32.6	31.8	31.5	31.3	29.9

a. Draw a scatter plot of the data.

b. Calculate the equation of the least squares regression line for the data.

c. Draw the least squares regression line into the graph of the scatter plot.

d. Draw the line of best fit on the scatter plot. Compare this line to the best fit line you drew in unit 2.

e. Use the regression equation to calculate the number of deaths from smoking predicted for 2022.

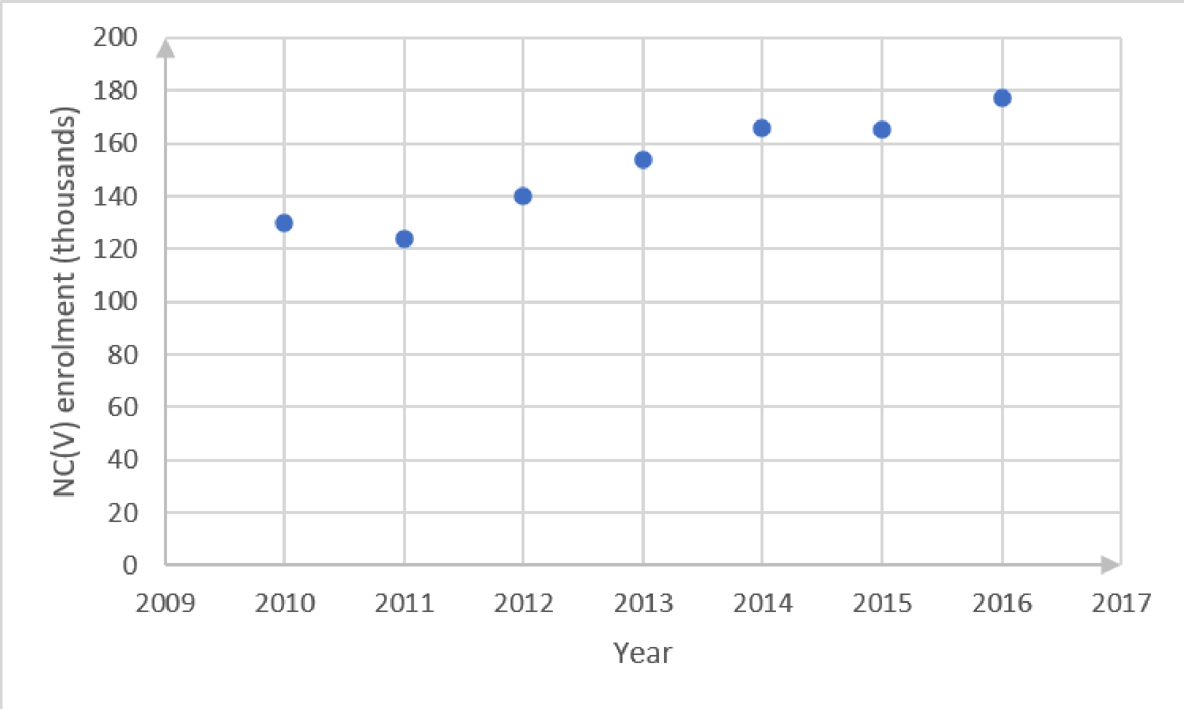
The full solutions are at the end of the unit.

Unit 3: Solutions

Exercise 3.1

1.

a.



b.

	Year x	Enrolment ('000)y	$x-\bar{x}$	$y-\bar{y}$	$(x-\bar{x})(y-\bar{y})$	$(x-\bar{x})^2$
	2010	130	-3	-20.86	62.58	9
	2011	124	-2	-26.86	53.72	4
	2012	140	-1	-10.86	10.86	1
	2013	154	0	3.14	0	0
	2014	166	1	15.14	15.14	1
	2015	165	2	14.14	28.28	4
	2016	177	3	26.14	78.42	9
Sums	14 091	1 056			249	28
Mean	2 013	150.86				

Gradient:

$$b = \frac{\sum(x-\bar{x})(y-\bar{y})}{\sum(x-\bar{x})^2}$$
$$= \frac{249}{28}$$
$$= 8.89$$

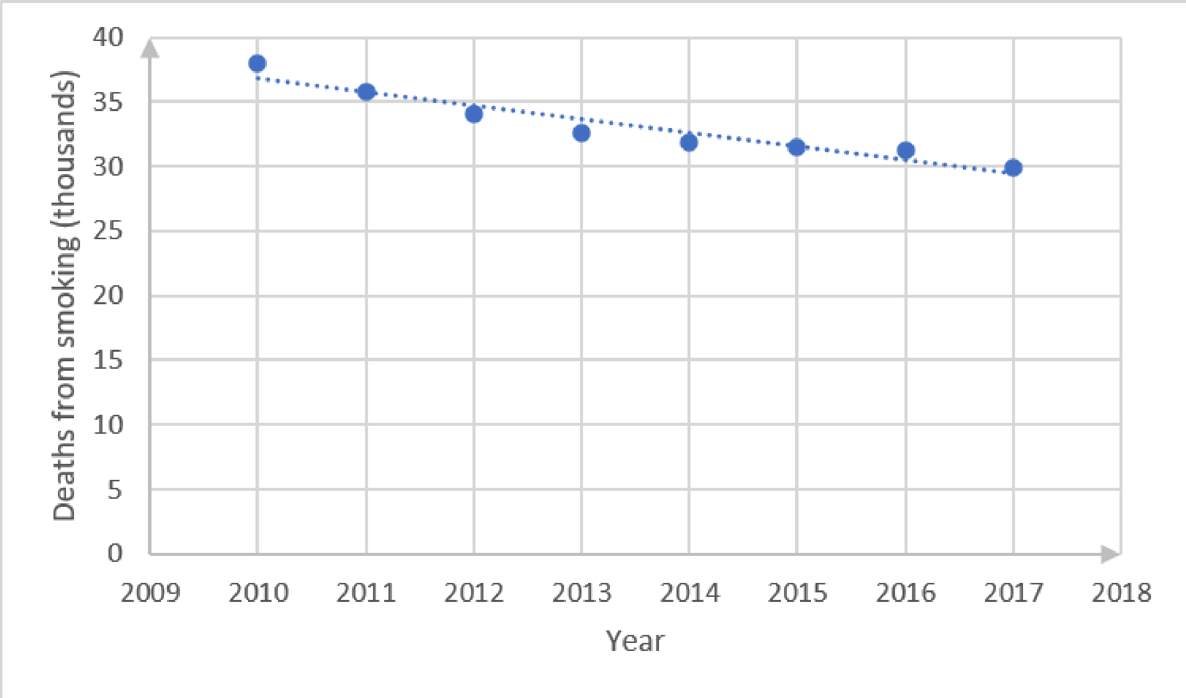
y-intercept:

$$a = \bar{y} - b\bar{x}$$
$$= 150.86 - (8.89)(2013)$$
$$= 150.86 - 17\ 895.57$$
$$= -17\ 744.71$$

Regression equation: substituting for b and a in $\hat{y} = a + bx$:

$$\hat{y} = a + bx$$
$$\hat{y} = -17\ 744.71 + 8.89x$$

c.



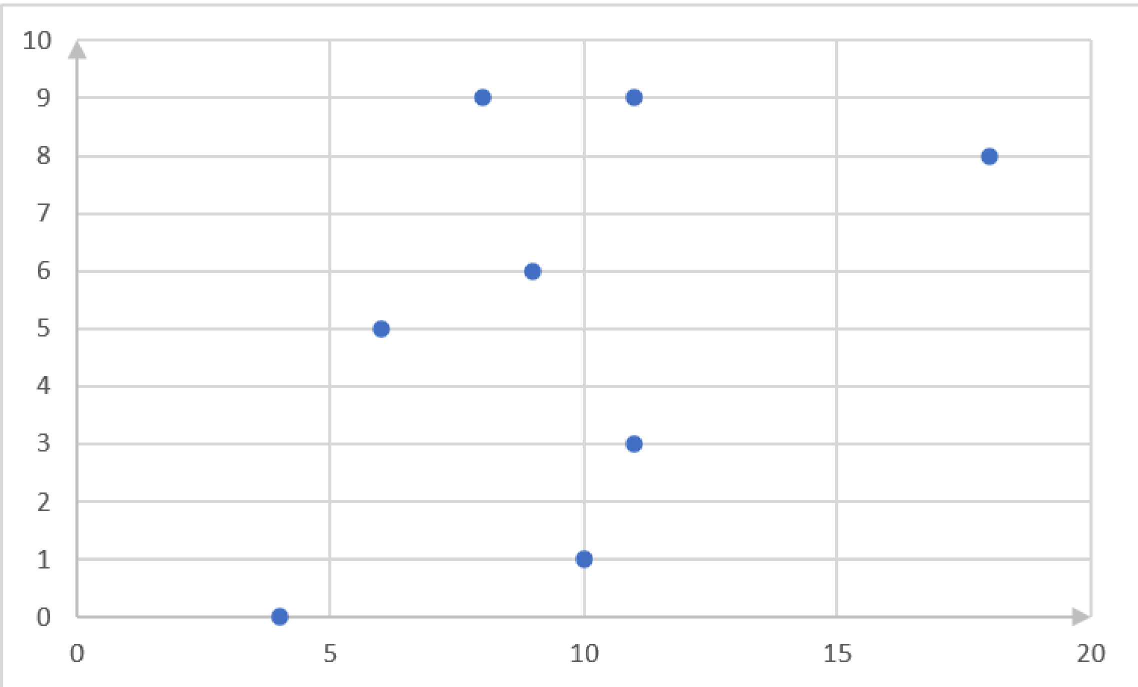
d. To find enrolment expected in 2022, substitute this value into the regression equation:
$\hat{y} = -17\ 744.71 + 8.89x$
$\hat{y} = -17\ 744.71 + 8.89(2022)$
$= -17\ 744.71 + 17\ 975.58$
$= 230.87$
So, enrolment in 2022 is expected to be 230.87 thousands, or $230\ 870$ learners.

2.

a.

x	10	4	9	11	11	6	8	18
y	1	0	6	3	9	5	9	8

i.



ii.

	x	y	$x-\bar{x}$	$y-\bar{y}$	$(x-\bar{x})(y-\bar{y})$	$(x-\bar{x})^2$
	10	1	0.37	-4.13	-1.53	0.14
	4	0	-5.63	-5.13	28.88	31.70
	9	6	-0.63	0.87	-0.55	0.40
	11	3	1.37	-2.13	-2.92	1.88
	11	9	1.37	3.87	5.30	1.88
	6	5	-3.63	-0.13	0.47	13.18
	8	9	-1.63	3.87	-6.31	2.66
	18	8	8.37	2.87	24.02	70.06
Sums	77	41			50.41	121.9
Mean	9.63	5.13				

Gradient:

$$b = \frac{\sum(x-\bar{x})(y-\bar{y})}{\sum(x-\bar{x})^2}$$
$$= \frac{50.41}{121.9}$$
$$= 0.41$$

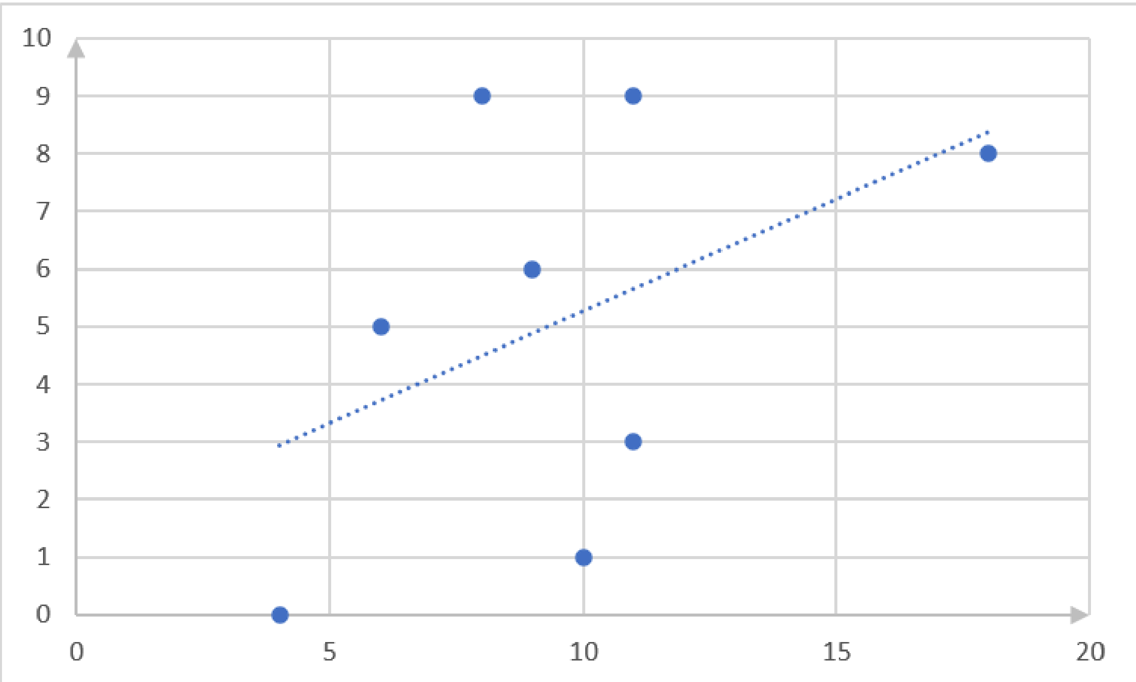
y-intercept:

$$a = \bar{y} - b\bar{x}$$
$$= 5.13 - (0.13)(9.63)$$
$$= 5.13 - 1.25$$
$$= 3.88$$

Regression equation: substituting for b and a in $\hat{y} = a + bx$:

$$\hat{y} = a + bx$$
$$\hat{y} = 3.88 + 0.41x$$

iii.



iv. To find y-value when $x = 25$, substitute this value into the regression equation:

$\hat{y} = 3.88 + 0.41x$

$\hat{y} = 3.88 + 0.41(25)$

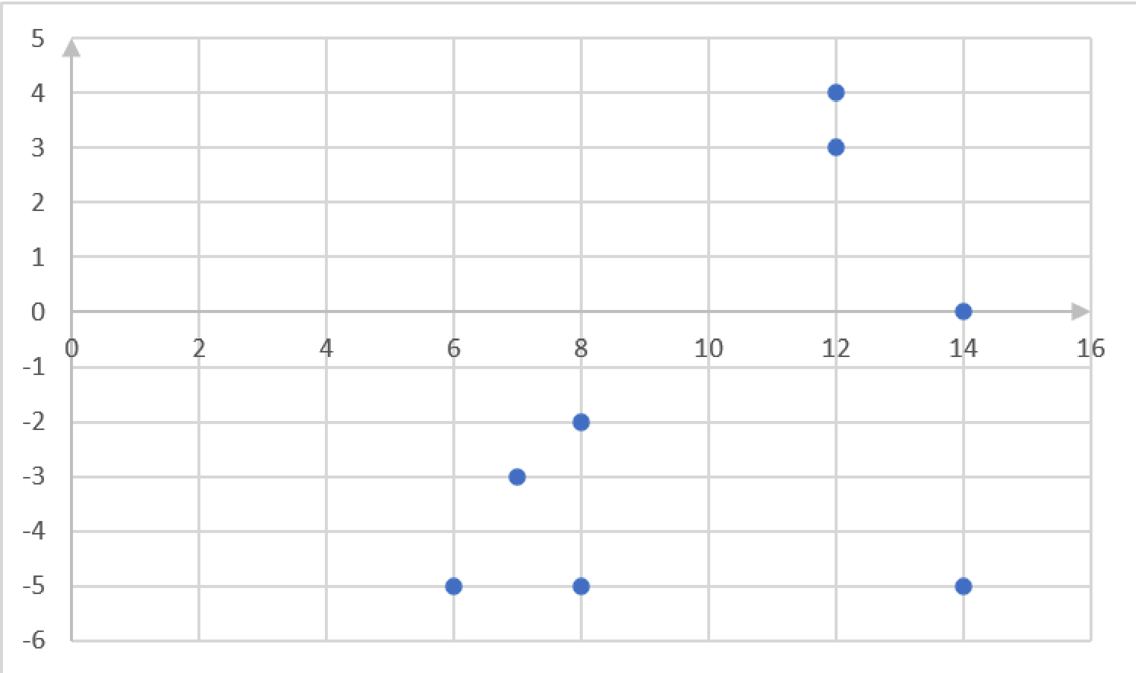
$= 3.88 + 10.25$

$= 14.13$

b.

x	8	12	12	7	6	14	8	14
y	-5	4	3	-3	-5	-6	-2	0

i.



ii.

	x	y	$x - \bar{x}$	$y - \bar{y}$	$(x - \bar{x})(y - \bar{y})$	$(x - \bar{x})^2$
	8	-5	-2.13	-3.25	6.92	4.54
	12	4	1.87	2.25	4.21	3.50
	12	3	1.87	4.75	8.88	3.50
	7	-3	-3.13	-1.25	3.91	9.80
	6	-5	-4.13	-4.5	18.59	17.06
	14	-6	3.87	-4.25	-16.45	14.98
	8	-2	-2.13	-0.25	0.53	4.54
	14	0	3.87	1.75	6.77	14.98
Sums	81	-14			33.36	72.9
Mean	10.13	-1.75				

Gradient:

$$b = \frac{\sum(x-\bar{x})(y-\bar{y})}{\sum(x-\bar{x})^2}$$
$$= \frac{33.36}{72.9}$$
$$= 0.46$$

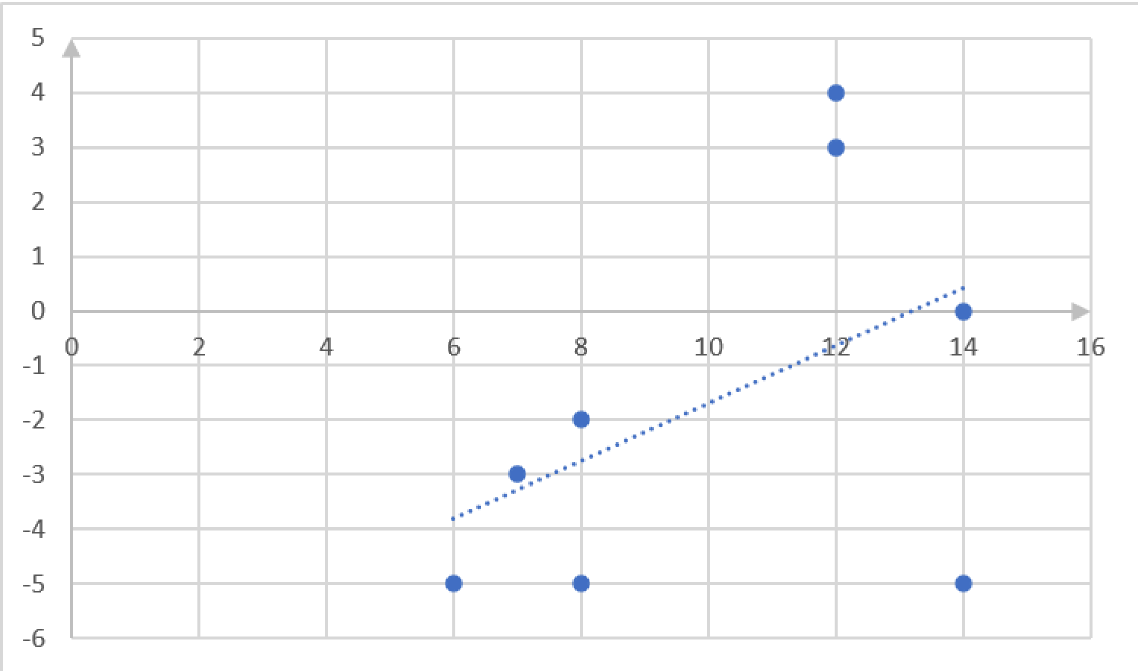
y-intercept:

$$a = \bar{y} - b\bar{x}$$
$$= -1.75 - (0.46)(10.13)$$
$$= -1.75 - 4.66$$
$$= -6.41$$

Regression equation: substituting for b and a in $\hat{y} = a + bx$:

$$\hat{y} = a + bx$$
$$\hat{y} = -6.41 + 0.46x$$

iii.



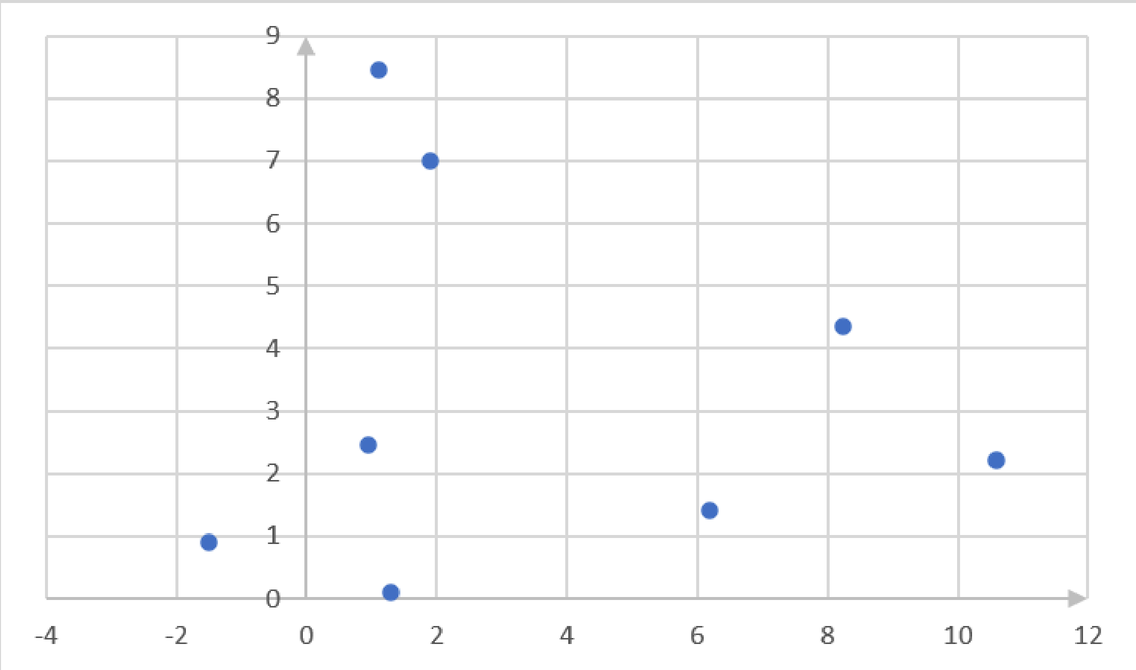
iv. To find y-value when $x = 25$, substitute this value into the regression equation:

$$\hat{y} = -6.41 + 0.46x$$
$$\hat{y} = -6.41 + 0.46(25)$$
$$= -6.41 + 11.5$$
$$= 5.09$$

c.

x	1.9	1.1	−1.5	1.3	0.95	8.25	10.6	6.2
y	7	8.45	0.9	0.1	2.45	4.35	2.2	1.4

i.



ii.

	x	y	$x-\bar{x}$	$y-\bar{y}$	$(x-\bar{x})(y-\bar{y})$	$(x-\bar{x})^2$
	1.9	7	-1.7	3.64	-6.19	2.89
	1.1	8.45	-2.5	5.09	-12.73	6.25
	-1.5	0.9	-5.1	-2.46	12.55	26.01
	1.3	0.1	-2.3	-3.26	7.50	5.29
	0.95	2.45	-2.65	-0.91	2.41	7.02
	8.25	4.35	4.65	0.99	4.60	21.62
	10.6	2.2	7	-1.16	-8.12	49
	6.2	1.4	2.6	-1.96	-5.10	6.76
Sums	28.8	26.85			-5.08	124.84
Mean	3.6	3.36				

Gradient:

$$b = \frac{\sum(x-\bar{x})(y-\bar{y})}{\sum(x-\bar{x})^2}$$
$$= \frac{-5.08}{124.84}$$
$$= -0.04$$

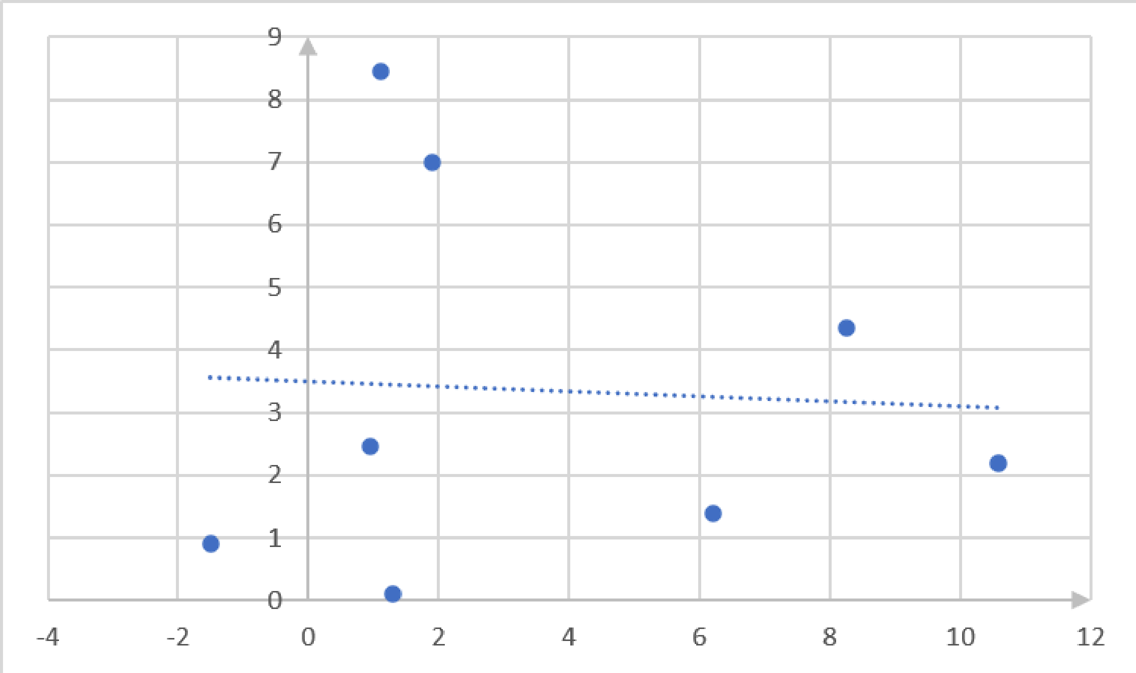
y-intercept:

$$a = \bar{y} - b\bar{x}$$
$$= 3.36 - (-0.04)(3.6)$$
$$= 3.36 + 0.144$$
$$= 3.50$$

Regression equation: substituting for b and a in $\hat{y} = a + bx$:

$$\hat{y} = a + bx$$
$$\hat{y} = 3.5 - 0.04x$$

iii.



iv. To find y-value when $x = 25$, substitute this value into the regression equation:

$\hat{y} = 3.5 - 0.04x$

$\hat{y} = 3.5 - 0.04(25)$

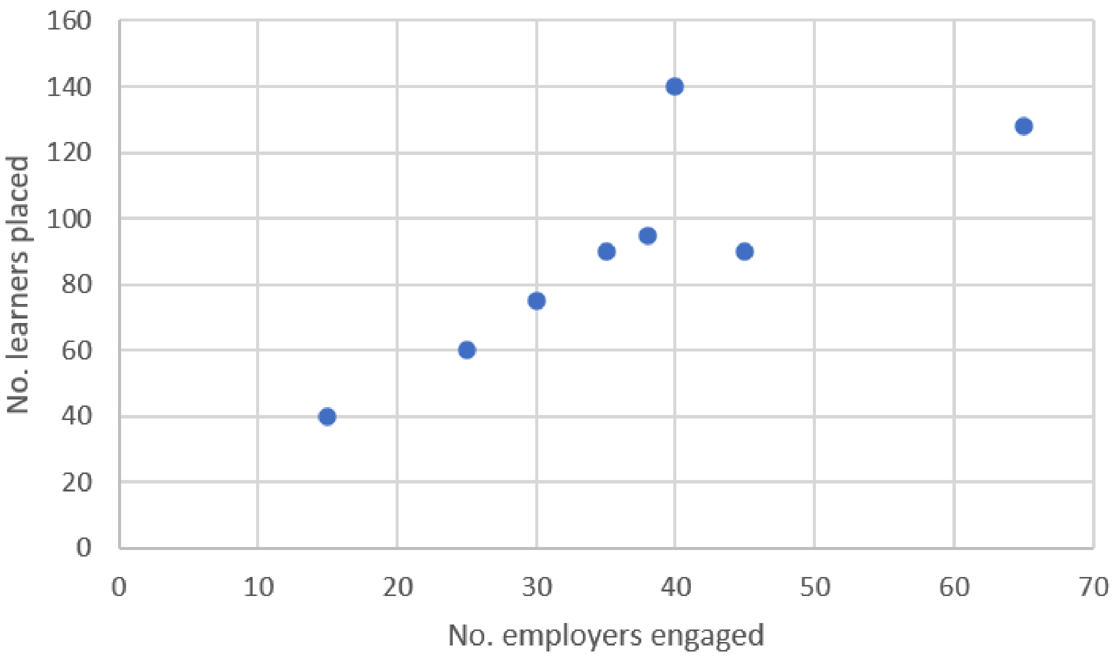
$= 3.5 - 1$

$= 2.5$

3.

No. employers engaged	15	45	65	35	38	25	40	30
No. learners placed	40	90	128	90	95	60	140	75

a.



b.

	No. employers engaged x	No. learners placed y	$x-\bar{x}$	$y-\bar{y}$	$(x-\bar{x})(y-\bar{y})$	$(x-\bar{x})^2$
	15	40	-21.63	-49.75	1 076.09	467.86
	45	90	8.37	0.25	2.09	70.06
	65	128	28.37	38.25	1 085.15	804.86
	35	90	-1.63	0.25	-0.41	2.66
	38	95	1.37	5.25	7.19	1.88
	25	60	-11.63	-29.75	345.99	135.26
	40	140	3.37	50.25	169.34	11.36
	30	75	-6.63	-14.75	97.79	43.96
Sums	293	718			2 783.23	1 537.9
Mean	36.63	89.75				

Gradient:

$$b = \frac{\sum(x-\bar{x})(y-\bar{y})}{\sum(x-\bar{x})^2}$$
$$= \frac{2\ 783.23}{1\ 537.9}$$
$$= 1.81$$

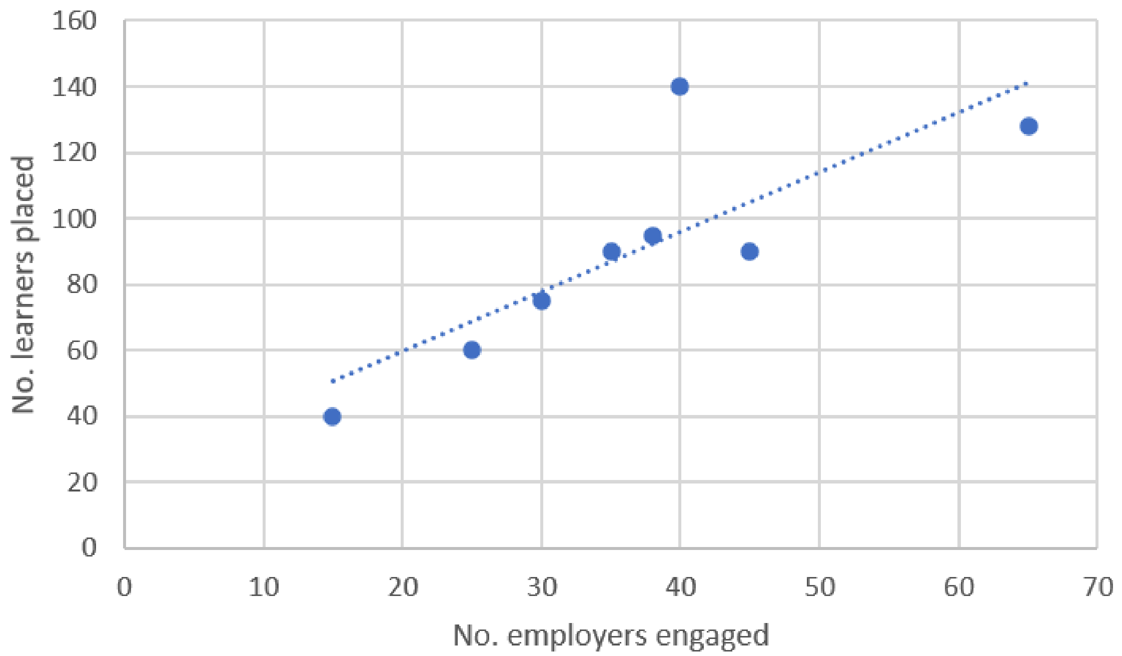
y-intercept:

$$a = \bar{y} - b\bar{x}$$
$$= 89.75 - (1.81)(36.63)$$
$$= 89.75 - 66.30$$
$$= 23.45$$

Regression equation: substituting for b and a in $\hat{y} = a + bx$:

$$\hat{y} = a + bx$$
$$\hat{y} = 23.45 + 1.81x$$

c.



d. To find x-value when $y = 175$, substitute this value into the regression equation:

$$\begin{aligned} \hat{y} &= 23.45 + 1.81x \\ 175 &= 23.45 + 1.81x \\ 1.81x &= 175 - 23.45 \\ x &= \frac{151.55}{1.81} \\ &= 83.73 \end{aligned}$$

At the current rate, approximately 84 employers would need to be engaged in order to find work placements for 175 learners.

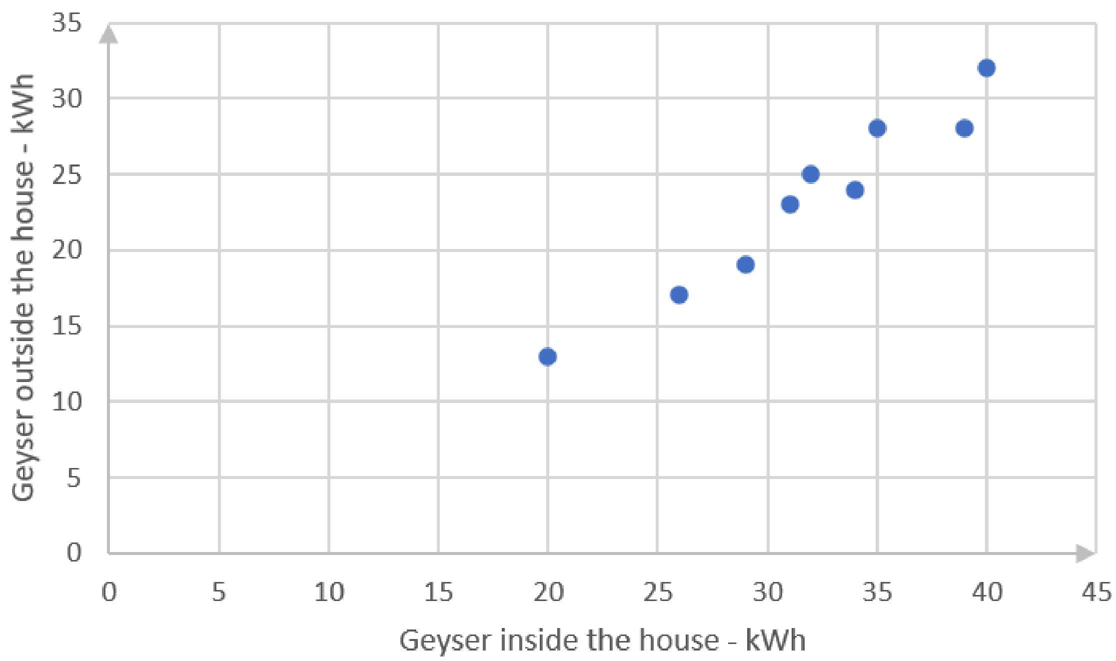
Back to Exercise 3.1

Unit 3: Assessment

1.

Inside the house (kWh)	29	31	20	40	26	39	32	34	35
Outside the house (kWh)	19	23	13	32	17	28	25	24	28

a.



b.

	Inside the house x	Outside the house y	$x-\bar{x}$	$y-\bar{y}$	$(x-\bar{x})(y-\bar{y})$	$(x-\bar{x})^2$
	29	19	-2.78	-4.22	11.73	7.73
	31	23	-0.78	-0.22	0.17	0.61
	20	13	-11.78	-10.22	120.39	138.77
	40	32	8.22	8.78	72.17	67.57
	26	17	-5.78	-6.22	35.95	33.41
	39	28	7.22	4.78	34.51	52.13
	32	25	0.22	1.78	0.39	0.05
	34	24	2.22	0.78	1.73	4.93
	35	28	3.22	4.78	15.39	10.37
Sums	286	209	-0.02	0.02	292.43	315.57
Mean	31.78	23.22				

Gradient:

$$b = \frac{\sum(x-\bar{x})(y-\bar{y})}{\sum(x-\bar{x})^2}$$
$$= \frac{292.43}{315.57}$$
$$= 0.93$$

y-intercept:

$$a = \bar{y} - b\bar{x}$$
$$= 23.22 - (0.93)(31.78)$$
$$= 23.22 - 29.56$$
$$= -6.34$$

Regression equation: substituting for b and a in $\hat{y} = a + bx$:

$$\hat{y} = a + bx$$
$$\hat{y} = -6.34 + 0.93x$$

c. To find the electricity usage inside the house when the outside usage is 40kWh, substitute into the equation:

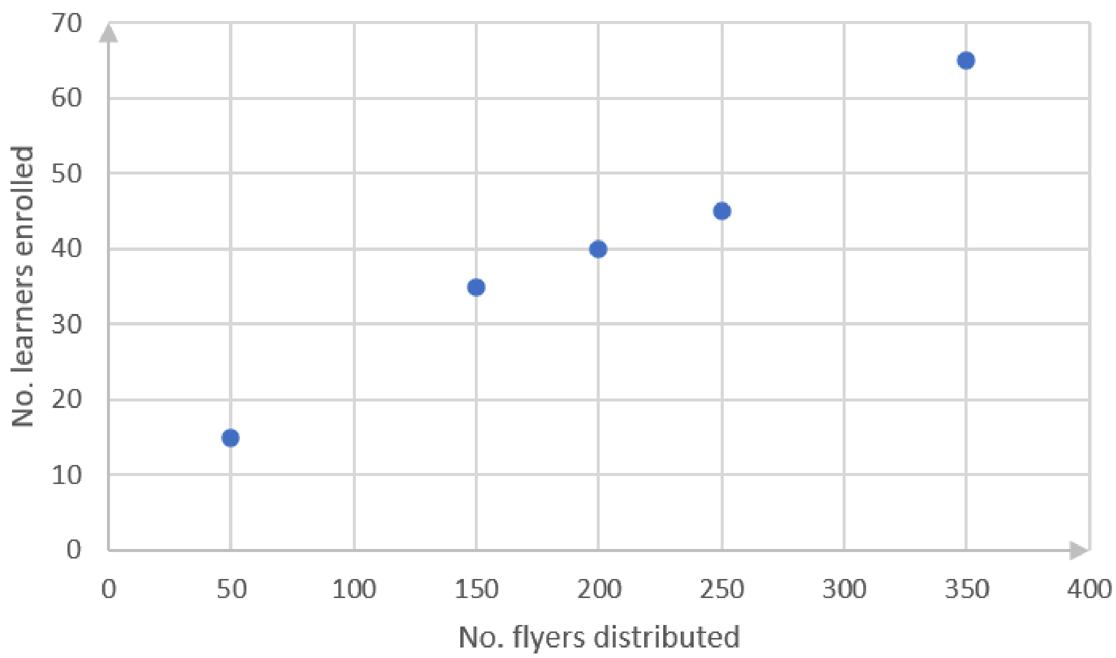
$$\hat{y} = -6.34 + 0.93x$$
$$0.93x = 6.34 + \hat{y}$$
$$x = \frac{6.34 + 40}{0.93}$$
$$x = 49.83$$

According to the regression equation, if the geyser outside the house uses 40kWh, the geyser inside the house will use 49.83kWh.

2.

Number of flyers distributed (x)	Number of learners enrolled (y)
50	15
250	45
200	40
350	65
150	35

a.



b.

	Number of flyers distributed x	Number of learners enrolled y	$x - \bar{x}$	$y - \bar{y}$	$(x - \bar{x})(y - \bar{y})$	$(x - \bar{x})^2$
	50	15	-150	-25	3 750	22 500
	250	45	50	5	250	2 500
	200	40	0	0	0	0
	350	65	150	25	3 750	22 500
	150	35	-50	-5	250	2 500
Sums	1 000	200		0	8 000	50 000
Mean	200	40				

Gradient:

$$b = \frac{\sum (x-\bar{x})(y-\bar{y})}{\sum (x-\bar{x})^2}$$
$$= \frac{8\ 000}{50\ 000}$$
$$= 0.16$$

y-intercept:

$$a = \bar{y} - b\bar{x}$$
$$= 40 - (0.16)(200)$$
$$= 40 - 32$$
$$= 8$$

Regression equation: substituting for b and a in $\hat{y} = a + bx$:

$$\hat{y} = 8 + 0.16x$$

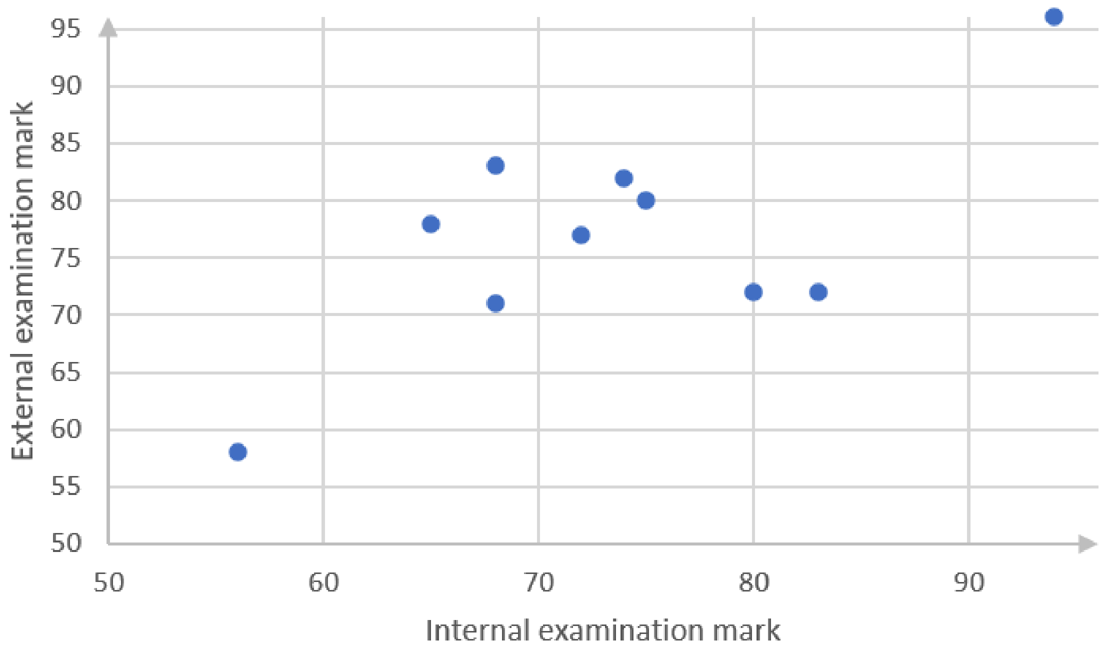
c. According to the equation, the number of learners that would be enrolled if 500 flyers were sent out would be:

$$\hat{y} = 8 + 0.16x$$
$$\hat{y} = 8 + 0.16(500)$$
$$= 8 + 80$$
$$= 88$$

3.

Internal examinations (x)	80	68	94	72	74	83	56	68	65	75
External examinations (x)	72	71	96	77	82	72	58	83	78	80

a.



b.

	Internal examinations % x	External examinations % y	$x-\bar{x}$	$y-\bar{y}$	$(x-\bar{x})(y-\bar{y})$	$(x-\bar{x})^2$
	80	72	6.5	-4.9	31.85	42.25
	68	71	-5.5	-5.9	32.45	30.25
	94	96	20.5	19.1	391.55	420.25
	72	77	-1.5	0.1	-0.15	2.25
	74	82	0.5	5.1	2.55	0.25
	83	72	9.5	-4.9	-46.55	90.25
	56	58	-17.5	-18.9	330.75	306.25
	68	83	-5.5	6.1	-33.55	30.25
	65	78	-8.5	1.1	-9.35	72.25
	75	80	1.5	3.1	4.65	2.25
Sums	735	769			640.5	996.5
Mean	73.5	76.9				

Gradient:

$$b = \frac{\sum (x-\bar{x})(y-\bar{y})}{\sum (x-\bar{x})^2}$$

$$= \frac{640.5}{996.5}$$

$$= 0.64$$

y-intercept:

$$a = \bar{y} - b\bar{x}$$

$$= 76.9 - (0.64)(73.5)$$

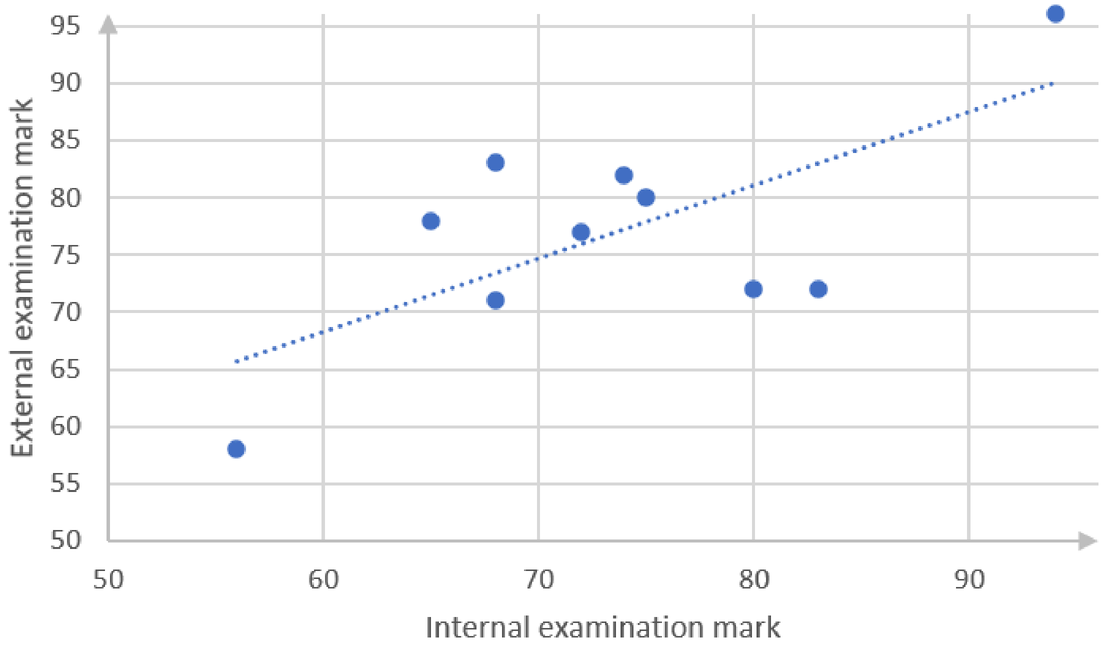
$$= 76.9 - 47.04$$

$$= 29.86$$

Regression equation: substituting for b and a in $\hat{y} = a + bx$:

$$\hat{y} = 29.86 + 0.64x$$

c.



d. The regression equation predicts that the final examination mark of a learner who scores 70 in the internal examination will be:

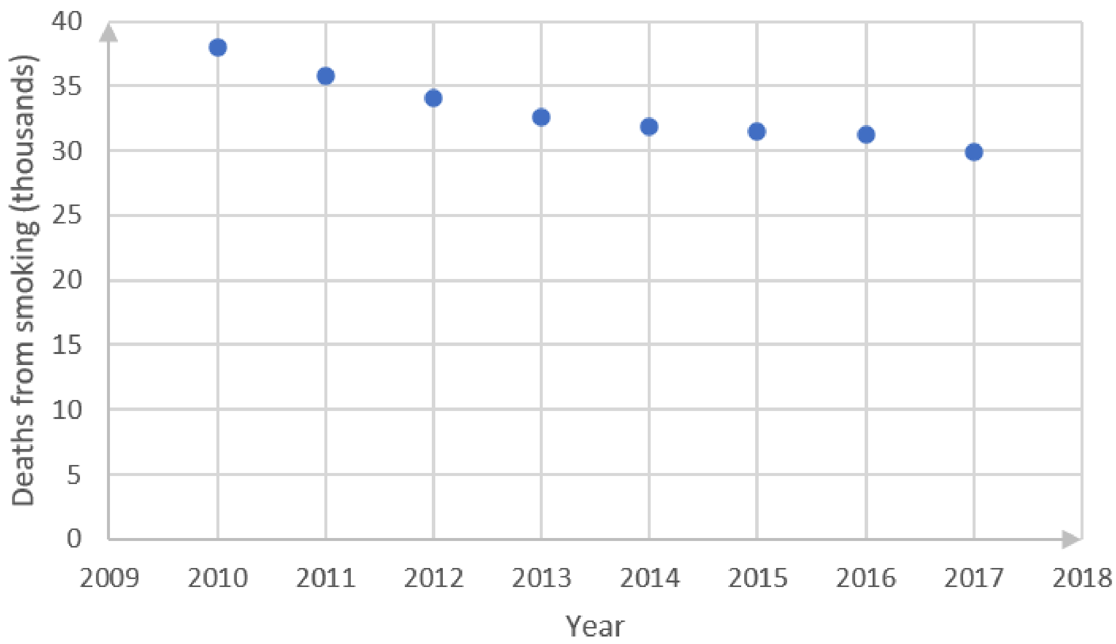
$$\begin{aligned}\hat{y} &= 29.86 + 0.64x \\ &= 29.86 + 0.64(70) \\ &= 29.86 + 44.8 \\ &= 74.66\end{aligned}$$

See question 2 in unit 2 assessment.

4.

Year	2010	2011	2012	2013	2014	2015	2016	2017
Deaths $('000)$	38.0	35.8	34.1	32.6	31.8	31.5	31.3	29.9

a.



b.

	Year x	Deaths from smoking $('000)$ y	$x-\bar{x}$	$y-\bar{y}$	$(x-\bar{x})(y-\bar{y})$	$(x-\bar{x})^2$
	2010	38	−3.5	4.87	−17.05	12.25
	2011	35.8	−2.5	2.67	−6.68	6.25
	2012	34.1	−1.5	0.97	−1.46	2.25
	2013	32.6	−0.5	−0.53	0.27	0.25
	2014	31.8	0.5	−1.33	−0.67	0.25
	2015	31.5	1.5	−1.63	−2.45	2.25
	2016	31.3	2.5	−1.83	−4.58	6.25
	2017	29.9	3.5	−3.23	−11.31	12.25
Sums	16 108	265			−43.93	42
Mean	2 013.5	33.13				

Gradient:

$$b = \frac{\sum(x-\bar{x})(y-\bar{y})}{\sum(x-\bar{x})^2} = \frac{-43.93}{42} = -1.05$$

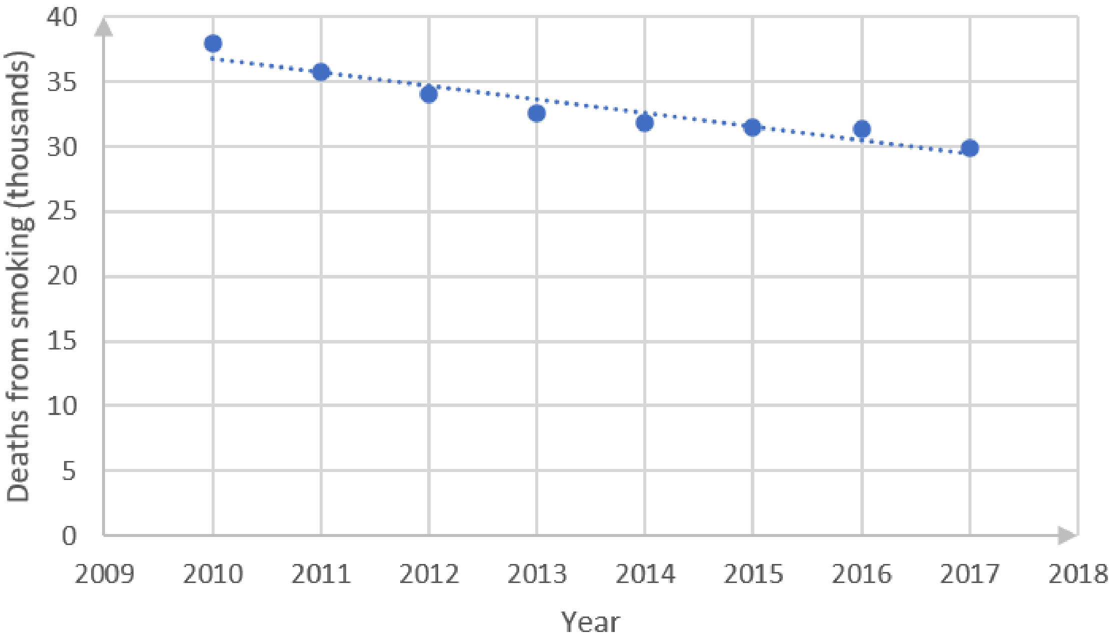
y-intercept:

$$a = \bar{y} - b\bar{x} = 33.13 - (-1.05)(2013.5) = 33.13 + 2114.18 = 2147.31$$

Regression equation: substituting for b and a in $\hat{y} = a + bx$:

$$\hat{y} = 2147.31 - 1.05x$$

c.



d. The regression equation indicates that the number of deaths (thousands) from smoking predicted for 2022 will be:

$$\begin{aligned}\hat{y} &= 2147.31 - 1.05x \\ &= 2147.31 - 1.05(2022) \\ &= 2147.31 - 2123.1 \\ &= 24.21\end{aligned}$$

Back to Unit 3: Assessment

Media Attributions

SUBJECT OUTCOME XIII

DATA HANDLING: USE EXPERIMENTS, SIMULATION AND PROBABILITY DISTRIBUTION TO SET AND EXPLORE PROBABILITY MODELS

Subject outcome

Subject outcome 4.3: Use experiments, simulation and probability distribution to set and explore probability models

Learning outcomes

- Explain and distinguish between the following terminology/events:
    - Probability
    - Dependent events
    - Independent events
    - Mutually exclusive
    - Mutually inclusive
    - Complementary events.
- Make predictions based on validated experimental or theoretical probabilities taking the following into account:
    - $P(\mathrm{S}) = 1$ (where S is the sample space)
    - Disjoint (mutually exclusive) events, and is therefore able to calculate the probability of either of the events occurring by applying the addition rule for disjoint events: $P(\mathrm{A} \text{ or } \mathrm{B}) = P(\mathrm{A}) + P(\mathrm{B})$
    - Complementary events and is therefore able to calculate the probability of an event not occurring
    - $P(\mathrm{A} \text{ or } \mathrm{B}) = P(\mathrm{A}) + P(B) - P(\mathrm{A} \text{ and } \mathrm{B})$ (where A and B are events within a sample space)
    - Correctly identify dependent and independent events (e.g. from two-way contingency tables or Venn diagrams) and therefore appreciate when it is appropriate to calculate the probability of two independent events occurring by applying the product rule for independent events: $P(\mathrm{A} \text{ and } \mathrm{B}) = P(\mathrm{A}) \cdot P(\mathrm{B})$.)
- Draw tree diagrams, Venn diagrams and complete contingency two-way tables to solve probability problems (where events are not necessarily independent).
  Range:
    - Venn diagrams to be limited to two subsets.
    - Tree diagrams where the sample space is manageable (not more than 15 possible outcomes).
- Interpret and clearly communicate results of the experiments correctly in terms of real context.

Unit 1 outcomes

By the end of this unit you will be able to:

- Understand the difference between independent and dependent events.
- Understand the difference between mutually inclusive and mutually exclusive events.
- Understand complementary events.
- Identify independent and dependent events using $P\ (\mathrm{A\,and\,B}) = P\ (\mathrm{A}) \,.\, P\ (\mathrm{B})$.
- Use the addition rule for mutually exclusive events $P\,(\mathrm{A\,or\,B}) = P\,(\mathrm{A}) + P\,(\mathrm{B})$.
- Use $P(\mathrm{A\ or\ B}) = P(\mathrm{A}) + P(\mathrm{B}) - P(\mathrm{A\ and\ B})$ when $P\,(\mathrm{A} \cap \mathrm{B}) \neq 0$.

Unit 2 outcomes

By the end of this unit you will be able to:

- Understand when to use Venn diagrams.
- Draw Venn diagrams.
- Interpret Venn diagrams.

Unit 3 outcomes

By the end of this unit you will be able to:

- Draw tree diagrams when appropriate.
- Use tree diagrams to solve probability problems.

Unit 4 outcomes

By the end of this unit you will be able to:

- Draw and complete contingency tables.
- Use contingency tables to solve probability problems.

Unit 1: Understand probability and make predictions

NATASHIA BEARAM-EDMUNDS

Unit outcomes

By the end of this unit you will be able to:

- Understand the difference between independent and dependent events.
- Understand the difference between mutually inclusive and mutually exclusive events.
- Understand complementary events.
- Identify independent and dependent events using $P\ (\mathrm{A\,and\,B})\ =\ P\ (\mathrm{A})\,.\,P\ (\mathrm{B})$.
- Use the addition rule for mutually exclusive events $P\,(\mathrm{A\,or\,B}) =\ P\,(\mathrm{A}) +\ P\,(\mathrm{B})$.
- Use $P(\mathrm{A\ or\ B}) = P(\mathrm{A}) + P(\mathrm{B}) - P(\mathrm{A\ and\ B})$ when $P\,(\mathrm{A} \cap \mathrm{B}) \neq 0$.

What you should know

There is no prior knowledge required for this unit.

Introduction

Probability is often referred to as chance. It is the study of how likely it is that some event will happen.

When your favourite sports team plays a game, you don't know whether they will win or not. When the weather report says there is a 40% chance of rain tomorrow, you may or may not end up getting wet. Uncertainty presents itself to some degree in every event that occurs around us and in every decision that we make.

Many events cannot be predicted with absolute certainty. The best we can do is to say how likely they are to happen, using the idea of probability.

Note

You can read about the discovery and history of probability at this link when you have an internet connection.

Experimental and theoretical probability

If we toss a coin 100 times, how many times will 'heads' come up? We can calculate this using theoretical probability or we can actually perform the experiment, and toss the coin 100 times and record the outcomes.

Theoretical probability uses logic and a formula to calculate the chance of an event. An event is one (or more) outcomes. The total number of possible outcomes is called the **sample space** and is shown using the symbol S. The number of elements in the sample space is denoted as $n(\mathrm{S})$.

The number of possible outcomes to an event E is denoted $n(\mathrm{E})$.

When we toss a coin there are two possible outcomes, H (heads) or T (tails). Logically we would expect that each outcome has a 50% chance of happening. We say that there is a theoretical probability of $\frac{1}{2}$ for each outcome. Using theoretical probability, 'heads' have half a chance of coming up, so we can expect 50 heads in 100 coin tosses.

When all possible outcomes of an experiment have an equal chance of occurring, we use the following formula to calculate the theoretical probability:

$$\begin{aligned}\mathrm{P(E)} &= \frac{\text{number of favourable outcomes}}{\text{number of possible outcomes}} \\ &= \frac{n(\mathrm{E})}{n(\mathrm{S})}\end{aligned}$$

Probability values are real numbers between and inclusive of 0 and 1, measured as a fraction or as a decimal. Probability can also be shown as a percentage between 0% and 100%. We use words such as impossible, unlikely, possible, chance, likely and certain when we describe probabilities.

Probabilities can range from impossible to certain. We can see the likelihood of an event on a probability scale.

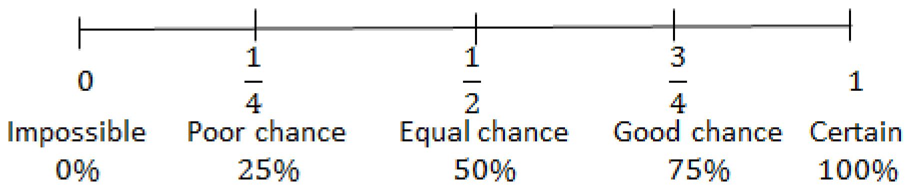


Figure 1: A probability scale

It is impossible, in a test out of 50 marks, to get a mark over 50.

It is equally likely that a pregnant woman will have a boy or a girl.

It is certain that the sun will rise tomorrow.

We have seen that by using theoretical probability the chance of getting a H when a coin is tossed 100 times is 50 but when we actually do the experiment we may record 48H or 55H ... or anything really, but in most cases it will be a number close to 50. **Experimental probability** is found from the results of an experiment repeated many times.

The more times the coin is tossed, the more accurate the probability of getting heads becomes and the closer the answer from the experimental probability will be to the theoretical probability.

Example 1.1

Write down the number of possible outcomes (sample space) $n(S)$ for each event below:

1. Throwing a die.
2. Tossing a coin.
3. Choosing a card from a pack of playing cards.

Solutions

1. When throwing a die, there are 6 possible outcomes in total $(1;\ 2;\ 3;\ 4;\ 5;\ 6)$, so $n(\text{S}) = 6$.
2. A coin has two sides: heads (H) and tails (T), so there are 2 possible outcomes $(\text{H};\ \text{T})$. Therefore, $n(\text{S}) = 2$.
3. There are 52 cards in a pack of cards so there are 52 possible outcomes to drawing a card from a pack. A pack of cards has four suits with 13 card values in each suit. So there are four cards of each value (i.e. there are 4 twos, threes, fours, etc. in a pack). The four suits are diamonds (red), spades (black), hearts (red) and clubs (black). The card values in each suit are $(2;\ 3;\ 4;\ 5;\ 6;\ 7;\ 8;\ 9;\ 10;\ \text{Jack};\ \text{Queen};\ \text{King};\ \text{Ace})$. Therefore, $n(\text{S}) = 52$.

Example 1.2

The letters of the word MATHEMATICS are written on separate cards of the same size. The cards are shuffled and dealt, face down, onto a table. A card is selected at random.

1. How many possible outcomes are there?
2. What is the probability that the card selected is:
   a. the letter I?
   b. the letter M?
   c. the letter O?

Solutions

1. There are 11 possible outcomes. $n(S) = 11$.
2. $\text{Probability} = \dfrac{\text{number of favourable outcomes}}{\text{number of possible outcomes}}$
   a. As there is only one letter 'I' there is only one favourable outcome.
   $\text{P(I)} = \dfrac{1}{11}$
   b. There are two Ms.

$$P(M) = \frac{2}{11}$$

c. There is no letter 'O'.

$$P(O) = \frac{0}{11}$$
$$= 0$$

Exercise 1.1

1. If you roll a die once:
   a. What is the lowest possible score?
   b. What is the highest possible score?
   c. What do you think is the most likely score?
2. If you roll a die once, what is the probability of throwing:
   a. four.
   b. a prime number.
   c. an odd number.
   d. an eight.
   e. a factor of six.
   f. a number less than eight.
3. A restaurant is having a raffle to raise funds. They sell a total of 500 tickets. What is the probability of Andy winning if he bought:
   a. five tickets?
   b. 50 tickets?
   c. one ticket?

The full solutions are at the end of the unit.

Dependent and independent events

Using probability, we can work out the chances of two or more events happening.

Events are **dependent** if the outcome of one event influences the outcome of the other. For example, if your lunchbox contains two apples and one banana, when you eat one of the fruit, this reduces the number of choices you have when deciding to eat a second fruit.

Independent events are not affected by previous events. For example, if you toss a coin and it comes up tails and you toss it again and it lands on heads, neither outcome influences the other. A coin does not 'know' it came up heads before, so each toss of the coin is independent.

You can use the product rule for independent events to test if events are independent and to calculate the probability of independent events.

Two events A and B are independent if and only if:
$P(\text{A and B}) = P(\text{A}) \cdot P(\text{B})$

Example 1.3

A bag contains five red balls and five blue balls. We remove a ball from the bag at random, record its colour and put it back into the bag. We then remove another ball from the bag and record its colour.

1. What is the probability that the first ball is red?
2. What is the probability that the second ball is blue?
3. What is the probability that the first ball is red and the second ball is blue?
4. Are the first ball being red and the second ball being blue independent events?

Solutions

1. Since there are a total of 10 balls, of which five are red, the probability of getting a red ball is:
$$\begin{aligned} P(\text{R}) &= \frac{5}{10} \\ &= \frac{1}{2} \end{aligned}$$
Note: always simplify fractions as far as possible when giving the answers to probability questions.
2. Since the first ball is placed back into the bag before we take the second ball this means that when we draw the second ball, there are still a total of 10 balls in the bag, of which five are blue. Therefore the probability of drawing a blue ball is:
$$\begin{aligned} P(\text{B}) &= \frac{5}{10} \\ &= \frac{1}{2} \end{aligned}$$
3. When drawing two balls from the bag, there are four possibilities. We can get:

   - a red ball and then another red ball
   - a red ball and then a blue ball
   - a blue ball and then a red ball
   - a blue ball and then another blue ball.

   We want to know the probability of the second outcome, where we have to get a red ball first. Since there are five red balls and 10 balls in total, there are $\frac{5}{10}$ ways to get a red ball first. Now we put the first ball back, so there are again five red balls and five blue balls in the bag. Therefore, there are $\frac{5}{10}$ ways to get a blue ball second if the first ball was red. This means that there are:
   $$\frac{5}{10} \times \frac{5}{10} = \frac{25}{100}$$
   ways to get a red ball first and a blue ball second. So, the probability of getting a red ball first and a blue ball second is $\frac{1}{4}$.

4. Events are independent if and only if: $P(\text{A and B}) = P(\text{A}) \cdot P(\text{B})$
   In this example:

   - $P(\text{first ball is red}) = \frac{1}{2}$
   - $P(\text{second ball is blue}) = \frac{1}{2}$
   - $P(\text{first ball red and second ball is blue}) = \frac{1}{4}$

   So we see that:

   $$\begin{aligned} P(\text{first ball red and second ball is blue}) &= P(\text{first ball is red}) \cdot P(\text{second ball is blue}) \\ &= \frac{1}{2} \cdot \frac{1}{2} \\ &= \frac{1}{4} \end{aligned}$$

   Therefore, the events are independent.

In the example above, we picked a random ball and put it back into the bag before continuing. This is called **sampling with replacement**. In the next example, we will follow the same process, except that we will not put the first ball back into the bag. This is called **sampling without replacement**.

Example 1.4

A bag contains five red balls and five blue balls. We remove a ball from the bag at random, record its colour. Then, without replacing the first ball, we remove another random ball from the bag and record its colour.

1. What is the probability that the first ball is red?
2. What is the probability that the second ball is blue?
3. What is the probability that the first ball is red and the second ball is blue?
4. Are the first ball being red and the second ball being blue independent events?

Solutions

1. Since there are five red balls and 10 balls in total, there are:
   $\frac{5}{10} = \frac{1}{2}$ ways to get a red ball first.
2. There are four possible outcomes when we remove the two balls:

   - a red ball and then another red ball
   - a red ball and then a blue ball
   - a blue ball and then a red ball
   - a blue ball and then another blue ball.

   Since the question asks for the probability that the second ball is blue, we will only consider outcomes two and four.

- a red ball and then a blue ball, AND
- a blue ball and then another blue ball.

For the first outcome (a red ball and then a blue ball), we get a red ball first. Since there are five red balls and 10 balls in total, there are $\frac{5}{10}$ ways to get a red ball first.

After we have taken out a red ball, there are now four red balls and five blue balls left so that is nine balls altogether. Therefore there are five ways to get a blue ball second if the first ball was red. Therefore, there are $\frac{5}{9}$ ways to get a blue ball second if a red ball was drawn first.

This means that there are $\frac{5}{10} \times \frac{5}{9} = \frac{25}{90}$ ways to get a blue ball second if a red ball was drawn first.

For the fourth outcome (a blue ball and then another blue ball), we get a blue ball first. Since there are five blue balls and 10 balls in total, there are $\frac{5}{10}$ ways to get blue ball first.

After we have taken out a blue ball, there are now five red balls and four blue balls left so that is nine balls altogether. Therefore there are four ways to get a blue ball second if the first ball was also blue. Therefore, there are $\frac{4}{9}$ways to get a blue ball second if a blue ball was drawn first.

This means that there are $\frac{5}{10} \times \frac{4}{9} = \frac{20}{90}$ ways to get a blue ball second if a blue ball was also drawn first.

To determine the probability of getting a blue ball on the second draw, we look at all of the outcomes that contain a blue ball second and add them.

$$\begin{aligned} P(\text{R and B}) + \text{P}(\text{B and B}) &= \frac{25}{90} + \frac{20}{90} \\ &= \frac{45}{90} \\ &= \frac{1}{2} \end{aligned}$$

This is the same as in the previous example! You might find it surprising that the probability of the second ball being blue is not affected by whether or not we replace the first ball. The reason why this probability is still $\frac{1}{2}$is that we are computing the probability that the second ball is blue without knowing the colour of the first ball. Because there are only two equal possibilities for the second ball (red and blue) and because we don't know whether the first ball is red or blue, there is an equal chance that the second ball will be one colour or the other.

3. We have already calculated the probability that the first ball is red and the second ball is blue. We saw that there are:

$$\begin{aligned} \frac{5}{10} \times \frac{5}{9} &= \frac{25}{90} \\ &= \frac{5}{18} \end{aligned}$$

ways to get a blue ball second if a red ball was drawn first.

4. Events are independent if and only if: $P(\text{A and B}) = P(\text{A}) \cdot P(\text{B})$
   In this example:

   - $P(\text{first ball is red}) = \frac{1}{2}$
   - $P(\text{second ball is blue}) = \frac{1}{2}$
   - $P(\text{first ball red and second ball is blue}) = \frac{5}{18}$

   So we see that:

   $$P(\text{first ball red and second ball is blue}) \neq P(\text{first ball is red}) \cdot P(\text{second ball is blue})$$
   $$\frac{5}{18} \neq \frac{1}{2} \cdot \frac{1}{2}$$
   Therefore the events are not independent, in other words the events are dependent.

Exercise 1.2

1. $P(\text{M}) = 0.45;\ P(\text{N}) = 0.35;\ P(\text{M and N}) = 0.1575$. Are M and N independent events?
2. I toss a coin and roll a die. What is the probability of getting a head and a two?
3. A box contains three black cards and four white cards. Two cards are randomly picked one after the other.
   a. Calculate the probability that the first card picked is black.
   b. Calculate the probability that the first card picked is white.
   c. If a black card is picked first and not put back into the box, what is the probability that a black card will be picked second?
4. If A and B are independent events and $P(\text{A}) = 0.2$ and $P(\text{A and B}) = 0.06$, find $P(\text{B})$.

The full solutions are at the end of the unit.

Union and intersection

When more than one event occurs at a time there can be a combination of outcomes, which results in unions and intersections of the events.

A union of events is the set of all outcomes that occur in the events, written as:
'A or B' or $\text{A} \cup \text{B}$ (A union B).

For example, the union of the sets $\text{A} = \{1;\ 3;\ 5;\ 7\}$ and $\text{B} = \{3;\ 6;\ 9;\ 12\}$ is the new set $\text{A} \cup \text{B} = \{1;\ 3;\ 5;\ 6;\ 7;\ 9;\ 12\}$ which contains all the elements from both sets with no elements repeated. Here $n(A) = 4$ and $n(B) = 4$, and $n(A \cup B) = 7$.

The intersection of events is the set of all outcomes that occur in all of the events. It is written as:

'$\text{A}$and B' or $\text{A} \cap \text{B}$ (A intersection B).

For example, the intersection of the sets $\text{A} = \{1;\ 3;\ 5;\ 7\}$ and $\text{B} = \{3;\ 6;\ 9;\ 12\}$ is the new set $\text{A} \cap \text{B} = \{3\}$, which contains all the elements that are common to both the sets. In other words, the intersection contains only the elements repeated in the sets. Here, $n(A \cap B) = 1$, meaning that there is only one element that is in both set A and set B.

We can show the relationships of unions and intersections of sets by using Venn diagrams. We will cover Venn diagrams in detail in unit 2 of this subject outcome.

Note

You can learn more about unions and intersections by watching the video "Venn diagram".

Venn diagram (Duration: 03.31)

We can calculate the probability of the union of two events using the addition rule for any two events:

$P(\text{A} \cup \text{B}) = P(\text{A}) + P(\text{B}) - P(\text{A} \cap \text{B})$

Mutually exclusive events

Two events are called mutually exclusive or disjoint if they cannot occur at the same time. For example, you cannot be 21-years-old and 25-years-old at the same time.

Another way of saying this is that the two events, A and B, cannot have any elements in common, so $n(A \cap B) = 0$. A set with no elements is called 'the empty set', and denoted by $\varnothing$. With no possible events in the sample space, it is clear that $P(\text{A} \cap \text{B}) = 0$.

The addition rule for two mutually exclusive events is $P(\text{A} \cup \text{B}) = P(\text{A}) + P(\text{B})$. This rule is a special case of the previous rule because the events are mutually exclusive, so $P(\text{A} \cap \text{B}) = 0$.

Mutually inclusive events are the opposite of mutually exclusive events and can occur at the same time. For example, you could choose a number that is both less than five and odd. There is a possibility of multiple outcomes, so mutually inclusive events cannot happen independently. The probability of the intersection of mutually inclusive events is greater than zero.

$P(\text{A} \cap \text{B}) > 0$

Complementary events

The complement of a set of events A is a separate set that consists of all elements that are not in A. The complement of any event is the event not (A). We write the complement of an event as A'. Since every element in the sample space (S) is in either A or A', the union of complementary events covers the entire sample space.

$A \cup A' = S$

Complementary events occur when there are only two outcomes. For example, passing an exam or not passing an exam.

Since every element in A is not in A', we know that complementary events are mutually exclusive:

$A \cap A' = 0$. All complementary events are mutually exclusive, but not all mutually exclusive events are complementary.

The probabilities of complementary events sum to 1.

$$\begin{aligned} P(A) + P(A') &= P(A \cup A') \\ &= 1 \\ \therefore P(A) &= 1 - P(A') \end{aligned}$$

Example 1.5

State if the following are mutually exclusive, mutually inclusive or complementary.

1. Day and night.
2. Getting a head and getting a tail on a single coin toss.
3. Getting a queen and getting a heart in a single draw from a pack of cards.
4. Pulling your ear and turning your head.
5. Getting a salary increase and not getting a salary increase.
6. Rolling a die and getting a five or six.

Solutions

1. It cannot be day and night at the same time. These are mutually exclusive and complementary events.
2. Getting a head and getting a tail on a single coin toss are mutually exclusive and complementary. You can get either a head or a tail on a single coin toss, but not both at the same time.
3. Getting a queen and getting a heart in a single draw from a pack of cards is mutually inclusive. You can get a queen of hearts.
4. Pulling your ear and turning your head are mutually inclusive. You can pull your ear and turn your head at the same time.
5. Getting a salary increase and not getting a salary increase are mutually exclusive and complementary events.
6. Rolling a die and getting a five or six are mutually exclusive events but NOT complementary since there are more than two possible outcomes.

Example 1.6

On a single roll of a die, find the probability of getting:

1. one or two .
2. A number less than three or a number greater than three.
3. An even number.
4. An odd number.
5. What type of events are those in questions 3 and 4 above called?

Solutions

1. The probability of rolling a one or two are mutually exclusive.
$$\begin{aligned} P(1 \text{ or } 2) &= P(1) + P(2) \\ &= \frac{1}{6} + \frac{1}{6} \\ &= \frac{2}{6} \\ &= \frac{1}{3} \end{aligned}$$
2. A number less than three or a number greater than three are mutually exclusive. There are two numbers less than three and three numbers greater than three.
$$\begin{aligned} P(\text{less than } 3 \cup \text{greater than } 3) &= P(\text{less than } 3) + P(\text{greater than } 3) \\ &= \frac{2}{6} + \frac{3}{6} \\ &= \frac{5}{6} \end{aligned}$$
3. There are three even numbers.
$$\begin{aligned} P(\text{even number}) &= \frac{3}{6} \\ &= \frac{1}{2} \end{aligned}$$
4. There are three odd numbers.
$$\begin{aligned} P(\text{odd number}) &= \frac{3}{6} \\ &= \frac{1}{2} \end{aligned}$$
5. Even and odd numbers are complementary events.

Example 1.7

A car dealership has 200 cars on its website for sale. If 50 of the cars are Toyotas, what is the probability that the next car to be sold is:

1. a Toyota?
2. not a Toyota?

Solutions

1. $$\begin{aligned} P(\text{Toyota}) &= \frac{50}{200} \\ &= \frac{1}{4} \end{aligned}$$
2. $$\begin{aligned} P(\text{Not Toyota}) &= 1 - \frac{1}{4} \\ &= \frac{3}{4} \end{aligned}$$

Example 1.8

A bowl contains three apples, four lemons and three bananas. If a fruit is selected at random, what is the probability that:

1. it is either a lemon or a banana?
2. it is not an apple?
3. it is a yellow fruit?

Solutions

1. These are mutually exclusive events.
$$\begin{aligned} P(\text{lemon or banana}) &= \frac{4}{10} + \frac{3}{10} \\ &= \frac{7}{10} \end{aligned}$$
2. $$\begin{aligned} P(\text{not an apple}) &= 1 - \frac{3}{10} \\ &= \frac{7}{10} \end{aligned}$$
3. Both the lemons and bananas are yellow and these are mutually exclusive.
$$\begin{aligned} P(\text{yellow fruit}) &= P(\text{banana}) + P(\text{lemon}) \\ &= \frac{7}{10} \end{aligned}$$

Example 1.9

A card is drawn at random from an ordinary pack of 52 playing cards. Find the probability that the card drawn is:

1. the three of diamonds
2. the three of diamonds or any heart
3. a diamond or a three

Solutions

1. There is only one three of diamonds in a pack of cards.
   $P(\text{3 of diamonds}) = \frac{1}{52}$
2. There are 13 hearts in a pack of cards and one three of diamonds. These are mutually exclusive events.
   $$\begin{aligned} P(\text{3 of diamonds or any heart}) &= P(\text{3 of diamonds}) + P(\text{heart}) \\ &= \frac{1}{52} + \frac{13}{52} \\ &= \frac{14}{52} \\ &= \frac{7}{26} \end{aligned}$$
3. There are four cards of each suit with a value of three. We must subtract the probability of getting a three of diamonds to avoid 'over counting'.

   $$\begin{aligned} P(\text{diamond or 3}) &= P(\text{diamond}) + P(3) - P(\text{diamond and 3}) \\ &= \frac{13}{52} + \frac{4}{52} - \frac{1}{52} \\ &= \frac{16}{52} \\ &= \frac{4}{13} \end{aligned}$$

Exercise 1.3

1. A bowl has pink and white sweets. The probability of taking out a pink sweet is $\frac{4}{9}$. What is the probability of taking out a white sweet?
2. You flip a coin two times.
   a. Write down the sample space.
   b. What is the probability of getting two heads?
   c. What is the probability of getting heads first and then tails?
3. Donald has 14 loose socks in a drawer. Six of these are white and two are red. Calculate the probability that the first sock taken out at random is:
   a. white
   b. red

c. not red

d. white or red

e. neither white nor red

4. A box contains six orange balls, five green balls and five yellow balls. You randomly pick balls one after the other.

   a. Calculate the probability of drawing two green balls if the first ball is put back into the box before drawing the second ball.

   b. Calculate the probability of drawing two green balls if the first ball is NOT put back into the box before drawing the second ball.

   c. Calculate the probability of drawing a yellow ball and an orange ball if the first ball is put back into the box before drawing the second ball.

5. There are 15 male (M) and 20 female (F) learners in an online lecture. The lecturer randomly chooses a learner to answer a question.

   a. Determine $P(\text{M})$ learner is chosen.

   b. Determine $P(\text{F})$.

   c. If he chooses two learners determine the probability that one is male and the other is female.

The full solutions are at the end of the unit.

Summary

In this unit you have learnt the following:

- How to define probability.
- How to calculate theoretical probability.
- How to identify dependent events.
- How to identify independent events.
- How to use the product rule for independent events.
- How to differentiate between mutually exclusive and mutually inclusive events.
- How to use the probability identity $P(\text{A or B}) = P(\text{A}) + P(\text{B}) - P(\text{A and B})$.
- How to calculate the probability of mutually exclusive events.
- How to calculate the probability of complementary events.

Unit 1: Assessment

Suggested time to complete: 35 minutes

1. You take all the hearts from a deck of cards. You then select a random card from the set of hearts.

   a. What is the sample space?

   b. Calculate the probability that the card is the ace of hearts.

   c. Calculate the probability that the card is a prime number.

   d. Calculate the probability that the card has a letter on it.

2. A and B are two events in a sample space where $P(\text{A}) = 0.3;\ \ P(\text{A or B}) = 0.8$ and $P(\text{B}) = k$. Determine

the value of k if:

a. A and B are mutually exclusive.

b. A and B are independent.

3. You roll two six-sided dice and are interested in the following two events:
A: the sum of the numbers on the dice equals eight
B: at least one of the dice shows a one
Show that these events are mutually exclusive.
4. The surface of a soccer ball is made up of 32 faces. 12 faces are regular pentagons, each with a surface area of about $37\ \text{cm}^2$. The other 20 faces are regular hexagons, each with a surface area of about $56\ \text{cm}^2$. You roll the soccer ball. What is the probability that it stops with a pentagon touching the ground?
5. A pack of 20 batteries contains two defective batteries. If two batteries are randomly chosen one after the other, what is the probability that:
   a. only one of the batteries will be defective?
   b. both batteries will be defective?
   c. neither battery will be defective?
6. A black bag contains four red beads, nine black beads and seven white beads. You randomly select two beads.
   a. Calculate the probability of choosing a red bead and then a black bead if the first bead is put back into the bag.
   b. Calculate the probability of choosing two black beads if the first bead is NOT replaced into the bag.

The full solutions are at the end of the unit.

Unit 1: Solutions

Exercise 1.1

1. If you roll a die, the total possible outcomes are $(1;\ 2;\ 3;\ 4;\ 5;\ 6)$.
   a. The lowest possible score is one
   b. The highest possible score is six.
   c. Are all outcomes just as likely or will some happen more often? All outcomes are equally likely so all have a probability of $\frac{1}{6}$ and no outcome is more likely than another to occur.
2. If you roll a die, what is the probability of throwing:
   a. $P(4) = \frac{1}{6}$
   b. There are three prime numbers $(2;\ 3;\ 5)$.
   $$\begin{aligned} P(\text{prime number}) &= \frac{3}{6} \\ &= \frac{1}{2} \end{aligned}$$
   c. There are three odd numbers $(1;\ 3;\ 5)$.
   $$\begin{aligned} P(\text{odd number}) &= \frac{3}{6} \\ &= \frac{1}{2} \end{aligned}$$

d.

$$P(8) = \frac{0}{6}$$
$$= 0$$

e. There are four factors of 6, $(1;\ 2;\ 3;\ 6)$.

$$P(\text{factor of } 6) = \frac{4}{6}$$
$$= \frac{2}{3}$$

f. All the numbers are less than 8 so this is a certainty.

$$P(\text{number less than } 8) = \frac{6}{6}$$
$$= 1$$

3.

a.

$$P(\text{winning}) = \frac{5}{500}$$
$$= 1\%$$

b.

$$P(\text{winning}) = \frac{50}{500}$$
$$= 10\%$$

c.

$$P(\text{winning}) = \frac{1}{500}$$
$$= 0.2\%$$

Back to Exercise 1.1

Exercise 1.2

1. $P(\text{M}) = 0.45;\ P(\text{N}) = 0.35;\ P(\text{M and N}) = 0.1575$
   $$P(\text{M}) \times P(\text{N}) = 0.45 \times 0.35$$
   $$= 0.1575$$
   Since $P(\text{M}) \times P(\text{N}) = P(\text{M and N})$, M and N are independent events.

2. Tossing a coin and rolling a die are independent events.
   $$P(\text{H}) = \frac{1}{2}$$
   $$P(2) = \frac{1}{6}$$
   $$P(\text{H and } 2) = \frac{1}{2} \times \frac{1}{6}$$
   $$= \frac{1}{12}$$

3.

a.

$$P(\text{B}) = \frac{\text{number of black cards}}{\text{total possible cards}}$$
$$= \frac{3}{7}$$

b.

$$P(W) = \frac{\text{number of white cards}}{\text{total possible cards}}$$
$$= \frac{4}{7}$$

c. If a black card is picked first and not put back into the box, then there will only be two black cards left in the box and six cards in total.

$$P(\text{B first and B second}) = \frac{3}{7} \times \frac{2}{6}$$
$$= \frac{6}{42}$$
$$= \frac{1}{7}$$

4. Since A and B are independent $P(\text{A and B}) = P(\text{A}) \times P(\text{B})$.

$P(\text{A}) = 0.2;\ P(\text{A and B}) = 0.06$

$$P(\text{A and B}) = P(\text{A}) \times P(\text{B})$$
$$0.2 \times P(\text{B}) = 0.06$$
$$\therefore P(\text{B}) = 0.3$$

Back to Exercise 1.2

Exercise 1.3

1.

$$P(\text{W}) = 1 - P(\text{P})$$
$$= 1 - \frac{4}{9}$$
$$= \frac{5}{9}$$

2. If you flip a coin two times there are four possible outcomes.
   a. $\text{S} = \{(\text{H;H}),\ (\text{H;T}),\ (\text{T;T}),\ (\text{T;H})\}$
   b. The probability of getting two heads:
   $$P(\text{H;H}) = \frac{1}{4}$$
   c. The probability of getting heads first and then tails:
   $$P(\text{H;T}) = \frac{1}{4}$$

3.
   a.
   $$P(\text{W}) = \frac{6}{14}$$
   $$= \frac{3}{7}$$
   b.
   $$P(\text{R}) = \frac{2}{14}$$
   $$= \frac{1}{7}$$
   c.
   $$P(R') = 1 - P(R)$$
   $$= 1 - \frac{1}{7}$$
   $$= \frac{6}{7}$$

d.

$$P(\mathrm{W} \cup \mathrm{R}) = P(\mathrm{W}) + P(\mathrm{R})$$
$$= \frac{3}{7} + \frac{1}{7}$$
$$= \frac{4}{7}$$

e.

$$P(\text{neither W nor R}) = 1 - P(\text{W or R})$$
$$= 1 - \frac{4}{7}$$
$$= \frac{3}{7}$$

4.

a.

$$P(\text{G and G}) = \frac{5}{16} \cdot \frac{5}{16}$$
$$= \frac{25}{256}$$

b. The probability of two green balls if the first ball is not replaced:

$$P(\text{G and G}) = \frac{5}{16} \cdot \frac{4}{15}$$
$$= \frac{1}{12}$$

c.

$$P(\text{Y or O}) = P(\text{Y;O}) + P(\text{O;Y})$$
$$= \left(\frac{5}{16} \cdot \frac{6}{16}\right) + \left(\frac{6}{16} \cdot \frac{5}{16}\right)$$
$$= \frac{15}{64}$$

5.

a.

$$P(\mathrm{M}) = \frac{15}{35}$$
$$= \frac{3}{7}$$

b.

$$P(\mathrm{F}) = \frac{20}{35}$$
$$= \frac{4}{7}$$

c.

$$P(\mathrm{M} \cap \mathrm{F}) = P(\mathrm{M}) \cdot P(\mathrm{F})$$
$$= \frac{3}{7} \cdot \frac{4}{7}$$
$$= \frac{12}{49}$$

Back to Exercise 1.3

Unit 1: Assessment

1.

a. $\mathrm{S} = \{\mathrm{A};\ 2;\ 3;\ 4;\ 5;\ 6;\ 7;\ 8;\ 9;\ 10;\ \mathrm{J};\ \mathrm{K};\ \mathrm{Q}\}$

b. $P(\text{A}) = \frac{1}{13}$

c. $P(2) + P(3) + P(5) + P(7) = \frac{4}{13}$

d. $P(\text{A}) + P(\text{J}) + P(\text{K}) + P(\text{Q}) = \frac{4}{13}$

2.

a.

$$\begin{aligned} P(\text{A or B}) &= 0.8 \\ P(\text{A} \cap \text{B}) &= 0 \text{ since A and B are mutually exclusive} \\ P(\text{A or B}) &= P(\text{A}) + \text{P(B)} \\ 0.3 + k &= 0.8 \\ \therefore k &= 0.5 \end{aligned}$$

b.

$$\begin{aligned} P(\text{A and B}) &= P(\text{A}) \cdot P(\text{B}) \text{ since A and B are independent} \\ &= 0.3 \times k \\ P(\text{A or B}) &= 0.8 \\ P(\text{A}) + P(\text{B}) - P(\text{A and B}) &= 0.8 \\ 0.3 + k - 0.3k &= 0.8 \\ 0.7k &= 0.5 \\ \therefore k &= \frac{5}{7} \end{aligned}$$

3. The sum of the dice must equal eight, so it is not possible for one die to show a one, since the other die will have to show a seven, which is not possible.
$n(1 \cap 8) = 0$ so these events are mutually exclusive.

4. The total surface area of 12 faces of pentagons is 444 cm^2. The total surface area of 20 hexagons is $1\ 120 \text{ cm}^2$. The total surface area of the soccer ball is $1\ 564 \text{ cm}^2$.

$$\begin{aligned} P(\text{pentagon surface}) &= \frac{444}{1\ 564} \\ &= \frac{111}{391} \\ &= 0.28 \end{aligned}$$

5. $\text{S} = \{(D;D),\ (D;D'),\ (D';D),\ (D';D')\}$ where D is defective and D' is not defective.

a.

$$\begin{aligned} &P(D \textit{ and } D') = P(1^{st} D) \cdot P(2^{nd} D') + P(1^{st} D') \cdot P(2^{nd} D) \\ &= \frac{2}{20} \cdot \frac{18}{19} + \frac{18}{20} \cdot \frac{2}{19} \\ &= \frac{18}{95} \end{aligned}$$

b.

$$\begin{aligned} P(\text{defective and defective}) &= P(\text{defective}) \cdot P(\text{defective}) \\ &= \frac{2}{20} \cdot \frac{1}{19} \\ &= \frac{1}{190} \end{aligned}$$

c.

$$\begin{aligned} &P(D') = 1 - P(D \text{ and } D) \\ &= 1 - \frac{1}{190} \\ &= \frac{189}{190} \end{aligned}$$

6.

a.

$$
\begin{aligned}
P(\text{R and B}) &= \frac{4}{20} \cdot \frac{9}{20} \\
&= \frac{9}{100}
\end{aligned}
$$

b.

$$
\begin{aligned}
P(\text{B and B}) &= \frac{9}{20} \cdot \frac{8}{19} \\
&= \frac{18}{95}
\end{aligned}
$$

Back to Unit 1: Assessment

Media Attributions

Unit 2: Draw Venn diagrams to solve probability problems

NATASHIA BEARAM-EDMUNDS

Unit outcomes

By the end of this unit you will be able to:

- Understand when to use Venn diagrams.
- Draw Venn diagrams.
- Interpret Venn diagrams.

What you should know

Before you start this unit, make sure you can:

- calculate the probability of an outcome. To revise probability calculations see unit 1 of this subject outcome.

Introduction

The outcomes of an experiment can be represented using sets, Venn diagrams, tree diagrams and contingency tables. Tree diagrams and contingency tables will be discussed in units 3 and 4. In this unit we use Venn diagrams to represent the sample space of compound events.

Draw and interpret Venn diagrams

A Venn diagram is a graphical way to represent the relationships between sets. A Venn diagram can be very helpful with probability calculations. In probability, a Venn diagram is used to show how two or more events are related to each other.

In a Venn diagram each event is represented by a shape, often a circle. The area inside the shape shows the outcomes included in the event and the region outside the shape shows the outcomes that are not in the event. The rectangle, drawn around the circles, shows all the outcomes contained in the sample space.

Consider two events, A and B, in a sample space S. Figure 1 shows the sample space S as a rectangle and the two events A and B as circles. The possible ways in which the events can overlap are represented using Venn diagrams.

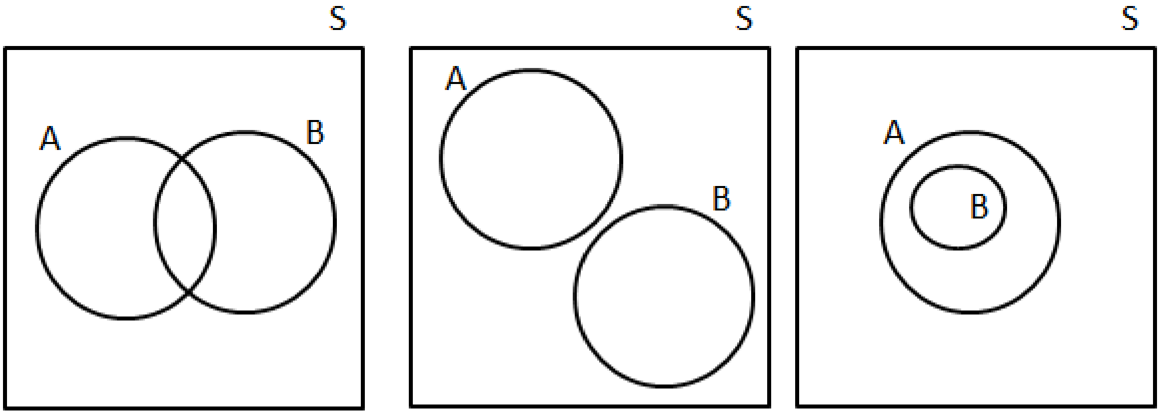


Figure 1: Venn diagrams showing possible outcomes for events A and B

In the first case the two events overlap partially and $n(A \cap B) > 1$: these are mutually inclusive events. In the second case the two events do not overlap at all. $n(A \cap B) = 0$: these are mutually exclusive events. In the third case event B is fully contained in event A, and $A \cap B = B$: these are mutually inclusive events.

Note: events will always appear inside the sample space since the sample space contains all possible outcomes of the experiment.

Note

This video shows how to draw a Venn diagram using a deck of playing cards as the sample space, "Probability with playing cards and Venn diagrams".

Probability with playing cards and Venn diagrams (Duration: 10.01)

Example 2.1

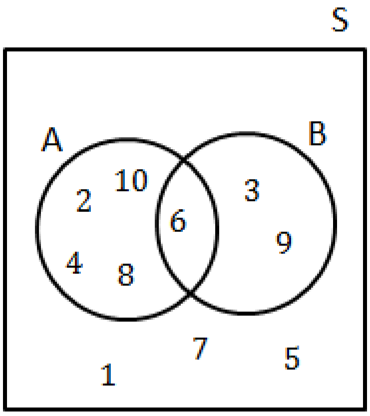
Anna thinks of a number between one and 10. Draw a Venn diagram to answer the following questions.

1. What is the probability that the number is a multiple of two?
2. What is the probability that the number is a multiple of three?
3. What is the probability that the number is a multiple of two or three?

4. What is the probability that the number is a multiple of two and three?
5. What is the probability that the number is NOT a multiple of two or three?

Solutions

Step 1: Draw the Venn diagram
The Venn diagram should show the sample space containing all numbers from one to 10. Let A be the event that contains all the multiples of two, $\mathrm{A} = \{2;\ 4;\ 6;\ 8;\ 10\}$. Let B be the event that contains all the multiples of three, $\mathrm{B} = \{3;\ 6;\ 9\}$.



We see that there are ten outcomes. The intersection of the two events is 6 and the outcomes $1;\ 7;\ 5$ are not part of either event.

Step 2: Calculate the probabilities

Remember that the probability of an event is the number of outcomes in the event set divided by the number of outcomes in the sample space.

$$\begin{aligned} \mathrm{P(E)} &= \frac{\text{number of favourable outcomes}}{\text{number of possible outcomes}} \\ &= \frac{n(\mathrm{E})}{n(\mathrm{S})} \end{aligned}$$

1. Since there are five outcomes in event A, the probability of a multiple of two is:
$$\begin{aligned} \mathrm{P(A)} &= \frac{5}{10} \\ &= \frac{1}{2} \end{aligned}$$
2. Since there are three outcomes in event B, the probability of a multiple of three is:
$$\mathrm{P(B)} = \frac{3}{10}$$
3. The event that the number is a multiple of two or three is the union of the above two event sets. There are seven elements in the union of the event sets, so the probability is $\frac{7}{10}$. This can be calculated as:

$$P(\text{A or B}) = P(\text{A}) + P(\text{B}) - P(\text{A} \cap \text{B})$$
$$= \frac{5}{10} + \frac{3}{10} - \frac{1}{10}$$
$$= \frac{7}{10}$$

4. The event that the number is a multiple of two and three is the intersection of the two event sets. There is one element in the intersection of the event sets, so $P(\text{A} \cap \text{B}) = \frac{1}{10}$.
5. The event 'NOT a multiple of two or three' is the set of all numbers not in events A or B. This is the complement of events A or B.

$$P(\text{not A or B}) = 1 - \frac{7}{10}$$
$$= \frac{3}{10}$$

Example 2.2

In a group of 50 learners, 35 take mathematics and 30 take science, while 12 take neither of the two subjects. Draw a Venn diagram representing this information. If a learner is chosen at random from this group, what is the probability that they take both mathematics and science?

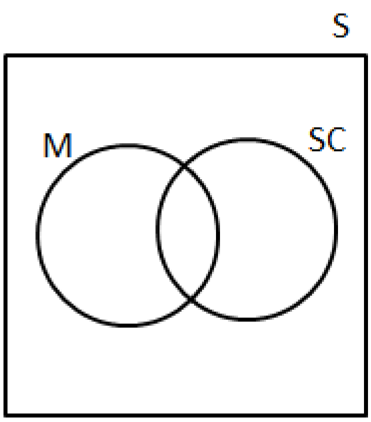
Solution

Step 1: Draw an outline of Venn diagram

Let M be the event 'takes mathematics'.

Let SC be the event 'takes science'.

We need to do some calculations before drawing the full Venn diagram, but with the information given we can already draw the outline.



Step 2: Write down sizes of the event sets, their union and intersection

We are told that 12 learners take neither of the two subjects. Graphically we can represent this outside the two events in the Venn diagram.

Since there are 50 learners in the sample space, we can see that there are $50 - 12 = 38$ elements in M or SC. So far we know:

$n(\text{M}) = 35$

$n(\text{SC}) = 30$

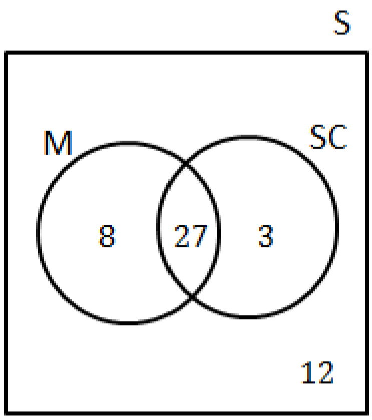
$n(\text{M or SC}) = 38$

We need to find the intersection of the two events to complete the Venn diagram.

From the addition rule:

$$\begin{aligned} n(\text{M} \cup \text{SC}) &= n(\text{M}) + n(\text{SC}) - n(\text{M} \cap \text{SC}) \\ \therefore n(\text{M} \cap \text{SC}) &= 35 + 30 - 38 \\ &= 27 \end{aligned}$$

Step 3: Draw the final Venn diagram



Step 4: Calculate the probability

$$P(\text{M} \cap \text{SC}) = \frac{27}{50}$$

Example 2.3

Use the following information to draw a Venn diagram. Then using the Venn diagram, find $P\left(\text{B and (not A)}\right)$.

$$\begin{aligned} P(\text{A}) &= 0.3 \\ P(\text{A and B}) &= 0.2 \\ P(\text{B}) &= 0.7 \end{aligned}$$

Solution

Step 1: Calculate the probabilities of each event only

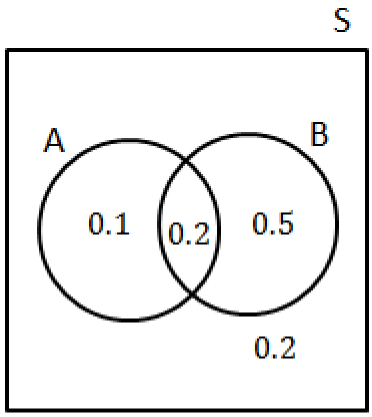
Start with the intersection and work out the probability of each event only.

Since $P(\text{A and B}) = 0.2$

$$\begin{aligned} P(\text{A only}) &= 0.3 - 0.2 \\ &= 0.1 \\ P(\text{B only}) &= 0.7 - 0.2 \\ &= 0.5 \end{aligned}$$

Therefore the probability of not being in event Aor B is $1 - (0.1 + 02 + 0.5) = 0.2$.

Step 2: Draw the Venn diagram showing the probabilities of the events



Step 3: Write down the solution

$P\,(\text{B and (not A)})$i s the same as probability of B only.

$P\,(\text{B and (not A)}) = 0.5$

Exercise 2.1

1. You are given the following information:
   $$\begin{aligned} P(\text{A}) &= 0.5 \\ P(\text{A and B}) &= 0.2 \\ P(\text{not B}) &= 0.6 \end{aligned}$$
   Draw a Venn diagram to represent this information and determine $P(\text{A or B})$.
2. In a group of 42 learners, all but three had a packet of chips or a cool drink or both. If 23 had a packet of chips and seven of these also had a cool drink, what is the probability that a learner chosen at random has:
   a. both chips and a cool drink?
   b. only cool drink?
3. In a survey at a college, 80 people were asked if they read the Daily News or the Newshound newspapers, or both. The survey revealed that 45 read the Daily News, 30 read the Newshound and 10 read neither. Use a Venn diagram to find the percentage of people who read:
   a. only the Daily News
   b. only the Newshound

c. both the Daily News and the Newshound.

The full solutions are at the end of the unit.

Summary

In this unit you have learnt the following:

- How to draw a Venn diagram.
- How to use a Venn diagram to answer probability questions.
- How to complete a Venn diagram.

Unit 2: Assessment

Suggested time to complete: 45 minutes

1. In a group of learners, 10% are left-handed, 8% are short-sighted and 2% are left-handed and short-sighted.
   a. Draw a Venn diagram to illustrate the above information.
   b. What is the probability that a randomly chosen learner from the group is left-handed or short sighted?
   c. What is the probability that a randomly chosen learner from the group is not left-handed?
2. In a group of 85 girls, 48 play hockey, 43 play tennis and 12 do not play hockey or tennis.
   a. Draw a Venn diagram to illustrate this information and use it to determine how many girls play both hockey and tennis.
   b. What is the probability that a girl chosen at random, does not play hockey?
3. There were 100 teenagers eating at a restaurant over the weekend, 68 of them had burgers, 50 ate chicken and 32 ate burgers and chicken.
   a. Draw a Venn diagram to illustrate the given data.
   b. Use the Venn diagram to find the probability that a randomly chosen person did not eat burgers or chicken.
4. NBE high school offers only two sporting activities; rugby and hockey.

   The following information is given:

   - There are 600 learners in the school.
   - 372 learners play hockey.
   - 288 learners play rugby.
   - 56 of the learners play no sport.
   - The number of learners that play both rugby and hockey is x.

   a. Represent the given information in a Venn diagram, in terms of x.
   b. Calculate the value of x.

c. Are the events playing rugby and playing hockey mutually exclusive? Justify your answer.

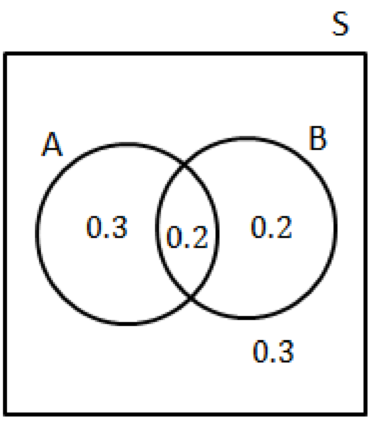
The full solutions are at the end of the unit.

Unit 2: Solutions

Exercise 2.1

1.

$$\begin{aligned} P(\text{A only}) &= 0.5 - 0.2 \\ &= 0.3 \\ P(\text{not B}) &= 0.6 \\ \text{since } P(\text{A only}) &= 0.3 \text{ then the probability of not A and not B is } 0.3 \\ \therefore P(\text{B only}) &= 1 - (0.3 + 0.2 + 0.3) \\ &= 0.2 \end{aligned}$$

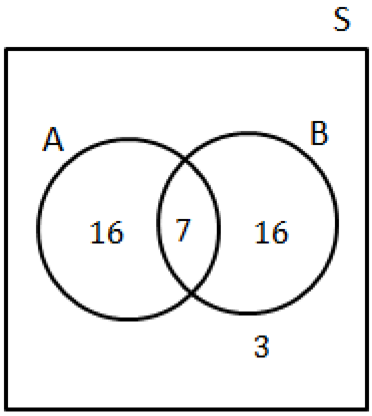


$$\begin{aligned} P(\text{A or B}) &= 0.3 + 0.2 + 0.2 \\ &= 0.7 \end{aligned}$$

2. Let A be the event 'packet of chips'
Let B be the event 'cool drink'.

$$\begin{aligned} n(A \cup B) &= 42 - 3 = 39 \\ n(A) &= 23 \\ n(A \cap B) &= 7 \\ n(A \cup B) &= n(A) + n(B) - n(A \cap B) \\ n(B) &= n(A \cup B) - n(A) + n(A \cap B) \\ &= 39 - 23 + 7 \\ &= 23 \end{aligned}$$

The Venn diagram:



a.

$$P(\text{A and B}) = \frac{7}{42} = \frac{1}{6}$$

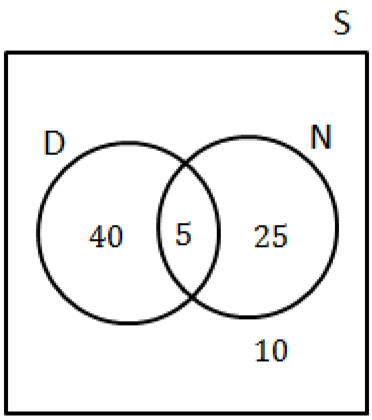
b.

$$P(\text{only B}) = \frac{16}{42} = \frac{8}{21}$$

3. Let D be the event read the Daily News
Let N be the event read the Newshound
70 people read the Daily News or Newshound newspaper or both.

$$n(D \cup N) = 80 - 10 = 70$$
$$n(D) = 45$$
$$N(N) = 30$$
$$n(D \cup N) = n(D) + n(N) - n(D \cap N)$$
$$N(D \cap N) = n(D) + n(N) - n(D \cup N) = 45 + 30 - 70 = 5$$

$45 + 30 - 70 = 5$ gives the intersection of the events.
The Venn diagram shows this information:



a. $\frac{40}{80} \times 100 = 50\%$ read only the Daily News

b. $\frac{25}{80} \times 100 = 31.25\%$read only the Newshound

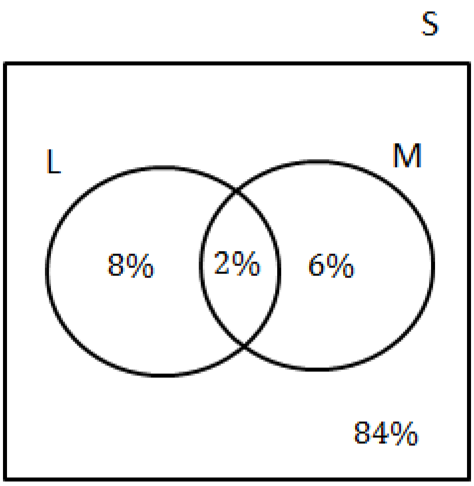
c. $\frac{5}{80} \times 100 = 6.25\%$ read both the Daily News and the Newshound

Back to Exercise 2.1

Unit 2: Assessment

1. Let L be 'left-handed'
   Let M be 'short-sighted'

   a.



   b.

$$\begin{aligned} P(\text{L or M}) &= P(\text{L}) + P(\text{M}) - P(\text{L and M}) \\ &= 10\% + 8\% - 2\% \\ &= 16\% \end{aligned}$$

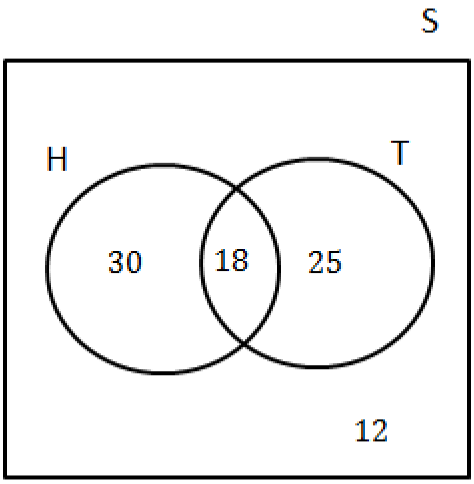
   c.

$$\begin{aligned} P(\text{not left handed}) &= 1 - P(L) \\ &= 90\% \end{aligned}$$

2. T = play tennis
   H = play hockey

   a.

$$\begin{aligned} n(H \cup T) &= 85 - 12 = 73 \\ n(H) &= 48 \\ n(T) &= 43 \\ n(H \cup T) &= n(H) + n(T) - n(H \cap T) \\ n(H \cap T) &= n(H) + n(T) - n(H \cup T) \\ &= 48 + 43 - 73 \\ &= 18 \end{aligned}$$



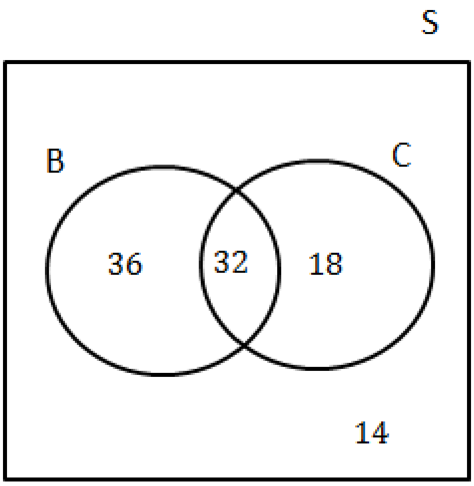
18 girls play both hockey and tennis.

b.

$$P(\{\text{H}\}') = 1 - \frac{48}{85}$$
$$= \frac{37}{85}$$

3. B = burgers
   C = chicken

   a.

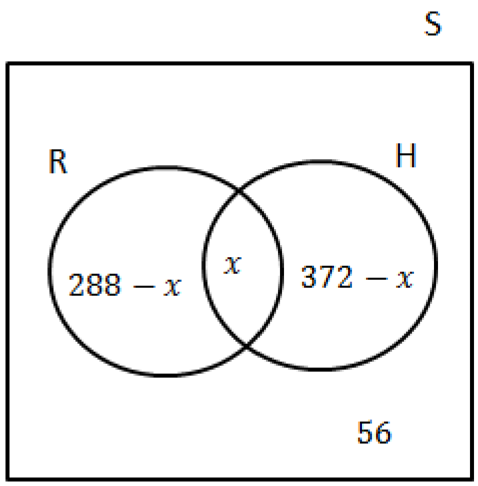


   b.

$$P(\text{not burgers or chicken}) = \frac{14}{100}$$
$$= \frac{7}{50}$$

4. R = rugby
   H = hockey

   a.



b. Calculate the value of x.

$$600 - 56 = 544 \text{ number of learners who play R or H}$$
$$288 + 372 - x = 544$$
$$\therefore x = 116$$

c. No, playing rugby and playing hockey are not mutually exclusive since $\mathrm{R} \cap \mathrm{H} = 116 \neq 0$.

Back to Unit 2: Assessment

Media Attributions

Unit 3: Draw tree diagrams to solve probability problems

NATASHIA BEARAM-EDMUNDS

Unit outcomes

By the end of this unit you will be able to:

- Draw tree diagrams when appropriate.
- Use tree diagrams to solve probability problems.

What you should know

Before you start this unit, make sure you can:

- Calculate the probability of equally likely outcomes. You can revise finding probabilities in unit 1 of this subject outcome.

Introduction

Tree diagrams are useful for organising and visualising the different possible outcomes of a sequence of events. For example, tossing a coin four times has sixteen outcomes, which can easily be organised and visualised using a tree diagram.

Draw and interpret tree diagrams

Each ‘branch’ or ‘arm’ in a tree diagram shows an outcome of an event, along with the probability of that outcome. The sum of each of the ‘branches’ will always be equal to one.

For each possible outcome of the first event, we draw a line where we write down the probability of that outcome. Then, for each possible outcome of the second event we do the same thing.

For example, if we toss a coin twice we get the following outcomes, shown on a tree diagram.

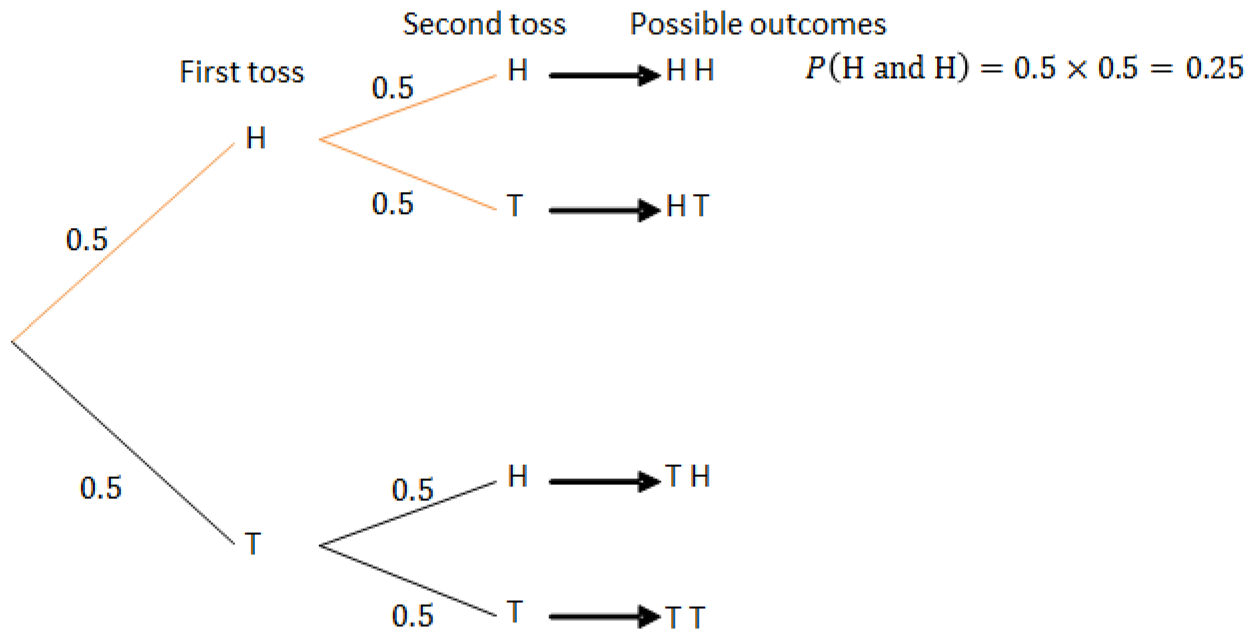


Figure 1: A tree diagram showing the outcomes from tossing a coin twice

The probability of a sequence of outcomes is calculated as the product of the probabilities along the branches of the sequence. For example, the probability of getting two heads is 0.25.

Tree diagrams are very useful for listing the outcomes of two or more combined events as we will see in the next example.

Example 3.1

A three-digit number is formed by choosing three digits separately from three sets of numbers. The first digit is chosen from the set $\{4;\ 5;\ 6\}$, the second digit from $\{2;\ 3\}$ and the third digit from $\{1;\ 2\}$. Draw a tree diagram illustrating the possible outcomes.

Use the tree diagram to determine the probability of forming:

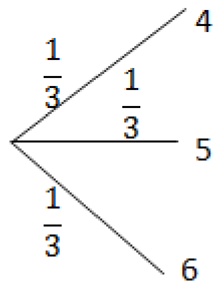
1. the number 631
2. a number consisting of two 2s
3. a number that contains digits with a sum of eight.

Solutions

There are three events; first digit, second digit and third digit.

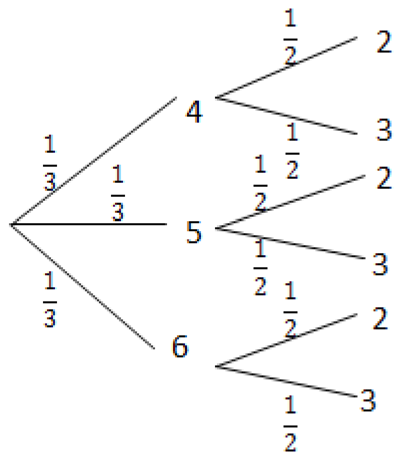
The first digit has three options, which are shown as three branches of the tree, each with a probability of $\frac{1}{3}$.

Draw the first level of the tree diagram.



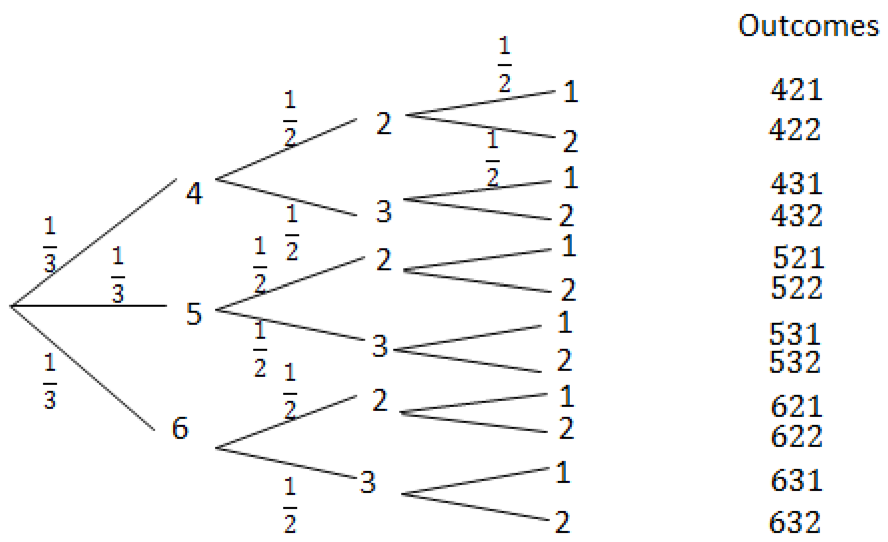
The second digit has two options, with a probability of $\frac{1}{2}$ for each of the branches.

Draw the second level of the tree diagram.



Finally, the third digit has two options with a probability of $\frac{1}{2}$ for each of the branches.

There are 12 total possible outcomes.



1. There is only one outcome with the number 631.
   $P(631) = \frac{1}{3} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{12}$
2. There are three outcomes with two 2s, $\{422;\ 522;\ 622\}$.

   $$\begin{aligned} P(\text{ 2 and 2}) &= \frac{1}{12} + \frac{1}{12} + \frac{1}{12} \\ &= \frac{3}{12} \\ &= \frac{1}{4} \end{aligned}$$
3. There are three numbers whose digits sum to eight, $\{422;\ 431;\ 521\}$.
   $$\begin{aligned} P(\text{sum of eight}) &= \frac{3}{12} \\ &= \frac{1}{4} \end{aligned}$$

In unit 1 you learnt about dependent and independent events. While tree diagrams can be used for both dependent and independent events, they are less useful for independent events since we can just multiply the probabilities of separate events to get the probability of the combined event as you saw in example 3.1. Remember that for independent events:

$P(\text{A and B}) = P(\text{A}) \cdot P(\text{B})$

Tree diagrams are very helpful for analysing dependent events. A tree diagram allows you to show how each possible outcome of one event affects the probabilities of the other events.

So if you already know that events are independent, it is usually easier to solve a problem without using tree diagrams. But if you are uncertain about whether events are independent, or if you know that they are not, you should use a tree diagram.

Example 3.2

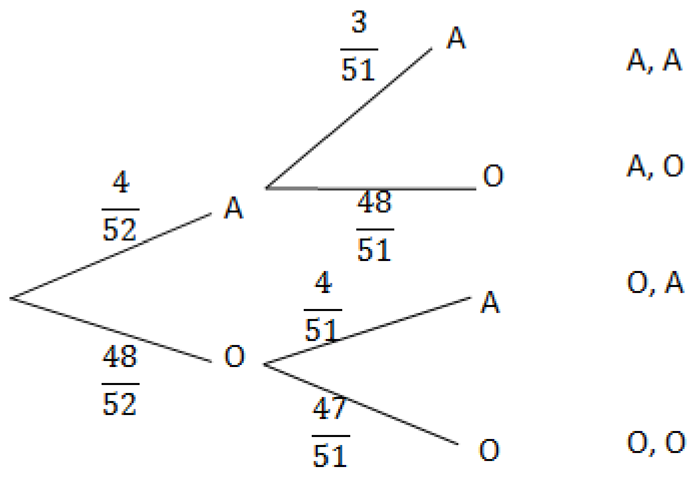
Jackie is a street magician and would like to draw aces one after the other from an ordinary deck of 52 playing cards. Use a tree diagram to find the probability he draws:

1. two aces
2. no aces
3. exactly one ace
4. at least one ace.

Solutions

Let A be the event drawing an ace. Let O be the event of getting any other card.

Since there are four aces in a deck of cards the probability of getting an ace on the first draw is $\frac{4}{52}$. Therefore the probability of getting any other card is $1 - \frac{4}{52} = \frac{48}{52}$.



After the first card is drawn there will be 51 cards left in the pack. If the first card drawn is an ace, then the probability of getting another ace is $\frac{3}{51}$ as shown in the second level of the tree diagram. If the first card drawn is any other card, then the probability of getting an ace on the second draw is $\frac{4}{51}$.

1.
$$\begin{aligned} P(\text{A and A}) &= \frac{4}{52} \cdot \frac{3}{51} \\ &= \frac{1}{221} \end{aligned}$$

2. The probability of getting no aces means that other cards were drawn both times.
$$\begin{aligned} P(\{\text{A}\}') &= P(\text{O and O}) \\ &= \frac{48}{52} \cdot \frac{47}{51} \\ &= \frac{188}{221} \end{aligned}$$

3. The probability of exactly one ace is the outcome (A,O) plus (O,A).

$$\begin{aligned} P(\text{exactly one A}) &= P(\text{A and O}) + P(\text{O and A}) \\ &= \frac{4}{52} \cdot \frac{48}{51} + \frac{48}{52} \cdot \frac{4}{51} \\ &= \frac{32}{221} \end{aligned}$$

4. The probability of at least one ace means one or more aces are drawn, which is the sum of 1. and 3. above.

$$\begin{aligned} P(\text{at least one ace}) &= \frac{1}{221} + \frac{32}{221} \\ &= \frac{33}{221} \end{aligned}$$

Exercise 3.1

1. What is the probability of throwing at least one five in three rolls of a regular six-sided die? Hint: do not show all possible outcomes of each roll of the die. We are interested in whether the outcome is five or not five only.
2. A bag contains 10 orange balls and seven black balls. You draw three balls from the bag without replacement. What is the probability that you will end up with exactly two orange balls? Represent this experiment using a tree diagram.
3. A person takes part in a medical trial that tests the effect of a medicine on a disease. Half the people are given medicine while the other half are given a sugar pill, which has no effect on the disease. The medicine has a 60% chance of curing the disease. But people who do not get the medicine still have a 10% chance of getting well. There are 50 people in the trial and they all have the disease. Tammy takes part in the trial, but we do not know whether she was given the medicine or the sugar pill. Draw a tree diagram of all the possible cases. What is the probability that Tammy gets cured?

The full solutions are at the end of the unit.

Summary

In this unit you have learnt the following:

- How to draw a tree diagram.
- When to use a tree diagram.
- How to calculate probabilities from a tree diagram.

Unit 3: Assessment

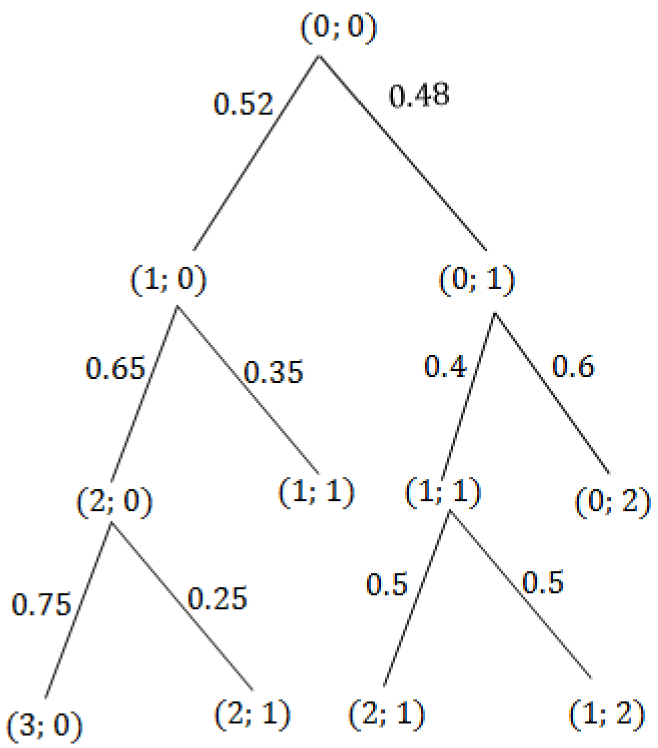
Suggested time to complete: 45 minutes

1. The probability that the floor of a supermarket will be wet when it opens in the morning is 30% and

there is a 10% probability of the floor being very wet. The probability that a person will slip and fall if the floor is dry is 12% and a person is three times more likely to fall if the floor is wet. If the floor is very wet, the probability that a person will fall is 0.6. Draw a tree diagram to represent the given information, showing the probabilities of each outcome, and use it to answer the following questions:

a. What is the probability that a person will fall on any given day?
b. What is the probability that a person will not fall on any given day?
c. Are the events of the floor being dry and a person falling independent? Justify your answer with a calculation.

2. Tandi has 10 bottles of nail polish in her handbag. Four of the bottles are red, and six are pink. She removes a bottle at random from the bag but does not replace it. She then chooses a second bottle at random and does not replace it and chooses a third bottle.
   a. Draw a tree diagram to show all possible outcomes.
   b. Determine the probability of her choosing a red bottle followed by another red bottle.
3. The following tree diagram represents points scored by two teams in a soccer game. At each level in the tree, the points are shown as (points for Team 1; points for Team 2).



Use this diagram to determine the probability that:

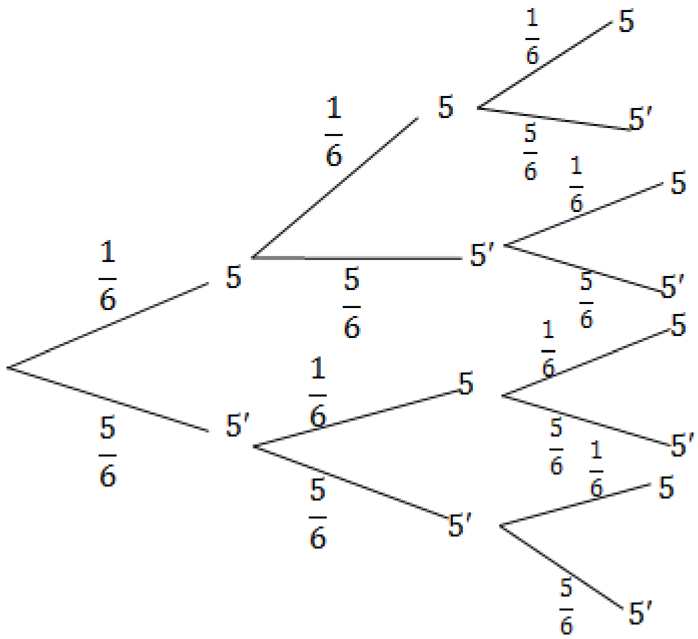
a. Team 1 will win
b. the game will be a draw
c. the game will end with an even number of total points.

The full solutions are at the end of the unit.

Unit 3: Solutions

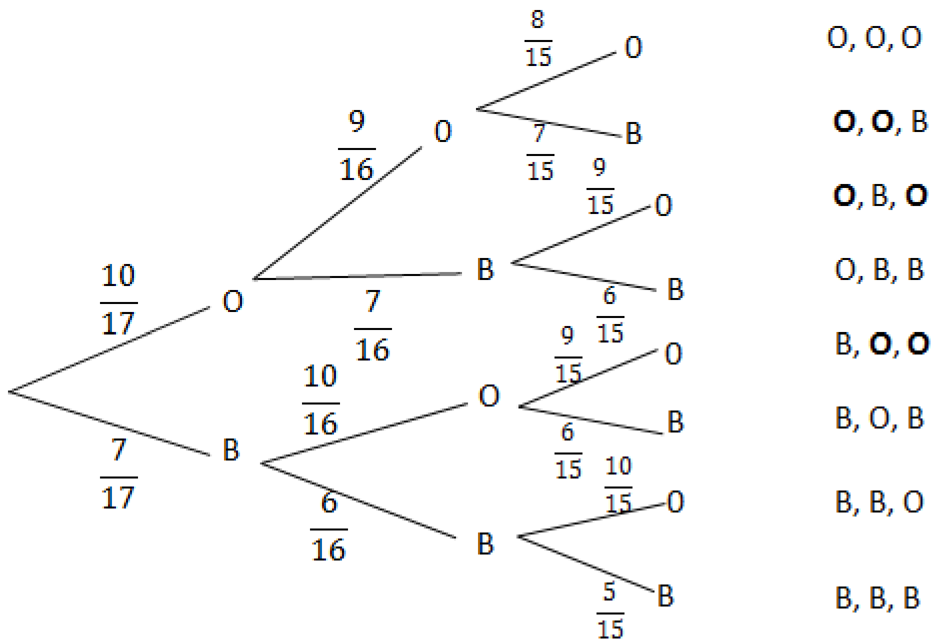
Exercise 3.1

1.



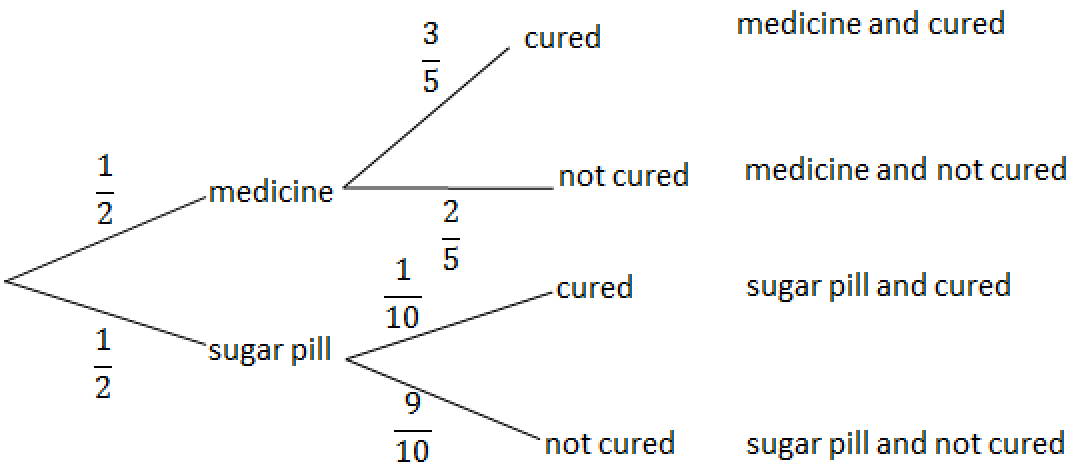
$$P(\text{at least one } 5) = P(5,5,5) + P(5,5,5') + P(5,5',5) + P(5,5',5') + P(5',5,5) + P(5',5,5') + P(5',5',5)$$
$$= \frac{1}{6}\cdot\frac{1}{6}\cdot\frac{1}{6} + \frac{1}{6}\cdot\frac{1}{6}\cdot\frac{5}{6} + \frac{1}{6}\cdot\frac{5}{6}\cdot\frac{1}{6} + \frac{1}{6}\cdot\frac{5}{6}\cdot\frac{5}{6} + \frac{5}{6}\cdot\frac{1}{6}\cdot\frac{1}{6} + \frac{5}{6}\cdot\frac{1}{6}\cdot\frac{5}{6} + \frac{5}{6}\cdot\frac{5}{6}\cdot\frac{1}{6}$$
$$= \frac{91}{216}$$

2. Let O be the event choosing an orange ball.
   Let B be the event choosing a black ball.



$$P(\text{exactly 2 orange balls}) = \left(\frac{10}{17}\cdot\frac{9}{16}\cdot\frac{7}{15}\right) + \left(\frac{10}{17}\cdot\frac{7}{16}\cdot\frac{9}{15}\right) + \left(\frac{7}{17}\cdot\frac{10}{16}\cdot\frac{9}{15}\right)$$
$$= \frac{21}{136} + \frac{21}{136} + \frac{21}{136}$$
$$= \frac{63}{136}$$

3. There are two uncertain events in this problem. Each person receives either medicine (probability $\frac{1}{2}$) or a sugar pill (probability $\frac{1}{2}$). Each person either gets cured (probability $\frac{3}{5}$ with medicine and $\frac{1}{10}$ without) or stays ill (probability $\frac{2}{5}$ with medicine and $\frac{9}{10}$ without).



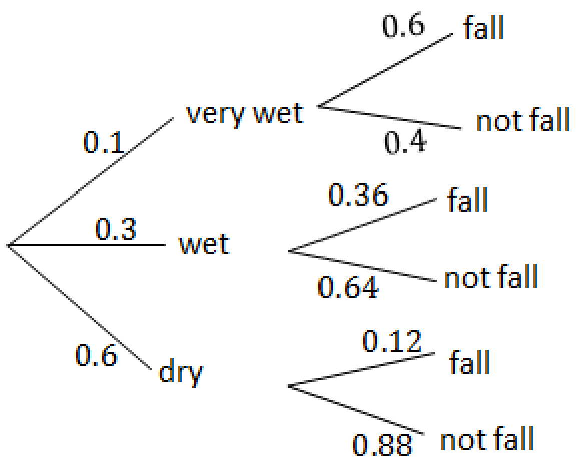
$$P(\text{cured}) = P(\text{medicine and cured}) + P(\text{sugar pill and cured})$$
$$= \frac{1}{2}\cdot\frac{3}{5} + \frac{1}{2}\cdot\frac{1}{10}$$
$$= \frac{7}{20}$$

The probability that Tammy is cured is $\frac{7}{20}$.

Back to Exercise 3.1

Unit 3: Assessment

1.



a.

$$\begin{aligned} P(\text{fall}) &= P(\text{very wet and fall}) + P(\text{wet and fall}) + P(\text{dry and fall}) \\ &= 0.1 \times 0.6 + 0.3 \times 0.36 + 0.6 \times 0.12 \\ &= 0.24 \end{aligned}$$

b.

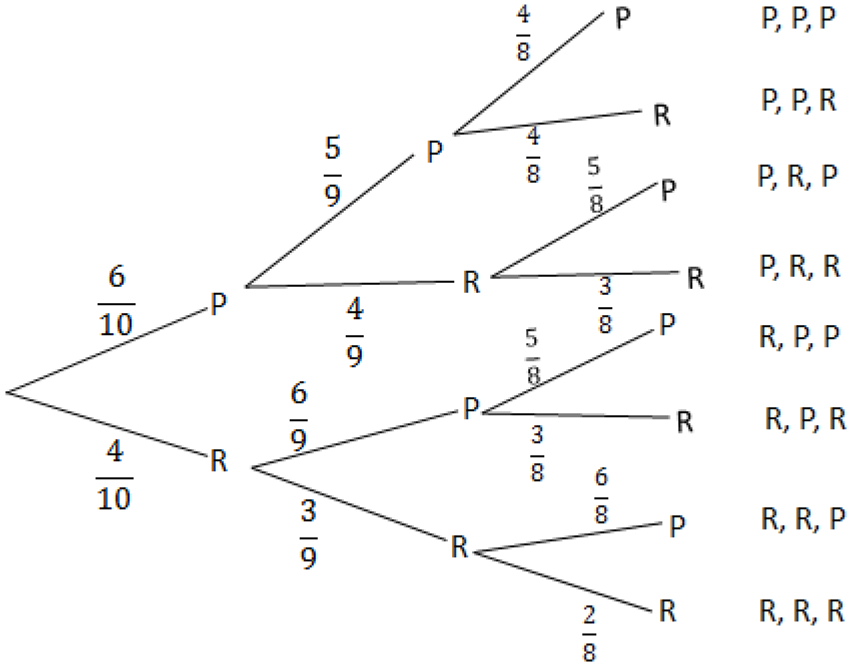
$$\begin{aligned} P(\text{not falling}) &= 1 - P(\text{fall}) \\ &= 1 - 0.24 \\ &= 0.76 \end{aligned}$$

c.

$$\begin{aligned} P(\text{dry and fall}) &= 0.072 \\ P(\text{dry}) \times P(\text{fall}) &= 0.6 \times 0.24 \\ &= 0.144 \\ P(\text{dry and fall}) &\neq P(\text{dry}) \times P(\text{fall}) \\ &\therefore \text{the floor being dry and a person falling are dependent events.} \end{aligned}$$

2.

a.



b.

$$\begin{aligned} P(\text{red followed by red}) &= P(\text{P, R, R}) + P(\text{R, R, P}) + P(\text{R, R, R}) \\ &= \left(\frac{6}{10} \cdot \frac{4}{9} \cdot \frac{3}{8}\right) + \left(\frac{4}{10} \cdot \frac{3}{9} \cdot \frac{6}{8}\right) + \left(\frac{4}{10} \cdot \frac{3}{9} \cdot \frac{2}{8}\right) \\ &= \frac{7}{30} \end{aligned}$$

3.

a.

$$\begin{aligned} P(\text{team one wins}) &= P(3;\ 0) + P(2;\ 1) + P(2;\ 1) \\ &= (0.52 \times 0.65 \times 0.75) + (0.52 \times 0.65 \times 0.25) + (0.48 \times 0.4 \times 0.5) \\ &= 0.434 \end{aligned}$$

b.

$$\begin{aligned} P(\text{draw}) &= P(1;\ 1) \\ &= 0.52 \times 0.35 \\ &= 0.182 \end{aligned}$$

c.

$$\begin{aligned} P(\text{even number}) &= P(1;\ 1) + P(0;\ 2) \\ &= 0.52 \times 0.35 + 0.48 \times 0.6 \\ &= 0.47 \end{aligned}$$

Back to Unit 3: Assessment

Media Attributions

Unit 4: Complete contingency tables to solve probability problems

NATASHIA BEARAM-EDMUNDS

Unit outcomes

By the end of this unit you will be able to:

- Draw and complete contingency tables.
- Use contingency tables to solve probability problems.

What you should know

Before you start this unit, make sure you can:

- Calculate the probability of equally likely outcomes. You can revise finding probabilities in unit 1 of this subject outcome.

Introduction

A two-way contingency table is also called a two-way table or a cross tabulation table. Contingency tables are used to examine relationships between categorical data. 'Contingency' means 'possibilities'. Contingency tables help you work out all the possible outcomes of combined events.

Constructing contingency tables

Contingency tables are especially helpful for figuring out whether events are dependent or independent.

In a two-way contingency table one event is written down the side of the table and the other event is written along the top of the table. The results are written in the cells of the table. The combined result of the activities is found by working two ways: across and then down.

We count the number of outcomes for two events and their complements, when working with two-way contingency tables, making four events in total. A two-way contingency table always shows the counts for the four possible combinations of events, as well as the totals for each event and its complement.

Example 4.1

A coin is tossed and a die is rolled simultaneously. Draw a table to show all possible pairs of outcomes.

From your table calculate the probability of getting a head and a two and state if these events are independent.

Solution

When you toss a coin there are two possible outcomes heads (H) or tails (T). When you roll a die there are six possible outcomes, $\{1;\ 2;\ 3;\ 4;\ 5;\ 6\}$.

We will draw up the contingency table with the event rolling a die at the top of the table and the event toss a coin along the side.

		Roll a die					
		1	2	3	4	5	6
Toss a coin	H	H; 1	H; 2	H; 3	H; 4	H; 5	H; 6
	T	T; 1	T; 2	T; 3	T; 4	T; 5	T; 6

In the cells of the table we list the possible combination of outcomes. There are 12 outcomes in total.

To find the probability of getting a head and a two, we find the row with H and the column with 2 and read across from the H and down from 2, the cell where they meet shows the outcome $\text{H};\ 2$. This outcome occurs once.

$$P(\text{H};\ 2) = \frac{1}{12}$$

For independent events:

$$P(\text{A and B}) = P(A) \cdot P(B)$$

From the table we see that:

$$\begin{aligned} P(\text{H}) &= \frac{6}{12} \\ &= \frac{1}{2} \end{aligned}$$

$$\begin{aligned} P(2) &= \frac{2}{12} \\ &= \frac{1}{6} \end{aligned}$$

$$\begin{aligned} P(\text{H}) \cdot P(2) &= \frac{1}{2} \cdot \frac{1}{6} \\ &= \frac{1}{12} \end{aligned}$$

We have shown that $P(\text{H};\ 2) = \frac{1}{12}$.

Since, $P(\text{H};\ 2) = P(\text{H}) \cdot P(2)$the events are independent.

Example 4.2

Example taken from from Siyavula Maths Grade 12

The table below shows the results of testing two different treatments on 240 fruit trees which have a disease causing the trees to die. Treatment A involves the careful removal of infected branches and treatment B involves removing infected branches as well as spraying the trees with antibiotics.

	Tree dies within four years	**Tree lives for more than four years**	**TOTAL**
Treatment A	70	50	
Treatment B			
TOTAL	90	150	

1. Fill in the missing values on the table.
2. What is the probability a tree received treatment B?
3. What is the probability that a tree will live beyond four years?
4. What is the probability that a tree is given treatment B and lives beyond four years?
5. Of the trees that were given treatment B, what is the probability that a tree lives beyond four years?
6. Are a tree given treatment B and living beyond four years independent events? Justify your answer with a calculation.

Solutions

1. Since each column has to add up to its total, we can work out the number of trees which fall into each category for treatments A and B. Then, we can add each row to get the totals on the right-hand side of the table.

	Tree dies within four years	**Tree lives for more than four years**	**TOTAL**
Treatment A	70	50	120
Treatment B	20	100	120
TOTAL	90	150	240

2. The probability that treatment B is given to a tree is the number of trees that received treatment B divided by the total number of trees.
$$\begin{aligned} P(\text{B}) &= \frac{120}{240} \\ &= \frac{1}{2} \end{aligned}$$
3. To find the probability that a tree lives beyond four years, we use the total of the column 'tree lives for more than four years' and divide that by the total number of trees.
$$\begin{aligned} P(\text{tree lives more than 4 years }) &= \frac{150}{240} \\ &= \frac{5}{8} \end{aligned}$$
4. To determine the probability that a tree receives treatment B and lives beyond four years, we must find the cell within the table that shows the combination of these events (row 3, column 3) and divide that by the total number of trees.

$$P(\text{B and tree lives more than 4 years }) = \frac{100}{240}$$
$$= \frac{5}{12}$$

5. Here, we are restricted to only the trees that received treatment B, living beyond four years. This means we no longer need to include the trees given treatment A, so the denominator needs to be adjusted accordingly.

$$P(\text{lives beyond 4 years having received B}) = \frac{100}{120}$$
$$= \frac{5}{6}$$

6.

$$P(\text{tree lives more than 4 years }) \times P(\text{B}) = \frac{5}{8} \times \frac{1}{2}$$
$$= \frac{5}{16}$$
$$P(\text{B and tree lives more than 4 years }) = \frac{5}{12}$$

So we see that $P(\text{B and tree lives more than 4 years }) \neq P(\text{B}) \cdot P(\text{lives more than 4 years})$. Therefore, the treatment of a tree with treatment B and living beyond four years are dependent events.

Exercise 4.1

1. Use the contingency table below to answer the following questions:

	Brown eyes	Not brown eyes	TOTAL
Black hair	50	30	80
Red hair	70	80	150
TOTAL	120	110	230

   a. What is the probability that someone with black hair has brown eyes?
   b. What is the probability that someone has black hair?
   c. What is the probability that someone has brown eyes?
   d. Are having black hair and having brown eyes dependent or independent events?
   e. What is the probability of having brown eyes or red hair?

2. You are given the following information:

   - Events A and B are independent.
   - $P(\text{not A}) = 0.3$
   - $P(\text{B}) = 0.4$

   Complete the contingency table below.

	A	Not A	TOTAL
B			
Not B			
TOTAL			50

3. A new treatment for influenza (the flu) was tested on a number of patients to determine if it was better than a placebo (a pill with no therapeutic value). The table below shows the results three days after treatment:

	Flu	No flu	TOTAL
Placebo	228	60	
Treatment			
TOTAL	240	312	

   a. Complete the table.
   b. Calculate the probability of a patient receiving the treatment.
   c. Calculate the probability of a patient having no flu after three days.
   d. Calculate the probability of a patient receiving the treatment and having no flu after three days.
   e. Using a calculation, determine whether a patient receiving the treatment and having no flu after three days are dependent or independent events.
   f. Calculate the probability that a patient receiving treatment will have no flu after three days.
   g. Calculate the probability that a patient receiving a placebo will have no flu after three days.
   h. Comparing you answers in f. and g., would you recommend the use of the new treatment for patients suffering from influenza?

The full solutions are at the end of the unit.

Summary

In this unit you have learnt the following:

- How to construct a two-way contingency table.
- How to complete a two-way contingency table.
- How to calculate probabilities from a contingency table.

Unit 4: Assessment

Suggested time to complete: 25 minutes

1. Researchers conducted a study to test how effective a certain inoculation is at preventing malaria. Part of their data is shown below:

	Malaria	No malaria	TOTAL
Male	A	B	216
Female	C	D	648
TOTAL	108	756	864

a. Calculate the probability that a randomly selected study participant will be female.
b. Calculate the probability that a randomly selected study participant will have malaria.
c. If being female and having malaria are independent events, calculate the value C.
d. Using the value of C, fill in the missing values on the table.

2. A rare kidney disease affects only one in $1\ 000$ people and the test for this disease has a 99% accuracy rate.
   a. Draw a two-way contingency table showing the results if $100\ 000$ of the general population are tested.
   b. Calculate the probability that a person who tests positive for this rare kidney disease is sick with the disease, correct to two decimal places.
3. The Clueless Club consists of 500 members. In order to be part of this club you have to be a lawyer, a teacher or an engineer. Given below is an incomplete contingency table that shows the distribution of 500 members, in terms of their profession and the type of hot drinks they drink.

	Tea	Coffee	Hot chocolate	TOTAL
Lawyer	52	41	80	173
Teacher	48	69	A	137
Engineer	100	B	10	190
TOTAL	200	190	110	500

   a. Determine the values of A and B.
   b. If a member is selected at random, what is the probability of selecting a teacher who drinks coffee?
   c. If a member is selected at random, what is the probability of selecting an engineer or a person who drinks hot chocolate?

The full solutions are at the end of the unit.

Unit 4: Solutions

Exercise 4.1

1.

a.

$$\begin{aligned} P(\text{with black hair has brown eyes}) &= \frac{50}{80} \\ &= \frac{5}{8} \end{aligned}$$

b.

$$P(\text{black hair}) = \frac{80}{230} = \frac{8}{23}$$

c.

$$P(\text{brown eyes}) = \frac{120}{230} = \frac{12}{23}$$

d.

$$P(\text{black hair}) \cdot P(\text{brown eyes}) = \frac{8}{23} \cdot \frac{12}{23} = \frac{96}{529}$$

$$P(\text{black hair and brown eyes}) = \frac{50}{230} = \frac{5}{23}$$

$$P(\text{black hair and brown eyes}) \neq P(\text{black hair}) \cdot P(\text{brown eyes})$$

The events are dependent.

e. The probability of having brown eyes or red hair is the union of the two events.

$$P(\text{brown eyes or red hair}) = P(\text{brown eyes}) + P(\text{red hair}) - P(\text{brown eyes and red hair}) = \frac{120}{230} + \frac{150}{230} - \frac{70}{230} = \frac{200}{230} = \frac{20}{23}$$

2.

$$P(\text{not A}) = 0.3$$
$$\therefore n(\text{not A}) = 0.3 \times 50 = 15$$
$$P(\text{B}) = 0.4$$
$$\therefore n(\text{B}) = 0.4 \times 50 = 20$$

$$P(\text{A and B}) = P(\text{A}) \cdot P(\text{B}) = \frac{35}{50} \times 0.4 = \frac{7}{25}$$
$$\therefore n(\text{A and B}) = \frac{7}{25} \times 50 = 14$$

	A	Not A	TOTAL
B	14	6	20
Not B	21	9	30
TOTAL	35	15	50

3.

a.

	Flu	No flu	TOTAL
Placebo	228	60	288
Treatment	12	252	264
TOTAL	240	312	552

b.

$$\begin{aligned} P(\text{receiving the treatment}) &= \frac{264}{552} \\ &= \frac{11}{23} \end{aligned}$$

c.

$$\begin{aligned} P(\text{no flu}) &= \frac{312}{552} \\ &= \frac{13}{23} \end{aligned}$$

d.

$$\begin{aligned} P(\text{receiving the treatment and having no flu}) &= \frac{252}{552} \\ &= \frac{21}{46} \end{aligned}$$

e.

$$\begin{aligned} P(\text{receiving the treatment}) \cdot P(\text{no flu}) &= \frac{11}{23} \cdot \frac{13}{23} \\ &= \frac{143}{529} \\ P(\text{receiving the treatment and having no flu}) &= \frac{21}{46} \end{aligned}$$

$P(\text{receiving the treatment and having no flu}) \neq P(\text{receiving the treatment}) \cdot P(\text{no flu})$

Therefore, a patient receiving the treatment and having no flu after three days are dependent events.

f.

$$\begin{aligned} P(\text{having no flu if received the treatment}) &= \frac{252}{264} \\ &= \frac{21}{22} \end{aligned}$$

g.

$$\begin{aligned} P(\text{having no flu if received the placebo}) &= \frac{60}{288} \\ &= \frac{5}{24} \end{aligned}$$

h. Yes, I would recommend the use of the new treatment for patients with the flu as there is a 95.5% chance of not getting the flu if they have been treated compared to the 20.8% of getting the flu if they were given the placebo.

Back to Exercise 4.1

Unit 4: Assessment

1.

a.

$$P(\text{female}) = \frac{648}{864}$$
$$= \frac{3}{4}$$

b.

$$P(\text{malaria}) = \frac{108}{864}$$
$$= \frac{1}{8}$$

c.

$$P(\text{female and malaria}) = P(\text{female}) \cdot P(\text{malaria}) \text{ since these are independent events}$$
$$= \frac{3}{4} \cdot \frac{1}{8}$$
$$\therefore \text{c} = \frac{3}{32} \times 864$$
$$= 81$$

d.

	Malaria	No malaria	TOTAL
Male	27	189	216
Female	81	567	648
TOTAL	108	756	864

2.

a.

$$\text{Number of people with disease } = \frac{1}{1\ 000} \times 100\ 000$$
$$= 100$$

Test for this disease has 99% accuracy rate:

$$\text{Number of people with disease and test is positive } = 0.99 \times 100$$
$$= 99$$
$$\text{Number of people with no disease and test is negative } = 0.99 \times 99\ 901$$
$$= 98\ 901$$

	Disease	No disease	TOTAL
Test is positive	99	999	1 098
Test is negative	1	98 901	98 902
TOTAL	100	99 900	100 000

b.

$$P(\text{have the disease if test is positive}) = \frac{99}{1\ 098}$$
$$= \frac{11}{122}$$
$$= 0.09$$

3.

a.

$$\text{A} = 137 - 48 - 69$$
$$= 20$$
$$\text{B} = 190 - 41 - 69$$
$$= 80$$

b.

$$P(\text{teacher and coffee}) = \frac{69}{500}$$

c.

$$\begin{aligned} P(\text{engineer or hot chocolate}) &= P(\text{engineer}) + P(\text{hot chocolate}) \\ &- P(\text{engineer and hot chocolate}) \\ &= \frac{190}{500} + \frac{110}{500} - \frac{10}{500} \\ &= \frac{290}{500} \\ &= \frac{29}{50} \end{aligned}$$

Back to Unit 4: Assessment

SUBJECT OUTCOME XIV

FINANCIAL MATHEMATICS: USE MATHEMATICS TO PLAN AND CONTROL FINANCIAL INSTRUMENTS

Subject outcome

Subject outcome 5.1: Use mathematics to plan and control financial instruments

Learning outcomes

- Use simple and compound growth formulae, $A = P(1 + in)$, $A = P(1 + i)^n$ and $A = P\left(1 + \frac{r}{100 \times m}\right)^{t \times m}$, to solve problems, including interest, hire-purchase and inflation.
- Understand, use and interpret tax tables.
- Use simple and compound decay formulae, $A = P(1 - in)$ and $A = P(1 - i)^n$, to solve problems (straight line depreciations and depreciation on a reducing balance).

Unit 1 outcomes

By the end of this unit you will be able to:

- Understand and apply the simple interest formula.
- Understand and apply the compound growth formulae.
- Understand and apply the compound growth formulae with interest compounded more than once a year.
- Calculate the values of A, P, i and n.

Unit 2 outcomes

By the end of this unit you will be able to:

- Understand the different tax categories.

- Use tax tables to answer questions.

Unit 3 outcomes

By the end of this unit you will be able to:

- Calculate straight-line depreciation.
- Calculate reducing-balance depreciation.
- Calculate the values of $A,\ P,\ i$ and n.

Unit 1: Work with simple and compound growth formulae

NATASHIA BEARAM-EDMUNDS

Unit outcomes

By the end of this unit you will be able to:

- Understand and apply the simple interest formula.
- Understand and apply the compound growth formulae.
- Understand and apply the compound growth formulae with interest compounded more than once a year.
- Calculate the values of A, P, i and n.

What you should know

Before you start this unit, make sure you can:

- do basic compound interest calculations. You can revise financial maths in level 3 subject outcome 5.1 and level 3 subject outcome 5.2.

Introduction

You have seen that financial mathematics has many practical applications. Some real-life examples include taking a bank loan, buying furniture on hire purchase and making monthly instalments on a car. You may have also heard of someone who lends money to others (referred to colloquially as 'Mashonisa').

What do all these situations have in common? Interest! Interest must be paid, or interest is being charged. Interest is earned by the lender and interest is paid by the borrower.

When a person borrows money they must pay an amount back, which includes interest added over time. Therefore, the cost of borrowing money is the interest you must pay back on the loan amount.

The benefit of investing money is the interest earned on the amount invested. When someone saves money in a bank account or invests money in an investment account, they earn interest at a given interest rate over the time that the money is saved or invested.

Let's have quick recap of the financial concepts you learnt in level 3. Remember that there are two different ways in which interest is calculated; simple interest or compound interest.

Simple interest

Simple interest is calculated on only the initial or principal amount invested or borrowed. The interest received or charged for each period will always be the same.

The formula for calculating simple interest is:

$A = P(1 + i \cdot n)$

$A\ = \text{accumulated(final) amount}$

$P = \text{initial or principal amount}$

$i = \text{interest rate (written as a decimal)}$

$n = \text{period (time in years for simple interest)}$

A hire purchase (HP) agreement, also known as an instalment plan, is an application of simple interest. In HP agreements a person agrees to buy an item at a certain interest rate over a stated period and will usually pay a deposit in order to secure the item.

Example 1.1

Ayesha buys a fridge on hire purchase for $\text{R}15\ 000$. She pays a 10% deposit. The store charges 20% simple interest per annum for two years. What are Ayesha's monthly instalments?

Solution

Write down the key information you have been given.

This is an HP agreement so we will use $A = P(1 + i \cdot n)$.

To work out Ayesha's monthly instalments we need to first calculate the accumulated amount after interest is added.

Deposit is 10% of $\text{R}15\ 000$.

$0.1 \times \text{R}15\ 000 = \text{R}1\ 500$

So the amount she must pay off is: $\text{R}15\ 000 - \text{R}1\ 500 = \text{R}13\ 500$.

$\therefore P = \text{R}13\ 500$

$$\begin{aligned} i &= \frac{20}{100} \\ &= 0.2 \end{aligned}$$

$n = 2$

Use your calculator and round off only at the very end of the question.

$$\begin{aligned} A &= 13\ 500(1 + (0.2) \cdot 2) \\ &= \text{R}18\ 900 \end{aligned}$$

To work out the monthly payments we must divide the final amount by the number of months in two years.

Monthly payments:

$$\frac{\text{R}18\ 900}{24} = \text{R}787.50$$

Exercise 1.1

1. Determine the value of an investment of $\text{R}13\ 000$ at 12% p.a. simple interest for three years.
2. The value of an investment grows from $\text{R}2\ 500$ to $\text{R}4\ 550$ in eight years. Determine the simple interest rate at which it was invested.
3. Jamie buys a sofa on an HP agreement to be paid off over three years. If the sofa costs $\text{R}6\ 000$ and he pays a deposit of 8%, and is charged 22% simple interest p.a., what are his monthly instalments? (Round off to the nearest rand).

The full solutions are at the end of the unit.

Compound growth

Compound growth allows interest to be earned on interest. The interest is calculated on the sum of the initial amount and the accumulated interest of an investment or loan. Compound interest makes the value of an investment or loan grow at a faster rate than simple interest does.

The formula for calculating compound growth is:

$A = P(1+i)^n$

$A\ = \text{accumulated(final) amount}$

$P = \text{initial or principal amount}$

$i = \text{interest rate (written as a decimal)}$

$n = \text{period}$

Let's look at some examples.

Example 1.2

Sandi wants to invest $\text{R}2\ 500$ for eight years. Safe Bank offers a savings account which pays simple interest at a rate of 15% per annum, and Buck Bank offers a savings account paying compound interest at a rate of 12% per annum. Which savings account would give Sandi the better bank balance at the end of the eight year period?

Solution

Final amount using the simple interest formula at Safe Bank:

$A = P(1 + i \cdot n)$

$$\begin{aligned} A &= 2\ 500(1+(0.15)(8)) \\ &= \text{R}5\ 500 \end{aligned}$$

Final amount using the compound interest formula at Buck Bank:

$A = P(1+i)^n$

$$\begin{aligned} A &= 2\ 500(1+(0.12))^8 \\ &= \text{R}6\ 190 \end{aligned}$$

The Buck Bank savings account would give Sandi a better bank balance at the end of the 8 year period.

Example 1.3

James decides to open an investment account with $\text{R}25\ 000$. What compound interest rate must the investment account achieve for him to double his money in 10 years? Give your answer correct to one decimal place.

Solution

Step 1: Write down the known variables and the compound interest formula

$$\begin{aligned} A &= 25\ 000 \times 2 \\ &= \text{R}50\ 000 \\ P &= \text{R}25\ 000 \\ n &= 10 \end{aligned}$$

$A = P(1+i)^n$

Step 2: Substitute the values and solve for i

$$\begin{aligned} 50\ 000 &= 25\ 000(1+i)^{10} \\ (1+i)^{10} &= 2 \\ i &= \sqrt[10]{2} - 1 \\ &= 0.0717\ldots \end{aligned}$$

We must round up to a rate of 7.2% p.a. so that James doubles his investment in the period.

Example 1.4

Calculate how much interest John will earn if he invests $\text{R}2\ 000$ for 4 years at 5.4% p.a. compound interest.

Solution

Step 1: Write down the known variables and the compound interest formula

$$\begin{aligned} P &= \text{R}2\ 000 \\ n &= 4 \\ i &= 0.054 \\ A &= ? \end{aligned}$$

$A = P(1+i)^n$

Step 2: Substitute the values and solve for A

$$\begin{aligned} A &= 2\ 000(1+0.054)^4 \\ &= \text{R}2\ 468.27 \end{aligned}$$

Step 3: Calculate the interest earned

$$\begin{aligned} \text{Interest} &= A - P \\ &= \text{R}2\ 468.27 - 2000 \\ &= \text{R}468.27 \end{aligned}$$

John earns $\text{R}468.27$ interest over the four years.

Exercise 1.2

1. Calculate the value of $\text{R}6\ 500$ invested at 8.6% p.a. compound interest for five years.
2. Bongani invested $\text{R}10\ 000$ for six years. If the value of his investment is $\text{R}18\ 500$, what compound interest rate did it earn?
3. If an investment is worth $\text{R}22\ 000$ at the end of five years at a compound interest rate of 11.5%, what was the initial amount invested?

The full solutions are at the end of the unit.

Compounding period

So far, we have looked at compound growth using annual interest rates only. However, interest is often compounded more than once a year. The compounding period tells us the number of times interest is charged or earned in a year.

Interest can be compounded daily, weekly, monthly, quarterly, half-yearly (semi-annually or biannually), annually, biennially (every 2 years) or even continuously. Generally, regardless of the compounding period, the interest rate is stated as an annual rate also called the **nominal rate**.

When the compounding period is not annual we must adjust the compound interest formula to reflect the different compounding period. We make the following adjustments to the compound interest formula:

$$A = P(1+\frac{i}{m})^{t\times m}$$

To take into account compounding that occurs more than once per annum, let's say m times a year, we multiply the number of years t in the compound interest formula by m and we also divide the interest rate i by m.

Some of the common compounding periods are listed below. For example, if compounding is half- yearly, interest is added two times in the year.

COMPOUNDING PERIOD	VALUE OF m
Monthly	12
Half-yearly/ biannually	2
Quarterly	4
Weekly	52
Daily	365
Biennially	$\frac{1}{2}$

Example 1.5

You invest $\text{R}6\ 000$ at 9% p.a. compounded monthly. After seven years you withdraw the full amount. How much will you be able to withdraw in total?

Solution

Since compounding occurs monthly, we must multiply t by 12, as there are 12 months in a year, and divide i by 12 in the compound interest formula.

$$\begin{aligned} A &= P(1 + \frac{i}{m})^{t \times m} \\ &= 6\ 000(1 + \frac{0.09}{12})^{7 \times 12} \\ &= 6\ 000(1.0075)^{84} \\ &= \text{R}11\ 239 \end{aligned}$$

You will be able to withdraw $\text{R}11\ 239$ at the end of seven years.

The amount of compound interest earned on an investment or paid for a loan depends on the frequency of compounding; the higher the number of compounding periods, the greater the compound interest.

Example 1.6

Nimrod and Mike each invest $\text{R}10\ 000$. Nimrod invests his money for five years at 8% p.a. compounded monthly. Mike invests his $\text{R}10\ 000$ for five years at 8% p.a. compounded half-yearly.
What is the difference in their accrued amounts at the end of the five-year period?

Solution

Nimrod:

$$A = P(1 + \frac{i}{m})^{t \times m}$$
$$= 10\ 000(1 + \frac{0.08}{12})^{5 \times 12}$$
$$= 10\ 000(1 + \frac{0.08}{12})^{60}$$
$$= \text{R}14\ 898$$

Mike:

$$A = P(1 + \frac{i}{m})^{t \times m}$$
$$= 10\ 000(1 + \frac{0.08}{2})^{5 \times 2}$$
$$= 10\ 000(1.04)^{10}$$
$$= \text{R}14\ 802$$

Nimrod earns $\text{R}14\ 898 - \text{R}14\ 802 = \text{R}96$ more than Mike. This makes sense since his investment has more frequent compounding than Mike's.

Example 1.7

Sarah invests $\text{R}5\ 000$ at 6.5% p.a. compounded monthly. Two years later she adds $\text{R}3\ 000$ to the savings account. Calculate the amount in her account six years after she invested the first amount.

Solution

First, calculate the amount she has at the end of two years:

$$A = 5\ 000(1 + \frac{0.065}{12})^{2 \times 12}$$
$$= \text{R}5\ 692.144665$$

Next, add $\text{R}3\ 000$ to the accumulated amount.

$\text{R}3\ 000 + \text{R}5\ 692.144665 = \text{R}8\ 692...$

The new amount will stay in her account for four more years at the same interest rate. So final amount will be:

$$A = 8\ 692...(1 + \frac{0.065}{12})^{4 \times 12}.$$
$$= \text{R}11\ 265.20$$

Exercise 1.3

1. Ntombi opens accounts at a number of clothing stores and spends freely. She gets herself into terrible debt and she cannot pay off her accounts. She owes Fashion World $\text{R}5\ 000$ and the shop

agrees to let her pay the bill at a nominal interest rate of 24% compounded monthly.

a. How much money will she owe Fashion World after two years?

b. What are her monthly instalments?

2. Jackson invests $\text{R}20\ 000$ for 18 months at 10% interest compounded quarterly. Calculate how much money he will have at the end of the period.
3. A financial advisor promises that she will treble the value of an investment at the end of six years. If the interest rate is fixed and compounded monthly, calculate the annual rate of interest that she offers.

The full solutions are at the end of the unit.

The effect of inflation

The price of a dozen eggs now is not the same as the price paid for a dozen eggs ten years ago. We all know that over time the prices of things increase. The average increase in the price of goods over time is called **inflation**.

Inflation changes continuously. The rate of inflation is quoted as a percentage per annum. We can use the compound growth formula to calculate the increase in the price of goods if we know the inflation rate.

Inflation affects the rate of interest that you receive from an investment or pay on a loan. If interest rates increase, money invested will increase. Similarly, interest on loans will increase and people will have to pay back more money.

If interest rates decrease you receive less on investments and loans will become cheaper.

Example 1.8

The average rate of inflation over the past five years was 4.5% per annum. The current price of a dozen eggs is $\text{R}35$.

1. How much did a dozen eggs cost five years ago?
2. Calculate the expected price of a dozen eggs in three years' time if the average rate of inflation stays the same.

Solutions

1. The current price of a dozen eggs is $\text{R}35 \therefore A = 35$ if we are looking back.

$$i = 0.045,\ n = 5$$
$$A = P(1.045)^5$$
$$35 = P(1.045)^5$$
$$P = \text{R}28.09$$

2. The current price of a dozen eggs is $\text{R}35 \therefore P = 35$ since we are looking forward in time.

$$i = 0.045,\ n = 3$$
$$A = 35(1.045)^3$$
$$= \text{R}39.94$$

Exercise 1.4

1. A painting costs $\text{R}25\ 000$ in April 2021. If the price of the painting increases at an average rate of 6.5% p.a. How much will the painting be worth in April 2030?
2. Kate's monthly salary is $\text{R}15\ 000$. Her monthly mortgage loan payment is $\text{R}4\ 500$. Her salary will increase this year by 6% and the interest rate on her mortgage will increase by 1%. Calculate:
   a. Her new monthly salary.
   b. The increase in her monthly loan payment.
3. David plans to replace his motorbike in five years' time with a similar make and type of motorbike. If his motorbike is currently worth $\text{R}109\ 000$ and prices of motorbikes increase by 7.5% a year, how much will a new bike cost in five years' time?

The full solutions are at the end of the unit.

Summary

In this unit you have learnt the following:

- How to calculate simple interest.
- How to calculate compound growth.
- How to calculate compound growth with different compounding periods.

Unit 1: Assessment

Suggested time to complete: 20 minutes

1. Greg enters into a five -year hire-purchase agreement to buy a computer for $\text{R}8\ 900$. The interest rate is quoted as 11% per annum based on simple interest. Calculate the required monthly payment for this contract.
2. Mrs. Brown retired and received a lump sum of $\text{R}200\ 000$. She deposited the money in a fixed deposit savings account for six years. At the end of the six years the value of the investment was $\text{R}265\ 000$. If the interest on her investment was compounded monthly, determine the nominal interest rate.
3. $\text{R}5\ 500$ is invested for a period of four years in a savings account. For the first year, the investment grows at a simple interest rate of 11%p.a. and then at a rate of 12.5%p.a. compounded quarterly for the rest of the period. Determine the value of the investment at the end of the four years.
4. It costs $\text{R}32\ 000$ per year to attend a private college. Determine the expected cost to study at this college in 10 years' time if the fees increase with inflation at a rate of 5% p.a.

The full solutions are at the end of the unit.

Unit 1: Solutions

Exercise 1.1

1.

$$\begin{aligned} A &= 13\ 000(1 + 0.12 \times 3) \\ &= \text{R}17\ 680 \end{aligned}$$

2.

$$\begin{aligned} 4\ 550 &= 2\ 500(1 + i \times 8) \\ (1 + i \times 8) &= \frac{4\ 550}{2\ 500} \\ i &= \left(\frac{4\ 550}{2\ 500} - 1\right) \div 8 \\ &= 0.1025 \\ \text{Interest rate} &= 0.1025 \times 100 \\ &= 10.25\% \end{aligned}$$

3.

$$\begin{aligned} \text{Deposit} &= 0.08 \times 6\ 000 \\ &= \text{R}480 \\ P &= 6\ 000 - 480 \\ &= \text{R}5\ 520 \\ A &= \text{R}5\ 520(1 + 0.22(3)) \\ &= \text{R}9\ 163.20 \\ \text{monthly pmts} &= \text{R}9\ 163.20 \div 36 \\ &= \text{R}255 \end{aligned}$$

Back to Exercise 1.1

Exercise 1.2

1.

$$\begin{aligned} A &= 6\ 500(1 + 0.086)^5 \\ &= \text{R}9\ 818.89 \end{aligned}$$

2.

$$\begin{aligned} 18\ 500 &= 10\ 000(1 + i)^6 \\ i &= \sqrt[6]{\frac{18\ 500}{10\ 000}} - 1 \\ &= 0.10797... \end{aligned}$$

Compound interest rate of 10.8%

3.

$$\begin{aligned} 22\ 000 &= P(1 + 0.115)^5 \\ P &= \text{R}12\ 765.81 \end{aligned}$$

Back to Exercise 1.2

Exercise 1.3

1.

a.

$$A = 5\ 000(1 + \frac{0.24}{12})^{2 \times 12}$$
$$= \text{R}8\ 042.19$$

b. Monthly instalments: $\text{R}8\ 042.19 \div 24 = \text{R}335.09$

2. Change 18 months to 1.5 years first, and then apply the compound interest formula taking into account the compounding period.

$$A = 20\ 000(1 + \frac{0.1}{4})^{1.5 \times 4}$$
$$= \text{R}23\ 193.87$$

3.

$$A = 3P$$
$$3P = P(1 + \frac{i}{12})^{6 \times 12}$$
$$(1 + \frac{i}{12})^{72} = 3$$
$$i = 12(\sqrt[72]{3} - 1)$$
$$= 0.1845...$$

Annual rate of interest that she offers is 18.45%.

Back to Exercise 1.3

Exercise 1.4

1.

$$A = 25\ 000(1 + 0.065)^9$$
$$= \text{R}44\ 064$$

2.

a.

Her new monthly salary:

$15\ 000 + 0.06(15\ 000) = \text{R}15\ 900$

OR

$\text{R}15\ 000 \times 1.06 = \text{R}15\ 900$

b. The increase in her monthly loan payment:

$\text{R}4\ 500 \times 0.01 = \text{R}45$

3. Cost of new bike:

$$\text{A} = 109\ 000(1{+}0.075)^5$$
$$= \text{R}156\ 484$$

Back to Exercise 1.4

Unit 1: Assessment

1.

$$A = 8\ 900(1 + 0.11 \times 5)$$
$$= \text{R}13\ 795$$

monthly payments:

$$\frac{\text{R}13\ 795}{60} = \text{R}229.92$$

2.

$$265\ 000 = 200\ 000(1 + \frac{i}{12})^{72}$$

$$(1 + \frac{i}{12})^{72} = \frac{53}{40}$$

$$i = 12\left(\sqrt[72]{\frac{53}{40}} - 1\right)$$

$$= 0.0469...$$

Nominal interest rate is 4.7%

3.

$$A = 5\ 500(1 + 0.11)(1 + \frac{0.125}{4})^{3 \times 4}$$

$$= \text{R}8\ 831.88$$

4.

$$A = 32\ 000(1 + 0.05)^{10}$$

$$= \text{R}52\ 124.63$$

Back to Unit 1: Assessment

Unit 2: Interpret tax tables

NATASHIA BEARAM-EDMUNDS

Unit outcomes

By the end of this unit you will be able to:

- Understand the different tax categories.
- Use tax tables to answer questions.

What you should know

There is no prior knowledge required for this unit.

Introduction

Tax is a compulsory financial charge that government imposes on individuals, companies and other organisations for goods and services. Generally, the money received from taxation is used for government spending and public services.

The main forms of taxation are:

- Income tax
- Corporate tax
- Capital gains tax
- Value-added tax (VAT) and
- Property taxes.

The South African Revenue Service (SARS) is responsible for collecting taxes in South Africa.

Note

You can read more about taxation in South Africa online.

Tax brackets

Most people, even those on social grants, pay taxes. Every time you buy an item, VAT, currently at a rate of 15%, is added to the total cost.

Personal income tax is South Africa's largest source of government revenue. Any person who receives an income within South Africa must be registered as a tax payer. Taxes are charged at different rates, depending on the income received by an individual or business.

Each year the Minister of Finance reviews and announces tax brackets and thresholds. Tax thresholds show the amount of income a person must receive to be liable to pay income tax.

SARS issues income tax return forms each year to those liable to pay tax. Tax returns must be completed and submitted through e-filing or manually at a SARS office to show income received. A tax return covers a financial tax year, which starts on 1 March and ends on the last day of February the next year.

Here is an example of the income tax brackets for individuals and Trusts for 2020/2021.

Taxable income	**Rate of tax**
$0 - \text{R}205\ 900$	18% of taxable income
$\text{R}205\ 901 - \text{R}321\ 600$	$\text{R}37\ 062 + 26\%$ of taxable income above $\text{R}205\ 900$
$\text{R}321\ 601 - \text{R}445\ 100$	$\text{R}67\ 144 + 31\%$ of taxable income above $\text{R}321\ 600$
$\text{R}445\ 101 - \text{R}584\ 200$	$\text{R}105\ 429 + 36\%$ of taxable income above $\text{R}445\ 100$
$\text{R}584\ 201 - \text{R}744\ 800$	$\text{R}155\ 505 + 39\%$ of taxable income above $\text{R}584\ 200$
$\text{R}744\ 801 - \text{R}1\ 577\ 300$	$\text{R}218\ 139 + 41\%$ of taxable income above $\text{R}744\ 800$
$\text{R}1\ 577\ 301$ and above	$\text{R}559\ 464 + 45\%$ of taxable income above $\text{R}1\ 577\ 300$

Note

Tax evasion is an illegal activity in which an individual or entity deliberately avoids paying a tax liability. Tax evasion often means taxpayers knowingly misrepresent the true state of their financial affairs to reduce their tax liability, and it includes dishonest tax reporting, such as declaring less income, profits or gains than the amounts actually earned, or overstating deductions. Those caught evading taxes are generally subject to criminal charges and substantial penalties and could face time in prison.

Income and deductions

Gross income is the money you earn before any deductions are made. To get to **taxable income** some deductions are allowed from gross income. These include pension or retirement contributions, donations and **interest received**.

Interest earned from investments or savings is treated as part of the taxpayer's total taxable income, with the following exemptions:

- For persons younger than 65, $\text{R}23\ 800$ of interest earned per annum is exempt from taxation
- For persons 65 years or older, $\text{R}34\ 500$ of interest earned per annum is exempt from taxation.

Calculating tax payable

An individual taxpayer is entitled to deductions that are subtracted from the tax payable. These are called tax rebates.

For example, for the 2022 financial year the following **tax** rebates are applicable:

- Primary rebate: R15 714 for all natural persons under 65 years old.
- Secondary rebate: R8 613 if the taxpayer is over 65 years old.
- Tertiary rebate: R2 871 if the taxpayer is 75 years of age or over.

A **medical tax credit** is available for taxpayers who pay medical scheme contributions. This rebate has replaced the medical expense deduction from gross income that was previously used up until 2012. Be careful when you revise this section from textbooks and exams published before 2012, as they will still deduct medical expenses using the old system. Calculations are now based on a fixed rate, and take into account the number of dependants covered by the scheme fees.

For the tax year commencing on 1 March 2021, the monthly rebates for medical scheme contributions were as follows:

- R332 for taxpayer.
- R332 for first dependant.
- R224 for each additional dependant.

Note

For more detail, the 2021 Tax Guide from the South African National Treasury can be found online.

Example 2.1

Onias is 30 years old and earned a gross salary of R15 500 per month before tax. He also received a 13th cheque as a bonus, which is equivalent to one month's salary. He received R500 interest from his savings account. No pension contributions were made. Use the 2020/2021 tax tables to calculate the following:

1. His gross income.
2. His taxable income.
3. The tax rate he must pay according to the tax table.
4. The tax payable for the year.

Solutions

1.

$$\begin{aligned}\text{Gross income} &= \text{yearly salary} + \text{bonus} + \text{interest}\\ &= 15\ 500 \times 12 + 15\ 500 + 500\\ &= \text{R}202\ 000\end{aligned}$$

2.

$$\begin{aligned}\text{Taxable income} &= \text{Gross income} - \text{interest less than R23 800} \ - \ \text{pension contributions}\\ &= \text{R}202\ 000 - \text{R}500\\ &= \text{R}201\ 500\end{aligned}$$

3.

Taxable income	Rate of tax
$0 - \text{R}205\ 900$	18% of taxable income
$\text{R}205\ 901 - \text{R}321\ 600$	$\text{R}37\ 062 + 26\%$ of taxable income above $\text{R}205\ 900$
$\text{R}321\ 601 - \text{R}445\ 100$	$\text{R}67\ 144 + 31\%$ of taxable income above $\text{R}321\ 600$
$\text{R}445\ 101 - \text{R}584\ 200$	$\text{R}105\ 429 + 36\%$ of taxable income above $\text{R}445\ 100$
$\text{R}584\ 201 - \text{R}744\ 800$	$\text{R}155\ 505 + 39\%$ of taxable income above $\text{R}584\ 200$
$\text{R}744\ 801 - \text{R}1\ 577\ 300$	$\text{R}218\ 139 + 41\%$ of taxable income above $\text{R}744\ 800$
$\text{R}1\ 577\ 301$ and above	$\text{R}559\ 464 + 45\%$ of taxable income above $\text{R}1\ 577\ 300$

$$\begin{aligned}\text{Tax} &= \frac{18}{100} \times \text{R}201\ 500\\ &= \text{R}36\ 270\end{aligned}$$

4.

$$\begin{aligned}\text{Tax payable} &= \text{tax rate} - \ \text{primary rebate}\\ &= \text{R}36\ 270 - \text{R}15\ 714\\ &= \text{R}36\ 270 - \text{R}15\ 714\\ &= \text{R}20\ 556\end{aligned}$$

Exercise 2.1

Use the 2020/2021 tax table and rebates below to answer the questions.

Taxable income	Rate of tax
$0 - \text{R}205\ 900$	18% of taxable income
$\text{R}205\ 901 - \text{R}321\ 600$	$\text{R}37\ 062 + 26\%$ of taxable income above $\text{R}205\ 900$
$\text{R}321\ 601 - \text{R}445\ 100$	$\text{R}67\ 144 + 31\%$ of taxable income above $\text{R}321\ 600$
$\text{R}445\ 101 - \text{R}584\ 200$	$\text{R}105\ 429 + 36\%$ of taxable income above $\text{R}445\ 100$
$\text{R}584\ 201 - \text{R}744\ 800$	$\text{R}155\ 505 + 39\%$ of taxable income above $\text{R}584\ 200$
$\text{R}744\ 801 - \text{R}1\ 577\ 300$	$\text{R}218\ 139 + 41\%$ of taxable income above $\text{R}744\ 800$
$\text{R}1\ 577\ 301$ and above	$\text{R}559\ 464 + 45\%$ of taxable income above $\text{R}1\ 577\ 300$

- Primary rebate: $\text{R}15\ 714$.
- Secondary rebate: $\text{R}8\ 613$ if the taxpayer is over 65 years old.
- Tertiary rebate: $\text{R}2\ 871$ if the taxpayer is over 75 years old.

The tax credit granted per month for medical aid contributions is as follows:

- $\text{R}332$ for taxpayer.
- $\text{R}332$ for first dependant.
- $\text{R}224$ for each additional dependant.

1. Mr Davids is 48 years old and his gross salary is $\text{R}35\ 000$ per month. He received a yearly bonus of $\text{R}20\ 000$ and interest of $\text{R}2\ 000$ from his saving account. His medical aid contribution is $\text{R}1\ 500$ p.m. and he is the only member. Calculate:
   a. His gross income.
   b. His taxable income.
   c. His tax rate using the 2020/2021 tax table.
   d. The tax he must pay.
2. Andrea's gross salary is $\text{R}27\ 000$ p.m. and she was paid a half-yearly bonus of $\text{R}13\ 500$ and an end of the year bonus of $\text{R}13\ 500$. She received $\text{R}10\ 000$ interest from her investment account. She pays $\text{R}2\ 500$ every month to a medical scheme for herself and her daughter. Calculate the tax she must pay for the year.

The full solutions are at the end of the unit.

Summary

In this unit you have learnt the following:

- How to calculate tax rates from tax tables.
- How to calculate tax payable.

Unit 2: Assessment

Suggested time to complete: 20 minutes

Use the 2021 tax table to answer the questions.

Taxable income	**Rate of tax**
$0 - \text{R}205\ 900$	18% of taxable income
$\text{R}205\ 901 - \text{R}321\ 600$	$\text{R}37\ 062 + 26\%$ of taxable income above $\text{R}205\ 900$
$\text{R}321\ 601 - \text{R}445\ 100$	$\text{R}67\ 144 + 31\%$ of taxable income above $\text{R}321\ 600$
$\text{R}445\ 101 - \text{R}584\ 200$	$\text{R}105\ 429 + 36\%$ of taxable income above $\text{R}445\ 100$
$\text{R}584\ 201 - \text{R}744\ 800$	$\text{R}155\ 505 + 39\%$ of taxable income above $\text{R}584\ 200$
$\text{R}744\ 801 - \text{R}1\ 577\ 300$	$\text{R}218\ 139 + 41\%$ of taxable income above $\text{R}744\ 800$
$\text{R}1\ 577\ 301$ and above	$\text{R}559\ 464 + 45\%$ of taxable income above $\text{R}1\ 577\ 300$

Sipho is 50 years old. His gross salary is $\text{R}64\ 000$ per month. He received a bonus of $\text{R}35\ 000$ for the year. His medical aid contribution is $\text{R}3\ 000$ per month for both himself and his wife.

The tax credit granted per month for medical aid contributions is as follows:

- $\text{R}332$ for taxpayer
- $\text{R}332$ for first dependant
- $\text{R}224$ for each additional dependant.

1. Calculate Sipho's taxable income for the year.
2. Write down the tax bracket in which Sipho's taxable income falls.
3. Calculate Sipho's normal tax (before deducting rebates and credits).
4. Determine Sipho's tax liability for the year.
5. If Sipho's employer deducted $\text{R}19\ 200$ PAYE per month from his salary, calculate the amount he will either receive from SARS or the amount he will have to pay to SARS.

The full solutions are at the end of the unit.

Unit 2: Solutions

Exercise 2.1

1.
   a. Gross income:
   $$\begin{aligned}\text{Gross income } &= \text{Annual salary } + \text{ bonus } + \text{ interest} \\ &= 35\ 000 \times 12 + 20\ 000 + 2\ 000 \\ &= \text{R}442\ 000\end{aligned}$$
   b. Taxable income:

$$\begin{aligned} \text{Taxable income } &= \text{Gross income } - \text{ interest less than R23 800} \\ &= \text{R442 000} - \text{R2 000} \\ &= \text{R440 000} \end{aligned}$$

c. $\text{R67 144} + 31\%$ of taxable income above R321 600
   $\text{R440 000} - \text{R321 600}$
   $= \text{R118 400}$
   $0.31 \times \text{R118 400}$
   $= \text{R36 704}$
   Tax rate:
   $\text{R67 144} + \text{R36 704}$
   $= \text{R103 848}$
d. Tax payable:
   $$\begin{aligned} \text{Tax payable } &= \text{Tax rate } - \text{ primary rebate } - \text{ medical tax credit} \\ &= \text{R103 848} - \text{R15 714} - (\text{R332} \times 12) \\ &= \text{R84 150} \end{aligned}$$

2.
   Taxable income:
   $\text{R27 000} \times 12 + \text{R27 000} - \text{R10 000}$
   $= \text{R341 000}$
   Tax rate:
   $\text{R67 144} + 31\%$ of taxable income above R321 600
   $= \text{R67 144} + 0.31(19\ 400)$
   $= \text{R73 158}$

   The tax payable by Andrea is:
   $$\begin{aligned} \text{Tax payable } &= \text{Tax rate } - \text{ primary rebate } - \text{ medical tax credit (taxpayer and 1st dependant)} \\ &= \text{R 73 158} - \text{R 15 714} - [(\text{R 332} + \text{R 332}) \times 12] \\ &= \text{R 73 158} - \text{R 15 714} - [\text{R 7 968}] \\ &= \text{R 49 476} \end{aligned}$$

Back to Exercise 2.1

Unit 2: Assessment

1. Taxable income for the year:
   $\text{R64 000} \times 12 + \text{R35 000}$
   $= \text{R803 000}$
2. Tax bracket:
   $\text{R744 800} - \text{R1 577 300}$
3. $\text{R218 139} + 41\%$ of taxable income above R744 800
   $\text{R218 139} + 0.41(\text{R58 200})$
   $= \text{R218 139} + \text{R23 862}$
   $= \text{R242 001}$
4. Tax liability:
   $$\begin{aligned} \text{Tax liability } &= \text{Tax rate - primary rebate - medical tax credit} \\ &= \text{R242 001} - \text{R15 714} - (\text{R332+R332}) \times 12 \\ &= \text{R218 319} \end{aligned}$$
5. Total PAYE:
   $\text{R19 200} \times 12$
   $= \text{R230 400}$
   He will not have to pay and will receive money backs from SARS:

$$\text{R}218\ 319 - \text{R}230\ 400$$
$$= -\text{R}\ 12\ 081$$

Back to Unit 2: Assessment

Unit 3: Work with simple and compound depreciation

NATASHIA BEARAM-EDMUNDS

Unit outcomes

By the end of this unit you will be able to:

- Calculate straight-line depreciation.
- Calculate reducing-balance depreciation.
- Calculate the values of A, P, i and n.

What you should know

Before you start this unit, make sure you can:

- calculate simple and compound interest using the correct formulae. To revise simple and compound interest you can go over level 3 subject outcome 5.2 and unit 1 of this subject outcome.

Introduction

When the value of an asset such as a vehicle, computer or appliance decreases due to usage, we say it has depreciated. In other words, the asset has lost value over time. An investment or shares in the stock exchange can also depreciate.

The importance of calculating depreciation is that it affects tax calculations for businesses. Businesses treat the depreciation amounts as an expense, and thereby reduce their taxable income. A lower taxable income means that the business will pay less tax to the Revenue Service.

There are two methods to calculate depreciation: simple or **straight-line depreciation**, and compound or **reducing-balance depreciation**.

The following terminology is often used when discussing depreciation.

Decay: Another word to describe depreciation.

Book value: The value of an asset after depreciation is taken into account.

Scrap value: The book value of an asset at the end of its useful life. Also called the salvage value.

Note

To see how businesses deal with depreciation you can click on this link when you have internet access.

Straight-line depreciation

In straight-line depreciation the value of an asset depreciates by a constant amount each year. The amount of depreciation is calculated each year as a percentage of the original or principal value of the asset. The asset is then reduced by that amount every year. Straight-line depreciation can be represented by a straight line graph.

The formula for straight-line depreciation looks very similar to the simple interest formula with a negative sign in the bracket to show the effect of depreciation.

$A = P(1 - i \cdot n)$

$$\begin{aligned} A &= \text{book value or scrap value of an asset} \\ P &= \text{initial or principal amount} \\ i &= \text{depreciation rate (written in decimal form)} \\ n &= \text{period (in years)} \end{aligned}$$

Example 3.1

Linda's mum buys her a new car, which costs $\text{R}200\ 000$, for her 18th birthday. The car depreciates in value at a rate of 10% per annum simple depreciation.

1. How much will the car be worth on Linda's 21st birthday?
2. Calculate the amount of depreciation each year.

Solutions

1. Write down the simple depreciation formula, list the values that you are given and then solve for the unknown value.

   $A = P(1 - in)$

   $$\begin{aligned} P &= \text{R}200\ 000 \\ i &= 0.1 \\ n &= 21 - 18 \\ &= 3 \\ A &= ? \end{aligned}$$

$$A = 200\ 000(1 - (0.1) \cdot 3)$$
$$= \text{R}140\ 000$$

2. In three years' time, Linda's car will be worth $\text{R}140\ 000$. That is a decrease in total value of $\text{R}60\ 000$. Depreciation per year:
$$\frac{\text{R}60\ 000}{3} = \text{R}20\ 000.$$

Example 3.2

A computer was valued at $\text{R}16\ 500$ when it was bought. Four years later, its value had depreciated to an amount of $\text{R}10\ 200$. Determine the rate at which the value depreciated using straight-line depreciation.

Solution

Write down the simple depreciation formula and list the values that you are given.

$$A = 10\ 200$$
$$P = 16\ 500$$
$$n = 4$$
$$i = ?$$

$$10\ 200 = 16\ 500(1 - i(4))$$
$$i = \left(\frac{10\ 200}{16\ 500} - 1\right) \div -4$$
$$= 0.0954...$$

Rate of depreciation:

$$0.09545... \times 100$$
$$= 9.55\%$$

Exercise 3.1

1. A new smartphone costs $\text{R}19\ 000$ and depreciates at 22% p.a. on a straight-line basis. Determine the book value of the smartphone at the end of each year over a three-year period.
2. A car is valued at $\text{R}350\ 000$. If it depreciates at 15% p.a. using straight-line depreciation, calculate the value of the car after five years.
3. Seven years ago, Rocco's drum kit cost him $\text{R}12\ 500$. It has now been valued at $\text{R}2\ 300$. What rate of simple depreciation does this represent?

The full solutions are at the end of the unit.

Reducing-balance depreciation

Reducing-balance depreciation or compound depreciation results in a higher depreciation expense in the earlier years of ownership of an asset. Compound depreciation is based on an asset's previous value. Every period, the depreciation will be a percentage of the reduced value (reducing-balance) of the asset. The value of the asset will depreciate by smaller amounts each period. At the end of the period the asset will still have some value; its value will never depreciate to zero.

Compound depreciation can be represented by a decreasing exponential graph.

The formula for compound depreciation looks very similar to the compound interest formula with a negative sign in the bracket to show that the asset is depreciating.

$A = P(1 - i)^n$

$$\begin{aligned} A &= \text{book value or scrap value of an asset} \\ P &= \text{initial or principal amount} \\ i &= \text{depreciation rate (written in decimal form)} \\ n &= \text{period (in years)} \end{aligned}$$

Example 3.3

Linda's mum buys her a new car, which costs $\text{R}200\ 000$, for her 18th birthday. The car depreciates in value at a rate of 10% per annum using reducing-balance depreciation.

1. How much will the car be worth on Linda's 21st birthday?
2. Calculate the amount of depreciation each year.

Solutions

1. Write down the reducing-balance depreciation formula and list the values that you are given; then solve for the unknown value.
   $$\begin{aligned} A &= P(1 - i)^n \\ P &= 200\ 000 \\ i &= 0.1 \\ n &= 3 \\ A &= ? \\ A &= 200\ 000(1 - 0.1)^3 \\ &= \text{R}145\ 800 \end{aligned}$$
2. In three years' time, Linda's car will be worth $\text{R}145\ 800$. That is a decrease in total value of $\text{R}54\ 200$.
   Depreciation at the end of year 1: $0.1 \times \text{R}200\ 000 = \text{R}20\ 000$
   Depreciation at the end of year 2:
   The car is now worth $\text{R}200\ 000 - \text{R}20\ 000 = \text{R}180\ 000$, so we must calculate the depreciation on the reduced value of the car.
   $0.1 \times \text{R}180\ 000 = \text{R}18\ 000$
   Depreciation at the end of year 3:
   $0.1 \times \text{R}162\ 000 = \text{R}16\ 200$

Compared to straight-line depreciation, we can see that in reducing-balance depreciation the depreciation amount decreases each year as the value of the car decreases.

Example 3.4

Simon bought a washing machine two years ago for $\text{R}9\ 999$ and sold it now for $\text{R}5\ 500$. At what rate did the value of the washing machine depreciate on a reducing-balance method? Give your answer correct to two decimal places.

Solution

Write down known variables and the compound decay formula.

$$\begin{aligned} P &= 9\ 999 \\ i &= ? \\ n &= 2 \\ A &= 5\ 500 \end{aligned}$$

$$A = P(1-i)^n$$

Substitute the values and solve for i.

$$\begin{aligned} 5\ 500 &= 9\ 999(1-i)^2 \\ (1-i)^2 &= \frac{5\ 500}{9\ 999} \\ 1-i &= \sqrt{\frac{5\ 500}{9\ 999}} \\ i &= 1-\sqrt{\frac{5\ 500}{9\ 999}} \\ &= 0.258... \end{aligned}$$

Rate of depreciation:

$$\begin{aligned} &0.2583... \times 100 \\ &= 25.83\% \end{aligned}$$

The next example is **for enrichment** only and is not part of the curriculum.

Example 3.5

Sam's car cost $\text{R}210\ 000$. After how many years will it be valued at $\text{R}80\ 000$ assuming a reducing-balance rate of depreciation of 15%.

Note

Although solving for n (the period of depreciation) is not required in the Assessment Guidelines for this unit, questions of this type have appeared in some past examination papers. For some explanation of how logs (logarithms) work, you could use the internet to read about "Exponents and Logarithms".

Solution

Write down known variables and compound decay formula.

$$\begin{aligned} P &= 210\ 000 \\ i &= 0.15 \\ n &= ? \\ A &= 80\ 000 \\ A &= P(1-i)^n \end{aligned}$$

Substitute the known values and solve for n.

$$80\ 000 = 210\ 000(1-0.15)^n$$

We see that n is a power and we cannot make the bases on either side of the equal sign the same. So we must use logs to solve for n.

$$\begin{aligned} (0.85)^n &= \frac{80\ 000}{210\ 000} \\ \log(0.85)^n &= \log(\frac{8}{21}) \\ n\log 0.85 &= \log(\frac{8}{21}) \text{ since } \log x^n = n\log x \\ n &= \log(\frac{8}{21}) \div \log 0.85 \end{aligned}$$

Use your calculator for this calculation, as follows:

So $n = 5.938 = 6$ years

Exercise 3.2

1. The number of pelicans at the Berg River mouth is decreasing at a compound rate of 12% p.a. If there are currently $3\ 200$ pelicans in the wetlands of the Berg River mouth, what will the population be in five years?
2. The population of Bonduel decreases at a reducing-balance rate of 9.5% per annum as people migrate to the cities. Calculate the decrease in population over a period of five years if the initial

population was $2\ 178\ 000$.

3. After 15 years, an aeroplane is worth $\frac{1}{6}$ of its original value. What is the annual rate of depreciation?

Question 4 is for enrichment only

4. Andy's car cost $\text{R}300\ 000$. After how many years will it be valued at $\text{R}120\ 000$ assuming a reducing-balance rate of depreciation of 12%.

The full solutions are at the end of the unit.

Summary

In this unit you have learnt the following:

- How to calculate straight-line depreciation.
- How to calculate reducing-balance depreciation.
- How to calculate the unknown variables in depreciation questions.

Unit 3: Assessment

Suggested time to complete: 20 minutes

1. Fiona buys a DStv satellite dish for $\text{R}3\ 000$. Due to weathering, its value depreciates simply at 15% per annum. After how long will the satellite dish have a book value of zero?
2. Harry's grandpa is very fashionable. Harry wants to buy his grandpa's jacket for $\text{R}1\ 000$. His grandpa is quite pleased with the offer, seeing that it only depreciated at a rate of 3% per year using the straight-line method. Grandpa bought the jacket five years ago. What did grandpa pay for the jacket then?
3. Steven invested $\text{R}50\ 000$ in an investment scheme. His investment did not perform well and depreciated on a reducing balance basis at the rate of 6% per annum each year for the first five years. At the end of the five years he withdrew $\text{R}10\ 000$ for personal reasons. After that his investment grew at a rate of 12% p.a. compounded quarterly.
   a. Determine the value of Steven's investment at the end of five years, before he withdrew the $\text{R}10\ 000$.
   b. Determine the value of his investment at the end of 10 years.
4. A 20 kg watermelon consists of 98% water. If it is left outside in the sun it loses 3% of its water each day. How much does it weigh after a month of 31 days?

The full solutions are at the end of the unit.

Unit 3: Solutions

Exercise 3.1

1. Calculate depreciation amount first.
   Depreciation:
   $19\ 000 \times \frac{22}{100} = 4\ 180$
   Therefore the smartphone depreciates by $\text{R}4\ 180$ every year.
   Then find the value of the phone at the end of each year.
   Book value at end of year 1: $\text{R}19\ 000 - \text{R}4\ 180 = \text{R}14\ 820$
   Book value at end of year 2: $\text{R}14\ 820 - \text{R}4\ 180 = \text{R}10\ 640$
   Book value at end of year 3: $\text{R}10\ 640 - \text{R}4\ 180 = \text{R}6\ 460$
2. $$\begin{aligned} A &= 350\ 000(1 - 0.15 \times 5) \\ &= \text{R}87\ 500 \end{aligned}$$
3. $$\begin{aligned} 2\ 300 &= 12\ 500(1 - 7i) \\ \frac{2\ 300}{12\ 500} - 1 &= -7i \\ i &= 0.116... \end{aligned}$$
   Rate of depreciation is 11.66%

Back to Exercise 3.1

Exercise 3.2

1. $$\begin{aligned} A &= 3\ 200(1 - 0.12)^5 \\ &\approx 1689 \end{aligned}$$
2. $$\begin{aligned} A &= 2\ 178\ 000(1 - 0.095)^5 \\ &\approx 1\ 322\ 211 \end{aligned}$$
3. $$\begin{aligned} \frac{1}{6}P &= P(1 - i)^{15} \\ i &= 0.112... \end{aligned}$$
   Annual rate of depreciation was 11.26%.
4. $$\begin{aligned} (1 - 0.12)^n &= \frac{120\ 000}{300\ 000} \\ \log{(0.88)^n} &= \log(\frac{2}{5}) \\ n\log 0.88 &= \log(\frac{2}{5}) \text{ since } \log x^n = n \log x \\ n &= \log(\frac{2}{5}) \div \log 0.88 \\ &= 7.167... \\ \therefore n &\approx 7\frac{1}{6} \text{ years} \end{aligned}$$
   Note: You cannot round down to seven years as the car will be valued at over $\text{R}120\ 000$ at seven years. So you need to include the fractional part of the year into your answer. $0.167... \times 12 = 2$ months and

$\frac{2}{12} = \frac{1}{6}$ of a year.

Back to Exercise 3.2

Unit 3: Assessment

1. $$\begin{aligned} A &= P(1 - in) \\ 0 &= 3\ 000(1 - 0.15(n)) \\ 0 &= 3\ 000 - 450n \\ n &= \frac{20}{3} \\ &= 6\frac{2}{3} \\ &= 6 \text{ years and } 8 \text{ months} \end{aligned}$$
2. $$\begin{aligned} A &= P(1 - in) \\ P(1 - 0.03(5)) &= 1\ 000 \\ P &= \text{R}1\ 176.47 \end{aligned}$$
3.
   a. The value of Steven's investment at the end of five years, before he withdrew the $\text{R}10\ 000$:
   $$\begin{aligned} A &= P(1 - i)^n \\ &= 50\ 000(1 - 0.06)^5 \\ &= \text{R}36\ 695.20 \end{aligned}$$
   b. At the end of 10 years:
   New principal amount is $\text{R}36\ 695.20 - \text{R}10\ 000 = \text{R}26\ 695.20$
   $$\begin{aligned} A &= 26\ 695.20(1 + \frac{0.12}{4})^{5\times 4} \\ &= \text{R}48\ 214.50 \end{aligned}$$
4. The amount of water at the start: $0.98 \times 20 = 19.6$. Therefore, 0.4 kg makes up the remainder of the watermelon weight.
   At the end of 31 days the amount of water:
   $$\begin{aligned} A &= P(1 - i)^n \\ &= 19.6(1 - 0.03)^{31} \\ &= 7.62 \text{ kg} \end{aligned}$$
   Watermelon in total weighs:
   $7.6 \text{ kg} + 0.4 \text{ kg} = 8.02 \text{ kg}$

Back to Unit 3: Assessment

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